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ABOUT BELLMAN PRINCIPLE AND SOLUTION PROPERTIES OF NAVIER–STOKES EQUATIONS IN THE 3D CAUCHY PROBLEM

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About Bellman Principle and Solution Properties of Navier–Stokes Equations in the 3d Cauchy Problem

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Abstract: The main purpose of this article is to consider the smoothness control of a weak solution after some moment if there is known solution regularity until this moment. The necessary tools can be varied. It is possible to control kinetic energy dissipation to fix moment or changing of velocity square or summability of acceleration square to fix at point time.

Keywords: Navier-Stokes equations; blow up solution; regular solution

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1. Introduction

The well-known principle of dynamic programming (Bellman principle) postulates: whatever the state of the system at any step and the control selected at this step, subsequent controls should be selected optimal relative to the state to which the system will arrive at the end of this step.

We would like to get something similar or close to it for the smoothness properties of solutions in the Cauchy problem for the spatial Navier–Stokes equations.

I.e. we would like to find out when given solution smoothness over a short time interval (See, [1], [2]) and some important parameters (control parameters) of this solution by a given point in time until which smoothness is still preserved will be able to ensure the existence of a global smooth extension of this solution, or at least, of a local smooth extension for a some guaranteed longer time interval. The first steps in this way were taken in the work [8], where numerical parameters were introduced that indirectly control the smoothness of the solution over a longer time interval.

This point of view differs from the already classical methods in studying the smoothness properties of weak solutions (See, [1,2,5,6] and also [3]) where the main tools are connected with embedding lemmas and multiplicative inequalities. In the fact, all this is the main tool up to now.

Here, another interesting aspect should be noted. It is related to the asymptotics of smooth solutions at infinity and integral identities for solenoid fields (See, [7]). It may be also a new tool for another a' priory estimates. We should not forget about such facts at least because the problem of solution regularity attracts attention of many mathematicians.

2. Notations

We consider a motion of ideal incompressible fluid and the simplest problem that is Cauchy problem in space $(n = 3)$ which is described by equations:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i} = \nu \Delta u - \nabla P, \;\; div\, u = 0, \; u(0, x) = \varphi(x), \tag{1}$$

where $u = u(t, x) = (u_1(t, x),\ u_2(t, x),\ u_3(t, x))$

is a fluid velocity, t , $x = (x_1,\ x_2,\ x_3)$ are time and spatial variables respectively. A function $P = P(t, x)$ is pressure function and $\varphi(x) = (\varphi_1(x),\ \varphi_2(x),\ \varphi_3(x))$ is an initial data, ν is a viscosity coefficient.

By symbols $\triangle$ and ∇ we denote Laplace operator and gradient operator on spatial variables, respectively. In particular, ∇u is Jacobi matrix on spatial variables.

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In addition, we suppose a mapping $\varphi \in C^{\infty}_{6/5,3/2}(R^3)$, i.e. φ is infinitely differentiable mapping, it belongs to Lebesgue class $L_{6/5}(R^3)$, the first partial derivatives $\nabla\varphi \in L_{3/2}(R^3)$ and the rest derivatives belong to class $L_r(R^3)$ for any $r > 1$. (Class $C^{\infty}_{6/5,3/2}(R^3)$, its properties and usefulness are described in [8]). In particular, it implies the following inclusions $\varphi \in L_2(R^3), \nabla\varphi \in L_2(R^3)$ (See, [8], Lemma 32). For us this class is interesting only because for any fixed t the solution $u(t,\cdot)$ of the Cauchy problem (1) belongs to class $C^{\infty}_{6/5,3/2}(R^3)$ (See, [8], Theorems 2, 6).

Well-known classical results belonging to Ladyzhenskaya O. A. [1] and Serrin J. [2] show an existence of time interval $[0,T)$ where the solution of problem (1) is regular in zone $[0,T) \times R^3$. Denote by $T_\star$ a least upper bound of these T. If $T_\star < \infty$ then the solution of problem (1) is called a blow up solution and the fluid flow describing of this solution is called a turbulent flow.

Let us introduce the following notations:

$$|\varphi| = \sqrt{\sum_{i=1}^{3} \varphi_i^2},\ |\nabla\varphi| = \sqrt{\sum_{i=1}^{3} |\nabla\varphi_i|^2},$$

$$\|u(t,\cdot)\|_p^p = \int_{R^3} |u(t,x)|^p dx,\ \|\nabla u(t,\cdot)\|_p^p = \int_{R^3} |\nabla u(t,x)|^p dx,\ p > 1.$$

Following [8] (See, formulae (68), (69), (87), (5)), we define numerical parameters $\lambda, \mu, \varepsilon$ as parameters which control solution smoothness and a number T_0 by equalities:

$$l(\varphi) = \|\varphi\|_2 \cdot \|\nabla\varphi\|_2,\ \lambda = \left(\frac{4\sqrt[4]{3}}{3a_1}\right)^2 \frac{\nu^2}{l(\varphi)} = \frac{81\nu^2}{8l(\varphi)},\ \ \lambda(t) = \frac{81\nu^2}{8l(u(t,\cdot))}, \tag{2}$$

where $u(t,x)$ is solution of the Cauchy problem (1).

$$\mu = \frac{T_*}{T_0}, \tag{3}$$

$$\|u(T_0,\cdot)\|_2^2 = \|\varphi\|_2^2(1-\varepsilon\lambda^2), \tag{4}$$

$$T_0 = \left(\frac{9}{4}\right)^4 \frac{\nu^3}{\|\nabla\varphi\|_2^4}. \tag{5}$$

Here, $[0,T_0)$ is that time interval (it is not necessarily optimal) where every weak solution (See definition in [1],[3], [5]) of problem (1) is regular (i.e. smooth) and it satisfies condition

$$\|\nabla u(t,\cdot)\|_2^2 \le \frac{\|\nabla\varphi\|_2^2}{\sqrt{1-\frac{t}{T_0}}}.$$

If $T_\star$ is finite, (See, [8], Lemma 50, Theorems 6–7) then for these parameters there are fulfilled inequalities $\lambda < 1, 0 < \varepsilon < 1$ and

$$\frac{1}{4}(\varepsilon + \frac{1}{\varepsilon})^2 < \mu < \lambda^{-4}. \tag{6}$$

From Leray's estimates (See, [4]) it follows that every blow up solution of problem (1) satisfies condition

$$\int_0^{T_*} \|\nabla u(t,\cdot)\|_2^4 dt = +\infty. \tag{7}$$

for finite T_*. Nevertheless, this weak solution for every $T < T_*, T > 0$, satisfies inequality:

$$\int_0^{T} \|\nabla u(t,\cdot)\|_2^4 dt < +\infty, \tag{8}$$

which implies solution smoothness on set $[0,T] \times R^3$ (see, [2].)

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3. Main results

If initial data $\varphi \in C^{\infty}_{6/5,3/2}(R^3)$ and parameter $\lambda \geq 1$ then the Cauchy problem (1) has a global regular solution (See [8], Theorem 7). Therefore, the following two results are to compare with formula (7).

Theorem 1. *Let be $\varphi \in C^{\infty}_{6/5,3/2}(R^3)$ an initial data in the problem (1). Parameter $\lambda < 1$ and mean T_0 are defined by formulae (7) and (5) respectively. Suppose that*

$$\frac{1}{T_0}\int_0^{T_0} \|\nabla u(t,\cdot)\|_2^4 dt \leq \|\nabla \varphi\|_2^4 \ln\frac{1}{1-\lambda^4},$$

where u is a smooth solution of problem (1) on the time interval $[0, T_0)$. Then this solution has a global regular extension on the set $[0, \infty) \times R^3$. Moreover, the following estimates are fulfilled

$$\|\nabla u(T_0,\cdot)\|_2^2 \leq \frac{\lambda^2\|\nabla\varphi\|_2^2}{\sqrt{1-\lambda^4}},\ \lambda(T_0) > 1,$$

$$\|\nabla u(t,\cdot)\|_2^2 \leq \frac{\lambda^2(T_0)}{\lambda^2(T_0)-1}\|\nabla u(T_0,\cdot)\|_2^2$$

for all $t > T_0$, where $\lambda(T_0)$ is defined in formulae (2).

Proof. For the first time we note that there exists a number $\xi \in (0,\ T_0]$ satisfying inequality

$$\|\nabla u(\xi,\cdot)\|_2^2 \leq \frac{\lambda^2\|\nabla\varphi\|_2^2}{\sqrt{1-\lambda^4\frac{\xi}{T_0}}}. \tag{9}$$

In the fact, let us suppose the opposite. Then on the interval $[0,\ T_0]$ we have the following inequality

$$\|\nabla u(t,\cdot)\|_2^4 > \frac{\lambda^4\|\nabla\varphi\|_2^4}{1-\lambda^4\frac{t}{T_0}}.$$

Integrating it over this interval we obtain the estimate which contradicts theorem condition for mean value.

Therefore, (9) is true. Rewriting it by following way

$$\frac{T_0}{\lambda^4} \leq \xi + \frac{c\nu^3}{\|\nabla u(\xi,\cdot)\|_2^4} = \tau_1(\xi), c = \left(\frac{9}{4}\right)^4, \tag{10}$$

we note that for all $t \geq \xi, t < T_*$, $\tau_1(\xi) \leq \tau_1(t)$ because function τ_1 is not decreasing (See Lemma 45, formula (85) from [8]).

Therefore, if T_* is finite then we obtain inequality $\frac{1}{\lambda^4} \leq \mu$. This contradicts to (6).

Hence, $\mu = \infty$ and solution u of problem (1) is global and regular.

From (10) and function monotonicity τ_1 it follows immediately that for all $t, \xi \leq t < T_*$ the next inequality is fulfilled

$$\|\nabla u(t,\cdot)\|_2^2 \leq \frac{\lambda^2\|\nabla\varphi\|_2^2}{\sqrt{1-\lambda^4\frac{t}{T_0}}}. \tag{11}$$

Hence, we have the first inequality of Theorem.

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Let us prove the second estimate. Suppose the opposite. Then $\lambda(T_0) \leq 1$. In this case for solution $u(t,x)$ function $\tau_2(t) = \|u(t,\cdot)\|_2^2(\lambda^2(t)-1)$ is not decreasing (See inequality (77) from [8]). Hence, for $0 \leq t \leq T_0$ we obtain inequalities:

$$\|u(t,\cdot)\|_2^2(\lambda^2(t)-1) \leq \|u(T_0,\cdot)\|_2^2(\lambda^2(T_0)-1) \leq 0.$$

Then $\lambda(t) \leq 1$.

Hence, for all $0 \leq t \leq T_0$ we have estimate (constant c from (10)):

$$4c\nu^4 \leq \|u(t,\cdot)\|_2^2\|\nabla u(t,\cdot)\|_2^2. \tag{12}$$

It is the strong inequality in a some neighbourhood of point $t = 0$ because $\lambda(0) = \lambda < 1$ and functions $\eta_1(t) = \|\nabla u(t,\cdot)\|_2$, $\eta_4(t) = \|\nabla u(t,\cdot)\|_2$ (See Lemma 36 in [8]) are continuous.

Therefore, integrating (12) over interval $[0,T_0]$ we extract a strong estimate:

$$4c\nu^4 T_0 < \int_0^{T_0} \|u(t,\cdot)\|_2^2\|\nabla u(t,\cdot)\|_2^2 dt = \frac{1}{4\nu}(\|\varphi\|_2^4\| - \|u(T_0,\cdot)\|_2^4).$$

Hence

$$\|u(T_0,\cdot)\|_2^4 < \|\varphi\|_2^4(1-\lambda^4). \tag{13}$$

Apply this inequality and inequality (11) for $t = T_0$. Then from (12) we obtain the strong estimate

$$4c\nu^4 < \|\varphi\|_2^2\|\nabla\varphi\|_2^2\lambda^2 = 4c\nu^4.$$

Contradiction. The second inequality is proved.

The third estimate follows from Theorem 10 (See, [8]) if we consider $u(t+T_0,x)$ as the Cauchy problem solution with the initial data $u(T_0,x)$. Theorem 1 is proved. □

The following statement is connected with a local extension.

Theorem 2. *Let be $\varphi \in C^\infty_{6/5,3/2}(R^3)$ an initial data in the problem (1). Parameter $\lambda < 1$ and mean T_0 are defined by formulae (7) and (5) respectively. Suppose that*

$$\frac{1}{T_0}\int_0^{T_0} \|\nabla u(t,\cdot)\|_2^4 dt \leq \|\nabla\varphi\|_2^4 \ln\frac{1}{1-\lambda^2},$$

where u is a smooth solution of problem (1) on the time interval $[0,T_0)$. Then this solution has a local smooth extension on set $[0,\frac{T_0}{\lambda^2})\times R^3$. In addition, it is true the following estimate

$$\|\nabla u(t,\cdot)\|_2^2 \leq \frac{\lambda\|\nabla\varphi\|_2^2}{\sqrt{1-\lambda^2\frac{t}{T_0}}} \tag{14}$$

for every moment t, $T_0 \leq t < \frac{T_0}{\lambda^2}$.

Proof. This theorem is proved by analogy with Theorem 1. Here, there exists a number $\xi \in (0,\ T_0]$ satisfying condition

$$\|\nabla u(\xi,\cdot)\|_2^2 \leq \frac{\lambda\|\nabla\varphi\|_2^2}{\sqrt{1-\lambda^2\frac{\xi}{T_0}}}.$$

As in Theorem above we prove it from the opposite. The monotonicity of function τ_1 from Theorem 1 implies inequality:

$$\frac{T_0}{\lambda^2} \leq \xi + \frac{c\nu^3}{\|\nabla u(\xi,\cdot)\|_2^4} = \tau_1(\xi) \leq \tau_1(t),\ \xi < t. \tag{15}$$

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Hence, we have estimate (14). Theorem is proved. □

Remark 1. *The number* $\frac{T_0}{\lambda^2} = \frac{\|\varphi\|_2^2}{4\nu\|\nabla\varphi\|_2^2}$ *is interesting because it doesn't depend on from constants in a'priory estimates for solutions. This is the first. The second. It influences on estimates for kinetic energy of turbulence flows at moment close to initial (see, [9], Theorem 1).*

Theorem 3. *Let be* $\varphi \in C^{\infty}_{6/5,3/2}(R^3)$ *an initial data in the problem* (1)*. Parameter* $\lambda < 1$ *and* $\sqrt{2}\lambda^2 \geq 1$*. In addition, number* T_0 *, parameter* ε *are defined by formulae* (4) *and* (5) *respectively. Suppose* $\varepsilon \leq \sqrt{2} - 1$*. Then a weak solution* u *of problem* (1) *is global and regular.*

Proof. If solution u of problem (1) is blow up solution then from (6) and theorem conditions we obtain for parameter μ following inequalities: $\mu < \lambda^{-4} \leq 2$. Hence, and still one estimate from (6) we have inequality: $\frac{1}{2}(\varepsilon + \frac{1}{\varepsilon}) < \sqrt{2}$.

Therefore $\varepsilon > \sqrt{2} - 1$. Contradiction. □

Theorem 4. *Let be* $\varphi \in C^{\infty}_{6/5,3/2}(R^3)$ *an initial data in the problem* (1)*. Parameters* $\lambda < 1$ *and* ε *are defined by formulae* (2) *and* (5) *respectively. Suppose that number* T_0 *from* (4) *and a weak solution* u *of problem* (1) *satisfies inequality*

$$\|u(\frac{T_0}{\lambda^4}, \cdot)\|_2^2 \geq \|\varphi\|_2^2(1 - \tau(\varepsilon)\lambda^2),$$

where $\tau(\varepsilon) = \frac{1}{2}\left(\varepsilon + \frac{1}{\varepsilon}\right)$*. Then solution* u *is global and regular.*

Proof. Suppose the opposite. Then $T_* < \frac{T_0}{\lambda^4}$ and

$$\|u(T_*, \cdot)\|_2^2 \geq \|\varphi\|_2^2(1 - \tau(\varepsilon)\lambda^2).$$

On the other hand, we have

$$\|u(T_*, \cdot)\|_2^2 \leq \|\varphi\|_2^2(1 - \sqrt{\mu}\lambda^2).$$

(See, [8], Lemma 49). Compare its we obtain contradiction with (6). Theorem is proved. □

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