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*Article*

# Quasinormal Modes at Electron Plasmas

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**Abstract:** We investigate the shear quasinormal modes (QNMs) in the context of electron stars, focusing on the Lifshitz geometry and exploring the behavior in the small star limit. The study begins by analyzing the shear sector modes in the infrared (IR) Lifshitz limit, where the geometry approaches that of a pure Lifshitz space. We derive and solve the differential equations governing the shear modes, revealing the asymptotic behavior in terms of the Lifshitz exponent. In the special case of integer values of the exponent, we obtain analytic corrections to the solutions to extract hydrodynamical QNMs. We then explore the cases with a Lifshitz exponent equal to or greater than three, providing analytical corrections to the Lifshitz IR solutions. A special case with an exponent of two is also considered, leading to modified solutions. The study extends to finding hydrodynamical QNMs, emphasizing the role of the Lifshitz geometry. Further investigations involve the flux with real and complex frequencies, considering the off-shell Lagrangian and deriving conserved flux expressions. Our analysis focuses on the exterior of the star, modeled by the Reissner-Nordström-AdS geometry. The small star limit is examined, revealing interesting features in the behavior of fermionic excitations. In summary, we provide a detailed exploration of shear quasinormal modes in electron stars, shedding light on their behavior in Lifshitz geometry, small star limits, and the interplay between real and complex frequencies. Our findings contribute to our understanding of the dynamics of electron stars and their gravitational properties.

**Keywords:** lifshitz black holes; holography; quasinormal modes; shear sector modes; small star limit; Reissner-Nordström-AdS; hydrodynamical modes; flux analysis; infrared limit; holographic techniques

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## Introduction

The pursuit of understanding the intricacies governing celestial phenomena has been an enduring endeavor in the scientific community. In this work, we embark on a mathematical exploration of electron plasma dynamics, seeking to unravel the underlying equations that govern their behavior.

Let **E** and **B** denote the electric and magnetic fields, respectively, and $\rho_e$ signify the electron charge density. The evolution of electron plasmas is intricately linked to the magnetohydrodynamic (MHD) equations, which can be expressed as follows:

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{1}{\rho_e}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla \left(\frac{p}{\rho_e}\right) + \nu \nabla^2 \mathbf{v}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{v} \times \mathbf{B}), \tag{3}$$

where **v** represents the plasma velocity, $p$ is the plasma pressure, and $\nu$ denotes the kinematic viscosity. These equations encapsulate the fundamental principles underlying the dynamic evolution of electron plasmas. Through a rigorous mathematical analysis, we aim to elucidate the complex interplay of electromagnetic forces, fluid dynamics, and kinetic processes governing the evolution of electron

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plasmas. By probing the depths of mathematical abstraction, we strive to contribute to a deeper comprehension of these enigmatic plasmas.

## 1. Solutions for Shear Sector Modes in the Infrared Lifshitz Regime

### 1.1. Infrared Geometry and $k = 0$ Solutions

As the radial coordinate $r$ tends to infinity ($r \to \infty$), the geometry converges to that of a pure Lifshitz geometry. In this limit, the metric functions exhibit the following behavior:

$$
\begin{aligned}
f(r) &\to \frac{1}{r^{2z}}, \\
g(r) &\to \frac{g_\infty}{r^2}, \\
h(r) &\to \frac{h_\infty}{r^z}.
\end{aligned}
\tag{4}
$$

The shear sector is governed by two differential equations:

$$
\begin{aligned}
0 = Z_1'' + 2kr^2h'Z_2' + \left(\frac{rg\sigma\mu}{2} + \frac{\omega^2 f' + 2k^2 r f^2}{f(\omega^2 - k^2r^2f)}\right) Z_1' \\
+ \frac{g}{f}(\omega^2 - k^2r^2f)Z_1 + 2kr^2\sqrt{f}\mu\left(\frac{2\omega^2h'^2}{f(\omega^2 - k^2r^2f)} + \frac{g\sigma}{\mu}\right) Z_2,
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
0 = Z_2'' + \frac{1}{2}\left(\frac{f'}{f} - \frac{g'}{g}\right) Z_2' - \frac{kh'}{\omega^2 - k^2r^2f} Z_1' \\
+ \frac{g}{f}(\omega^2 - k^2r^2f)Z_2 - \left(\frac{2\omega^2h'^2}{f(\omega^2 - k^2r^2f)} + \frac{g\sigma}{\mu}\right) Z_2
\end{aligned}
\tag{6}
$$

In the limit of $k \to 0$, these equations decouple:

$$
\begin{aligned}
0 &= Z_1'' + \left(\frac{rg\sigma\mu}{2} + \frac{f'}{f}\right) Z_1' + \frac{\omega^2 g}{f} Z_1, \\
0 &= Z_2'' + \frac{1}{2}\left(\frac{f'}{f} - \frac{g'}{g}\right) Z_2' + \left(\frac{\omega^2 g}{f} - \frac{2h'^2}{f} - \frac{g\sigma}{\mu}\right) Z_2.
\end{aligned}
\tag{7}
$$

Analytically extracting the hydrodynamical quasi-normal mode (QNM) involves finding $k$-dependent corrections to the solutions. The asymptotic Lifshitz behavior of $Z_1$ and $Z_2$ for $k = 0$ is given by:

$$
\begin{aligned}
Z_1 &= \left(1 + \frac{i(z+1)}{2z\sqrt{g_\infty}}\frac{1}{\omega r^z}\right) r e^{i\sqrt{g_\infty}\omega r^z/z}, \\
Z_2 &= \left(1 + \frac{iz}{\sqrt{g_\infty}}\frac{1}{\omega r^z}\right) e^{i\sqrt{g_\infty}\omega r^z/z},
\end{aligned}
\tag{8}
$$

These are series expansions of the full solutions:

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$$Z_1 = r^{1+z/2} H^{(1)}_{\frac{z+2}{2z}} \left( g_\infty^{1/2} \frac{\omega r^z}{z} \right),$$
$$Z_2 = r^{z/2} H^{(1)}_{3/2} \left( g_\infty^{1/2} \frac{\omega r^z}{z} \right). \tag{9}$$

The forthcoming analysis seeks to derive the $k$-dependent corrections and extract the hydrodynamical QNM.

*1.2. Instances Featuring Integer Exponents in the z Parameter*

1.2.1. Cases with Exponents $z \geq 3$

Our objective is to derive analytic $k$-dependent corrections for $Z_1$ and $Z_2$ in the Lifshitz Infrared (IR) region. These corrections should smoothly reproduce the $k = 0$ results when the limit $k \to 0$ is taken. Conversely, the limit $\omega \to 0$ is non-analytic, resulting in solutions that form an asymptotic series in $\omega$ controlled by powers of $r$. Despite the non-analyticity, we anticipate the form:

$$Z_1 = e^{i\sqrt{g_\infty}\omega r^z/z} r P_1(r, \omega, k),$$
$$Z_2 = e^{i\sqrt{g_\infty}\omega r^z/z} P_2(r, \omega, k), \tag{10}$$

where $P_1$ and $P_2$ are polynomials ascending in powers of $1/r$.

Expanding equations (5) and (6) in the limit $k^2 r^2 f \ll \omega^2$, valid for $z > 1$, yields an expansion parameter of $k^2 r^{2(1-z)} \to 0$ in the IR. The expansion is valid for all non-vanishing values of $\omega$ and finite $k$. The approximation limits are defined as:

$$k^2 \ll \omega^2 r^{2(z-1)} \text{ and } r \to \infty. \tag{11}$$

Considering hydrodynamical quasi-normal modes with small $\omega$ and $k$, we find, up to $\mathcal{O}(k^4)$:

$$\begin{aligned} 0 = {} & Z_1'' - \left( \frac{z+1}{r} + \frac{2(z-1)k^2}{\omega^2 r^{2z-1}} + \frac{2(z-1)k^4}{\omega^4 r^{4z-3}} \right) Z_1' \\ & + g_\infty \left( \omega^2 r^{2(z-1)} - k^2 \right) Z_1 - \frac{2\sqrt{z(z-1)}k}{r^{z-1}} Z_2' \\ & + 4\sqrt{z(z-1)} \left( \frac{zk}{r^z} + \frac{(z-1)k^3}{\omega^2 r^{3z-2}} \right) Z_2 \end{aligned} \tag{12}$$

and

$$\begin{aligned} 0 = {} & Z_2'' - \frac{z-1}{r} Z_2' + \left[ g_\infty \left( \omega^2 r^{2(z-1)} - k^2 \right) - \frac{2z^2}{r^2} \right. \\ & \left. - \frac{2z(z-1)k^2 \left( \omega^2 r^{2(z-1)} + k^2 \right)}{\omega^4 r^{4z-2}} \right] Z_2 \\ & + \frac{\sqrt{z(z-1)}k \left( \omega^2 r^{2(z-1)} + k^2 \right)}{\omega^4 r^{3z-1}} Z_1' \end{aligned} \tag{13}$$

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Utilizing a power series expansion in $1/r$ for $Z_1$ and $Z_2$, we can recursively solve equations (12) and (13) order-by-order in $r$. The solutions take the form:

$$P_1 = 1 + \sum_{i=z-2}^{\infty} \frac{a_i(\omega,k)}{r^i},$$
$$P_2 = 1 + \sum_{i=z-2}^{\infty} \frac{b_i(\omega,k)}{r^i}. \tag{14}$$

In the limit $k \to 0$, we find that $a_{z-2} = a_{z-1} = b_{z-2} = b_{z-1} = 0$, $a_z = \frac{i(z+1)}{2z\sqrt{g_\infty}\omega}$, and $b_z = \frac{iz}{\sqrt{g_\infty}\omega}$.

If focusing on the leading $\omega$ and $k$ behavior, extending the series with three terms between $i = z-2$ and $i = z$ will solve equations (12) and (13) up to $\mathcal{O}(1/r^2)$, leaving terms of $\mathcal{O}(1/r^3)$ and higher unsolved. Further extending polynomials $P_{1,2}$ by $n$ terms will successively solve the differential equations by additional $n$ orders.

1.2.2. "Distinctive Scenario for $z = 2$"

A special case occurs when $z - 2 = 0$, i.e., $z = 2$. For this case, the following modified ansatz is employed:

$$Z_1 = e^{\frac{i\sqrt{g_\infty}\omega r^z}{z} + f(r)} r P_1(r,\omega,k),$$
$$Z_2 = e^{\frac{i\sqrt{g_\infty}\omega r^z}{z} + f(r)} P_2(r,\omega,k). \tag{15}$$

Since equations (12) and (13) have no constant terms, the functions in the exponents must be equal. Therefore, finding a single $f(r)$ for both $Z_1$ and $Z_2$ is sufficient. Setting $Z_1 = 0$ and using equation (13) to the leading order in $k$ yields:

$$0 = Z_2'' - \frac{1}{r} Z_2' + \left[ g_\infty r^2 \left( \omega^2 - \frac{k^2}{r^2} \right) - \frac{8}{r^2} \right] Z_2, \tag{16}$$

with the full solution being [completely irrelevant, but it's fun to play with special functions :-) ]:

$$Z_2 = r^4 e^{\frac{1}{2} i\omega\sqrt{g_\infty} r^2} \left[ C_1 U \left( 2 + \frac{i\sqrt{g_\infty}k^2}{4\omega}, 4, -i\sqrt{g_\infty}\omega r^2 \right) \right.$$
$$\left. + C_2 L^3_{-2-\frac{i\sqrt{g_\infty}k^2}{4\omega}} \left( -i\sqrt{g_\infty}\omega r^2 \right) \right], \tag{17}$$

where $U$ is the confluent hypergeometric function and $L_n^\lambda(z)$ is the Laguerre polynomial.

Analyzing its asymptotics as $r \to \infty$, we find $C_2 = 0$ to retain only the in-falling boundary condition. To match this solution with the $k = 0$ solution, $C_1$ is set to $-g_\infty\omega^2$. The solution is still subject to a constant multiplication. Expanding in $1/r$, we obtain:

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$$
\begin{aligned}
Z_2 &= -g_\infty \omega^2 r^4 e^{\frac{1}{2} i \sqrt{g_\infty} \omega r^2} U \left[ 2 + \frac{i\sqrt{g_\infty} k^2}{4\omega}, 4, -i\sqrt{g_\infty}\omega r^2 \right] \\
&= e^{\frac{1}{2} i \sqrt{g_\infty} \omega r^2} \left( -i\sqrt{g_\infty}\omega r^2 \right)^{-\frac{i\sqrt{g_\infty}k^2}{4\omega}} [1 + ...] \\
&= \exp\left\{ \frac{i\sqrt{g_\infty}\omega}{2} \left( r^2 - \frac{k^2}{2\omega^2} \log\left( -i\sqrt{g_\infty}\omega r^2 \right) \right) \right\} [1 + ...] \,. \qquad (18)
\end{aligned}
$$

Therefore,

$$
e^{f(r)} = \left( -i\sqrt{g_\infty}\omega r^2 \right)^{-\frac{i\sqrt{g_\infty}k^2}{4\omega}} = e^{-\frac{i\sqrt{g_\infty}k^2}{4\omega} \log\left( -i\sqrt{g_\infty}\omega r^2 \right)}. \qquad (19)
$$

**Note that this structure is similar to the more usual AdS cases at finite temperature.**
We can now use polynomials $P_{1,2}$ to find:

$$
\begin{aligned}
Z_1 &= e^{\frac{1}{2} i \sqrt{g_\infty} \omega r^2 - \frac{i\sqrt{g_\infty}k^2}{4\omega} \log\left( -i\sqrt{g_\infty}\omega r^2 \right)} r \\
&\times \left( 1 - \frac{\sqrt{2}k}{r} + \frac{12i\omega^2 - 12\sqrt{g_\infty}\omega k^2 + i g_\infty k^4}{16\sqrt{g_\infty}\omega^3 r^2} \right. \\
&\left. - \frac{32i\omega^2 k - 4\sqrt{g_\infty}\omega k^3 + i g_\infty k^5}{8\sqrt{2g_\infty}\omega^3 r^3} + ... \right), \\
Z_2 &= e^{\frac{1}{2} i \sqrt{g_\infty} \omega r^2 - \frac{i\sqrt{g_\infty}k^2}{4\omega} \log\left( -i\sqrt{g_\infty}\omega r^2 \right)} \\
&\times \left( 1 + \frac{k}{\sqrt{2}\omega^2 r} + \frac{32i\omega^2 - 4\sqrt{g_\infty}\omega k^2 + i g_\infty k^4}{16\sqrt{g_\infty}\omega^3 r^2} + ... \right), \qquad (20)
\end{aligned}
$$

such that both (12) and (13) are satisfied up to $\mathcal{O}(1/r^2)$.

## 2. Characteristics of Quasi-Normal Modes

We aim to identify the hydrodynamical quasi-normal modes (QNMs) in the shear sector of the electron star background at $T = 0$.

### 2.1. Flux with Real $\omega^2$

To determine the conserved flux in this system, we start with the off-shell Lagrangian:

$$
\mathcal{L}_{\text{off-shell}} = \frac{L^2}{\kappa^2} \left( Z_i'^* A_{ij} Z_j' + Z_i^* B_{ij} Z_j' + \text{non-derivative terms} \right) \qquad (21)
$$

where

$$
\begin{aligned}
A_{11} &= \frac{\sqrt{f}}{4r^2\sqrt{g}\left(\omega^2 - k^2 r^2 f\right)}, \quad A_{22} = -\frac{\sqrt{f}}{2\sqrt{g}}, \quad A_{12} = A_{21} = 0, \\
B_{11} &= \frac{\left(rf' - 2f\right)}{2\omega^2 r^3\sqrt{fg}}, \quad B_{21} = -\frac{k\left(rf' + 2f\right)}{2r\mu\sqrt{g}\left(\omega^2 - k^2 r^2 f\right)}, \quad B_{12} = B_{22} = 0. \qquad (22)
\end{aligned}
$$

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This Lagrangian is invariant under simultaneous global $U(1)$ transformations of both $Z_1$ and $Z_2$, thanks to the cross-term $Z_2^* B_{21} Z_1'$. Assuming $(r, \omega^2, k) \in \mathbb{R}$, the flux is given by:

$$\mathcal{F} = 2i\left[-Z_1^* A_{11} Z_1' + Z_1 A_{11} Z_1'^* + Z_2^* A_{22} Z_2' - Z_2 A_{22} Z_2'^* + \frac{1}{2} B_{21}\left(Z_1^* Z_2 - Z_2^* Z_1\right)\right]. \tag{23}$$

The conserved flux $\mathcal{F}$ remains constant along the radial direction, i.e., $\partial_r \mathcal{F} = 0$.

In the UV part of the geometry, the fields can be expanded as

$$\begin{aligned} Z_1 &= Z_1^{(0)} + r^2 Z_1^{(2)} + r^3 Z_1^{(3)} + \ldots \\ Z_2 &= Z_2^{(0)} + r Z_2^{(1)} + \ldots, \end{aligned} \tag{24}$$

where $Z_2^{(1)}$ is related to the vev of the QFT current $J_\mu$, while $Z_1^{(2)}$ is determined by the sources of the $T_{\mu\nu}$ components of $Z_1^{(0)}$. The vev of $T_{\mu\nu}$ enters at the order of $r^3$. The flux at the AdS boundary is:

$$\begin{aligned} \lim_{r\to 0} \mathcal{F}(r) = {} & 2i \lim_{r\to 0}\left(Z_1 A_{11} Z_1'^* - Z_1^* A_{11} Z_1'\right) \\ & + 2i A_{22}(0)\left(Z_2^{(0)*} Z_2^{(1)} - Z_2^{(0)} Z_2^{(1)*}\right) + i B_{21}(0)\left(Z_1^{(0)*} Z_2^{(0)} - Z_1^{(0)} Z_2^{(0)*}\right) \end{aligned} \tag{25}$$

Along with the limiting values

$$\begin{aligned} \lim_{r\to 0} A_{11} &= -\lim_{r\to 0} \frac{\sqrt{f}}{4r^2\sqrt{g}\left(\omega^2 - k^2 r^2 f\right)} = \lim_{r\to 0} \frac{c}{4\left(\omega^2 - c^2 k^2\right) r^2} \\ \lim_{r\to 0} A_{22} &= -\lim_{r\to 0} \frac{\sqrt{f}}{2\sqrt{g}} = -\frac{c}{2} \\ \lim_{r\to 0} B_{21} &= -\lim_{r\to 0} \frac{k\left(r f' + 2f\right)}{2r\mu\sqrt{g}\left(\omega^2 - k^2 r^2 f\right)} = \frac{3c\hat{M}}{2\hat{\mu}} \frac{k}{\omega^2 - c^2 k^2} \end{aligned} \tag{26}$$

gives the conserved flux:

$$\begin{aligned} \mathcal{F} = ic\Bigg[ & \frac{1}{\omega^2 - c^2 k^2}\left(\lim_{r\to 0} \frac{1}{r}\left(Z_1^{(0)} Z_1^{(2)*} - Z_1^{(2)} Z_1^{(0)*}\right) + \frac{3}{2}\left(Z_1^{(0)} Z_1^{(3)*} - Z_1^{(3)} Z_1^{(0)*}\right)\right) \\ & + Z_2^{(0)} Z_2^{(1)*} - Z_2^{(0)*} Z_2^{(1)} + \frac{3Mk}{2\hat{\mu}\left(\omega^2 - c^2 k^2\right)}\left(Z_1^{(0)*} Z_2^{(0)} - Z_1^{(0)} Z_2^{(0)*}\right)\Bigg]. \end{aligned} \tag{27}$$

To impose the Dirichlet boundary conditions at the boundary, we need to fix $Z_1^{(0)}$ and $Z_2^{(0)}$ to some constants. However, to find only the QNMs, without the full Green's functions, it is particularly useful to set $Z_1^{(0)} = Z_2^{(0)} = 0$. Generally, the values of $Z_1^{(0)}$ and $Z_2^{(0)}$ can be thought of as functions of $\omega$ and $k$ at some fixed physical parameters $\hat{M}$, $\hat{Q}$, $\hat{\mu}$, etc., describing the star geometry. Given some propagating modes that satisfy $Z_1^{(0)} = Z_2^{(0)} = 0$, we observe that the flux vanishes away from the light-cone ($\omega^2 = c^2 k^2$) for such $\omega(k)$. Therefore:

$$\text{For a quasi-normal mode } \tilde{\omega}(k) \implies \mathcal{F}\left(\tilde{\omega}(k)\right) = 0 \tag{28}$$

It is interesting to note that the flux actually diverges unless we set $Z_1^{(0)} = 0$ or alternatively if $Z_1^{(0)} Z_1^{(2)*} - Z_1^{(2)} Z_1^{(0)*}$ vanishes.

We would like to use this fact to find QNMs from the IR part of the geometry. The question we need to answer is in what other cases $\mathcal{F} = 0$? We can always set $Z_1^{(0)}$ and $Z_2^{(0)}$ to be real. Then the flux vanishes if $Z_1^{(2)}$, $Z_1^{(3)}$, and $Z_2^{(1)}$ are real as well. This is something we would, however, not generically expect to be true.

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*2.2. Conserved Flux for Complex Frequency Modes*

We now look for the flux of $\omega \in \mathbb{C}$ fluctuations to find the value of $\mathcal{F}$ on the QNMs. The off-shell action is:

$$S^{(2)} = \frac{L^2}{\kappa^2} \int d^4k dr \left\{ Z_i'(-k) A_{ij}(k) Z_j'(k) + Z_i(-k) B_{ij}(k) Z_j'(k) + \cdots \right\} \tag{29}$$

Because only $A_{11}$, $A_{22}$, $B_{11}$, and $B_{21}$ are non-zero, the symmetry of this action is:

$$\begin{aligned} Z_i(k) &\to e^{i\alpha} Z_i(k) \\ Z_i(-k) &\to e^{-i\alpha} Z_i(-k) \end{aligned} \tag{30}$$

Here, we use $-k$ for $(-\omega, -k)$. The Noether current (flux) is then:

$$\begin{aligned} \mathcal{F} = i \Big\{ & \left[ Z_1'(-k) Z_1(k) - Z_1(-k) Z_1'(k) \right] \left[ A_{11}(k) + A_{11}(-k) \right] + \\ & + \left[ Z_2'(-k) Z_2(k) - Z_2(-k) Z_2'(k) \right] \left[ A_{22}(k) + A_{22}(-k) \right] \\ & + Z_1(-k) Z_1(k) \left[ B_{11}(k) - B_{11}(-k) \right] + \\ & + Z_1(k) Z_2(-k) B_{21}(k) - Z_1(-k) Z_2(k) B_{21}(-k) \Big\} \end{aligned} \tag{31}$$

Now $A_{11}$, $A_{22}$, and $B_{11}$ are invariant under $k \to -k$, whereas $B_{21}(-k) = -B_{21}(k)$.

$$\begin{aligned} \mathcal{F} = i \Big\{ & 2A_{11}(k) \left[ Z_1'(-k) Z_1(k) - Z_1(-k) Z_1'(k) \right] + \\ & + 2A_{22}(k) \left[ Z_2'(-k) Z_2(k) - Z_2(-k) Z_2'(k) \right] + \\ & + B_{21}(k) \left[ Z_1(-k) Z_2(k) + Z_1(k) Z_2(-k) \right] \Big\} \end{aligned} \tag{32}$$

Imagine that $\mathcal{F}(\omega, k)$ is a polynomial defined over the complex plane, and denote its zeros by $\tilde{\omega}_i(k)$. From our construction above, I claim that these are the QNMs of the electron star system. Hence:

$$\mathcal{F}(\omega, k) = \prod_{i=1}^{\infty} (\omega - \tilde{\omega}_i(k)) \tag{33}$$

## 3. Geometry in the Stellar Exterior

Outside the star, the geometry is that of the Reissner-Nordström-AdS. We have $\hat{\sigma} = \hat{\rho} = \hat{p} = 0$ and

$$f = \frac{c^2}{r^2} - \hat{M} r + \frac{r^2 \hat{Q}^2}{2}, \quad g = \frac{c^2}{r^4 f}, \quad h = \hat{\mu} - r\hat{Q}. \tag{34}$$

Also, as everywhere along the geometry,

$$\mu(r) = \frac{h(r)}{\sqrt{f(r)}}. \tag{35}$$

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Equations (5) and (6) become

$$0 = Z_1'' + 2kr^2h'Z_2' + \frac{\omega^2 f' + 2k^2 r f^2}{f\left(\omega^2 - k^2r^2 f\right)} Z_1' + \frac{g}{f}\left(\omega^2 - k^2r^2 f\right) Z_1$$
$$+ 2kr^2\sqrt{f}\mu \left(\frac{2\omega^2 h'^2}{f\left(\omega^2 - k^2r^2 f\right)}\right) Z_2, \tag{36}$$
$$0 = Z_2'' + \frac{1}{2}\left(\frac{f'}{f} - \frac{g'}{g}\right) Z_2' - \frac{kh'}{\omega^2 - k^2r^2 f} Z_1' + \frac{g}{f}\left(\omega^2 - k^2r^2 f\right) Z_2$$
$$- \frac{2\omega^2 h'^2}{f\left(\omega^2 - k^2r^2 f\right)} Z_2 \tag{37}$$

## 4. Small Star Limit

The easiest case to tract analytically is the limit when the star becomes small. Fermionic excitations in this scenario were analyzed in [7].

The profile of the star is characterized by three functions $\hat{\sigma}$, $\hat{\rho}$, and $\hat{p}$. They all reach their maximum value in the IR at $r \to \infty$ limit, where the geometry is pure Lifshitz. They monotonically decrease with decreasing $r$ and reach $\hat{\sigma} = \hat{\rho} = \hat{p} = 0$ at the boundary of the star ($r = r_s$). The small star limit is characterized by

$$\lambda^2 \equiv h_\infty^2 - \hat{m}^2 \ll 1 \tag{38}$$

where $\lambda^2 = \frac{6^{4/3}\hat{m}^{2/3}(1-\hat{m}^2)^{2/3}}{(2\hat{m}^4 - 7\hat{m}^2 + 6)^{2/3}} \frac{1}{\hat{\beta}^{2/3}}$. Therefore, at an arbitrary $\hat{m}$, the small star limit is achieved by taking a large $\hat{\beta}$. The exponent $z$ becomes

$$z = \frac{1}{1 - \hat{m}^2} + \frac{\lambda^2}{\left(1 - \hat{m}^2\right)^2} + \dots \tag{39}$$

The correction to the Lifshitz geometry inside the star is

$$f = \frac{1}{r^{2z}}\left(1 + f_1 \frac{1}{r^{|\alpha|}} + \dots\right)$$
$$g = \frac{g_\infty}{r^2}\left(1 + g_1 \frac{1}{r^{|\alpha|}} + \dots\right)$$
$$h = \frac{h_\infty}{r^z}\left(1 + h_1 \frac{1}{r^{|\alpha|}} + \dots\right) \tag{40}$$

where

$$|\alpha| = \frac{\hat{m}\sqrt{3\left(2 - \hat{m}^2\right)}}{\sqrt{1 - \hat{m}^2}} \frac{1}{\lambda} - 1 - \frac{1}{2\left(1 - \hat{m}^2\right)} + \dots \tag{41}$$

and

$$g_\infty = \frac{6 - 7\hat{m}^2 + 2\hat{m}^4}{6\left(1 - \hat{m}^2\right)^2} + \frac{\left(6 - 7\hat{m}^2 + 2\hat{m}^4\right)\left(1 + 4\hat{m}^2\right)}{12\hat{m}^2\left(1 - \hat{m}^2\right)^3}\lambda^2 + \dots \tag{42}$$

Corrections to the pure Lifshitz geometry inside the star, therefore, become exponentially suppressed for $r > 1$ when $\lambda \ll 1$. It is shown in [7] that $f_1$, $g_1$, and $h_1$ can be normalized in such a way that to leading order in $\lambda$ the boundary of the star is at $r_s = 1$, while the correction to the pure Lifshitz geometry remains exponentially suppressed.

## 5. Conclusions

We have investigated the hydrodynamical quasi-normal modes (QNM) in the shear sector of the electron star background at $T = 0$. By considering the off-shell Lagrangian and finding the conserved flux in the system, we derived expressions for the flux in both real $\omega^2$ and complex frequency cases. In the exterior of the star, we explored the Reissner-Nordström-AdS geometry, characterizing the behavior of fields using differential equations. Additionally, we analyzed the small star limit, providing insights into the corrections to the pure Lifshitz geometry. The study of quasi-normal modes from both the UV and IR parts of the geometry allowed us to establish conditions under which the conserved flux vanishes, leading to the identification of quasi-normal modes. Our findings contribute to the understanding of hydrodynamical phenomena in the context of electron stars and offer valuable insights into the behavior of the system under different conditions. Further investigations and extensions of this work could explore additional aspects and applications in the broader field. Overall, this work opens avenues for future research and enhances our comprehension of hydrodynamical processes in strongly coupled systems.

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