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Introducing Novel Geometric Insights and Three-Dimensional Depictions of the Pythagorean Theorem for any Triangles

Marco Aurélio Dos Santos Bernardes *

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Introducing novel Geometric Insights and Three-Dimensional Depictions of the Pythagorean Theorem for any Triangles

Marco Aurélio dos Santos Bernardes

Department of Mathematics, University of Hamburg, Bundesstraße 55, 20146, Hamburg, Germany (latest affiliation); dr.ing.dossantosbernardes@gmail.com

Abstract: The paper primarily demonstrates that the three internal triangles formed by connecting any point on a midsegment with the vertices of a generic triangle satisfy the geometric Proof of the Pythagorean theorem in Euclidean geometry. Moreover, this study elucidates the Pythagorean relationships inherent within three-dimensional geometric constructs resulting from the arrangement of three-dimensional spatial triangles. This geometric relationship, akin to a generalized extension of the Pythagorean theorem, unveils a unique spatial region characterized by this harmonious area interrelation among triangles.

Keywords: pythagorean theorem; three-dimensional space; triangles; three-dimensional geometric shape; triangle midsegment theorem

MSC: 00-XX; 00Axx; 00A05

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1. Introduction

Pythagorean theorem stands as one of the most crucial theorems in elementary mathematics, with numerous known proofs and applications. The available literature on this topic is vast, and the selection discussed here is far from being comprehensive. In the interest of brevity, we begin with the established book [1], delving into the historical development of the Pythagorean theorem, tracing its origins in different cultures and civilizations. It also provides more than 370 proofs of the theorem, showcasing different approaches and techniques used to demonstrate its validity. Additionally, the book discusses the implications and applications of the Pythagorean theorem beyond its basic form. This includes exploring how the theorem can be extended to non-right triangles (such as acute and obtuse triangles) through trigonometry and other geometric concepts. The book also touches on applications of the theorem in fields like geometry, physics, and engineering.

Kassie Smith in reference [2] inquires about a more geometric proof for these concepts. Consequently, starting with the equation, they deduce that the combined area of two regular n-gons with side lengths a and b respectively, equates to the area of a regular n-gon with side length c. This can be shown by multiplying both sides of the equation by a specific constant, which is essentially the area of a regular n-gon with a side length of 1. Interestingly, for any given $n \geq 3$, this assertion also serves as a way to derive Pythagorean theorem in reverse. Within this context, the Wallace-Bolyai-Gerwien decomposition theorem ([3]) becomes applicable. This theorem suggests the existence of a decomposition of the smaller n-gons into polygonal components, which can be rearranged to form the larger n-gon. However, it's important to note that the number of individual pieces involved in this process might be extensive. The reference [4] proposes still another proof which is nonstandard as he does not use neither squares nor similarity of triangles. The author introduces a geometric demonstration of the aforementioned assertion concerning equilateral triangles employing arithmetical operations of triangular shapes.

In the following discussion, we present two theorems concerning the application of the Geometric Pythagorean relationship. The first theorem addresses triangles formed by connecting a point in the

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midsegment to the vertices, while the second theorem pertains to non-coplanar triangles formed by connecting the sides of the original triangle to a point in space.

2. Pythagorean concept for coplanar triangles formed from a point in the midsegment and the vertices of a triangle

Theorem 1 (Geometric Pythagorean Relationships within Triangular Midsegment Compositions)**.** *In Euclidean geometry, for any triangle $\triangle ABC$ and a point M lying on the midsegment connecting the midpoints of sides AC and BC, as depicted in Figure 1, the relationship between the areas of the three triangles $\triangle AMC$, $\triangle AMB$, and $\triangle BMC$, is governed by the Geometric Proof of the Pythagorean theorem. Specifically, the sum of the areas of the triangles $\triangle AMC$ and $\triangle BMC$ is equal to the area of the triangle $\triangle AMB$.*

2.1. Definitions and Proof

Consider a generic triangle $\triangle ABC$, where AB is the base, and DE is the midsegment connecting the midpoints of sides AC and BC, as illustrated in Figure 1.

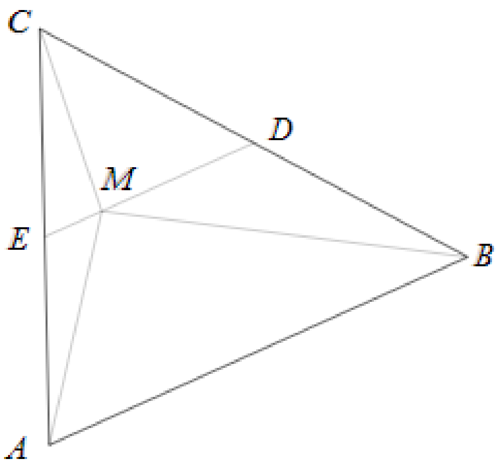

Figure 1. Generic triangle $\triangle ABC$.

The Triangle Midsegment Theorem elucidates the properties of triangles and the relationship between their midpoints. According to this theorem, when the midpoints of the sides of a triangle are connected, four identical smaller triangles are formed.

Let M be any point lying on the midsegment DE. If the area of each triangle, formed by connecting the midpoints of the sides of a given triangle, is one-fourth the area of the original triangle $\triangle ABC$, the following statements for the internally constructed triangles further align with the Triangle Midsegment Theorem and its underlying principles. Thus, it holds true that

$$ar(\triangle CDE) = \frac{1}{4}ar(\triangle ABC) \tag{1}$$

$$ar(\triangle CDE) = ar(\triangle CME) + ar(\triangle CMD) \tag{2}$$

$$ar(\triangle AMC) = 2ar(\triangle CME) = 2ar(\triangle AME) \tag{3}$$

$$ar(\triangle BMC) = 2ar(\triangle BMD) = 2ar(\triangle CMD) \tag{4}$$

The summation of the areas of the triangles $\triangle AMC$ and $\triangle BMC$ is given by Eq. 5

$$ar(\triangle AMC) + ar(\triangle BMC) = 2ar(\triangle CME) + 2ar(\triangle CMD) = \frac{1}{4}ar(\triangle ABC) + \frac{1}{4}ar(\triangle ABC) = \frac{1}{2}ar(\triangle ABC) \tag{5}$$

Since

$$ar(\triangle AMB) = \frac{1}{2}ar(\triangle ABC), \tag{6}$$

it can be demonstrated that, regardless of the size or shape of a triangle and for any point situated on the midsegment, the relationship between the areas of the three resultant internal triangles follows the principles of the Geometric Proof of the Pythagorean theorem, i.e. the sum of the areas of two smaller internal triangles given by Eq. 5 is always equivalent to the area of the larger internal triangle provided by Eq. 6. Hence,

$$ar(\triangle AMC) + ar(\triangle BMC) = ar(\triangle AMB) \tag{7}$$

2.2. *Special case for $M = D$*

For the case shown in Figure 2 where $M = D$, it holds true that

$$ar(\triangle BMC) = 0 \tag{8}$$

Unsurprisingly, the relationship between the internal triangle continues to hold Eq. 9 assertion.

$$ar(\triangle AMB) = ar(\triangle AMC) \tag{9}$$

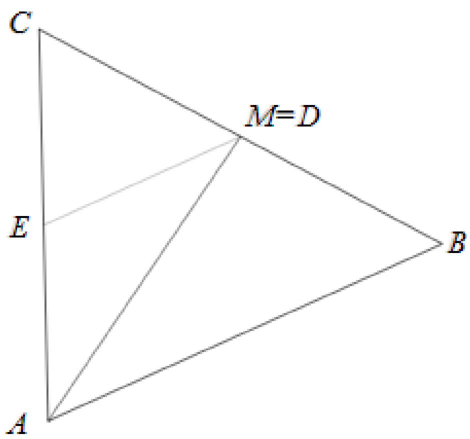

Figure 2. Special case for $M = D$.

2.3. *Theorem special case for $M \notin \triangle ABC$*

In the situation depicted in Figure 3, where M is not inside the $\triangle ABC$, the Pythagorean structure undergoes a transformation such that the Eq. 7 becomes the relationship shown in Eq. 10.

$$ar(\triangle AMC) - ar(\triangle BMC) = ar(\triangle AMB) \tag{10}$$

Precisely, the triangle $\triangle AMC$ has been enlarged to become the largest among the three triangles, ensuring the preservation of the Pythagorean composition.

$$ar(\triangle AMB) + ar(\triangle BMC) = ar(\triangle AMC) \tag{11}$$

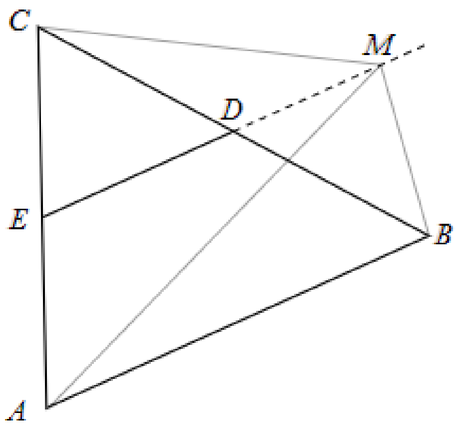

Figure 3. Theorem special case for $M \notin \triangle ABC$.

3. Extrapolating the Pythagorean structure into the 3D Space

Theorem 2 (Pythagorean Relationships in Three-Dimensional Geometric Structures Formed by Three-Dimensional Spatial Triangular Compositions)**.** *In a specific curvature-domain shape within three-dimensional space, a geometric configuration is established wherein the summation of the areas of two constituent triangles created by connecting the vertices of a generic triangle $\triangle ABC$ with a point $P(x, y, z)$ in 3D space, specifically triangle $\triangle APB$ and triangle $\triangle BPC$, is invariably equivalent to the area of the encompassing triangle $\triangle APC$ for any selection of points A, B, and C that form said triangles, equivalent to the geometric demonstration of the Pythagorean Theorem.*

3.1. Definitions and Proof

Now let's examine the general triangle $\triangle ABC$ depicted in Fig. 4, in which AB, AC, and BC represent its sides. The $\triangle APB$, $\triangle BPC$ and $\triangle APC$ arise by connecting the generic point $P(x, y, z)$ in three-dimensional space with AB, AC, and BC.

The altitude h and the altitude distance l to the axis y of the triangle $\triangle ABC$ are here determined by the its area, computed using the compact Heron's formula.

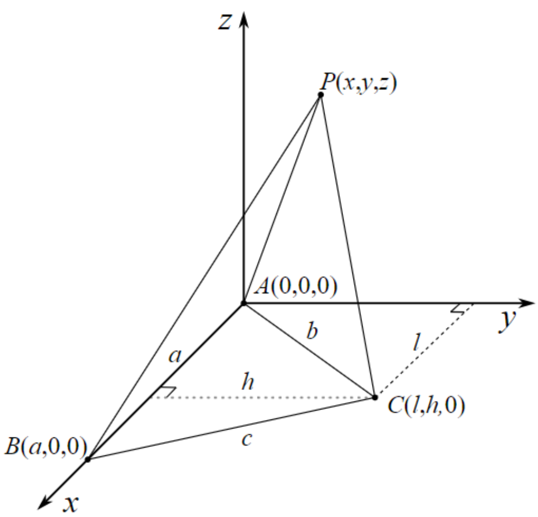

Figure 4. 3D Pythagorean composition from $\triangle ABC$.

$$ar(\triangle ABC) = \frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2} \tag{12}$$

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$$h = \frac{2ar(\triangle ABC)}{a} \tag{13}$$

$$l = \sqrt{b^2 - h^2)} \tag{14}$$

Hence, h and l may be expressed in the following manner:

$$h = \frac{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}}{2a} \tag{15}$$

$$l = \frac{c^2 - b^2 - a^2}{2a} \tag{16}$$

We express the demonstration through a computationally efficient equation that necessitates only a single square root operation. The area of a triangle in coordinate geometry can calculated by the Eq. 17, where (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are the vertices of a triangle in three-dimensional space.

$$ar(\triangle) = \frac{1}{2}\sqrt{\begin{matrix} ((x_2y_1) - (x_3y_1) - (x_1y_2) + (x_3y_2) + (x_1y_3) - (x_2y_3))^2 \\ +((x_2z_1) - (x_3z_1) - (x_1z_2) + (x_3z_2) + (x_1z_3) - (x_2z_3))^2 \\ +((y_2z_1) - (y_3z_1) - (y_1z_2) + (y_3z_2) + (y_1z_3) - (y_2z_3))^2 \end{matrix}} \tag{17}$$

In the subsequent iterations, we use Eq. 17 to calculate the area of the triangles $\triangle APB$, $\triangle BPC$ and $\triangle APC$ respectively. For the triangle $\triangle APB$ where $x_1 = 0, y_1 = 0, z_1 = 0, x_2 = a, y_2 = 0, z_2 = 0, x_3 = x, y_3 = y, z_3 = z$

$$ar(\triangle APB) = \frac{a}{2}\sqrt{y^2 + z^2} \tag{18}$$

Regarding the triangle $\triangle BPC$ where $x_1 = l, y_1 = h, z_1 = 0, x_2 = a, y_2 = 0, z_2 = 0, x_3 = x, y_3 = y, z_3 = z$

$$ar(\triangle BPC) = \frac{\sqrt{\beta z^2 + \left(2\,a^2 y + \alpha y - a\sqrt{\beta} + \sqrt{\beta}x\right)^2 + \left(2\,a^2 z + \alpha z\right)^2}}{4\,a} \tag{19}$$

with

$$\alpha = a^2 + b^2 - c^2 \tag{20}$$

$$\beta = 4\,a^2b^2 - \alpha^2 \tag{21}$$

Concerning the triangle $\triangle APC$ where $x_1 = l, y_1 = h, z_1 = 0, x_2 = 0, y_2 = 0, z_2 = 0, x_3 = x, y_3 = y, z_3 = z$

$$ar(\triangle APC) = \frac{\sqrt{\alpha^2 z^2 + \beta z^2 + \left(\alpha y + \sqrt{\beta}x\right)^2}}{4\,a} \tag{22}$$

Utilizing the Pythagorean geometric proof concept within the context of the irregular tetrahedron $ABCP$ yields three distinct solutions, as elucidated by Eq. 23.

$$\begin{matrix} ar(\triangle APB) = ar(\triangle BPC) + ar(\triangle APC) \\ ar(\triangle BPC) = ar(\triangle APB) + ar(\triangle APC) \\ ar(\triangle APC) = ar(\triangle APB) + ar(\triangle BPC) \end{matrix} \tag{23}$$

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By substituting the terms from Eq. 18, 19 and 22 into 23, we derive three respective implicit solutions for the hypothesized theorem tailored specifically to tetrahedrons named here as ***Pythagorean Tetrahedron***:

$$\begin{gathered} 2a^2\sqrt{y^2+z^2} \\ -\sqrt{\beta z^2+\left(2\,a^2y+\alpha y-a\sqrt{\beta}+\sqrt{\beta}x\right)^2+\left(2\,a^2z+\alpha z\right)^2} \\ -\sqrt{\alpha^2z^2+\beta z^2+\left(\alpha y+\sqrt{\beta}x\right)^2}=0 \end{gathered} \tag{24}$$

$$\begin{gathered} -2a^2\sqrt{y^2+z^2} \\ +\sqrt{\beta z^2+\left(2\,a^2y+\alpha y-a\sqrt{\beta}+\sqrt{\beta}x\right)^2+\left(2\,a^2z+\alpha z\right)^2} \\ -\sqrt{\alpha^2z^2+\beta z^2+\left(\alpha y+\sqrt{\beta}x\right)^2}=0 \end{gathered} \tag{25}$$

$$\begin{gathered} -2a^2\sqrt{y^2+z^2} \\ -\sqrt{\beta z^2+\left(2\,a^2y+\alpha y-a\sqrt{\beta}+\sqrt{\beta}x\right)^2+\left(2\,a^2z+\alpha z\right)^2} \\ +\sqrt{\alpha^2z^2+\beta z^2+\left(\alpha y+\sqrt{\beta}x\right)^2}=0 \end{gathered} \tag{26}$$

It is important to note that the concepts introduced here should not be confused with the concept of the Trirectangular Tetrahedron or with De Gua's theorem, [5]. De Gua's theorem asserts that when a tetrahedron contains a right-angle corner (similar to a corner on a cube), the sum of the squares of the areas of the three faces adjacent to the right-angle corner is equal to the square of the area of the face positioned opposite to that corner.

3.2. Graphic representation of the resultant curvature

In this segment, we present some illustrative three-dimensional graphical depictions of the solution of the implicit equations denoted as Eq. 24, 25, and 26, corresponding to the conjectured theorem outlined in section 3 given the assistance of a computer algebra system (CAS).

We initiate the graphical depiction of resultant curvature using the archetypal right 3-4-5 triangle. By graphically representing Eq. 24, 25, and 26, three bell-shaped tri-dimensional curvatures emerge outward from the midsegments of the triangle $\triangle ABC$, as portrayed in the visual representation labeled as Figure 5. It is anticipated that these bell-curvatures will continue to grow indefinitely. Void spaces comprise the internal triangle enclosed by the midsegments, as well as three additional regions formed by extending these midsegments in a way that avoids intersecting the vertices of the prototypical triangle.

Figures 6–8 illustrate the projection of the resulting surface onto the xy-plane, with the original triangle depicted using bold black lines, and the contour levels represented by green lines for 3-4-5 right, 3-3-3 isosceles, and 3-4-6 scalene triangles. The conjectured Theorem 1, as outlined in Section 2, can be readily confirmed by observing the midsegments.

Therefore, as proposed, we have shown that the three inner triangles created by connecting a point on a midsegment with the triangle's vertices adhere to the geometric verification of the Pythagorean theorem. Moreover, we have clarified the geometric connection, resembling an expanded version of the Pythagorean theorem, which reveals a distinct spatial surface defined by the Pythagorean interrelation of triangle areas for any triangle. Further endeavors should aim to develop an explicit formula for the newly introduced concepts in the current context.

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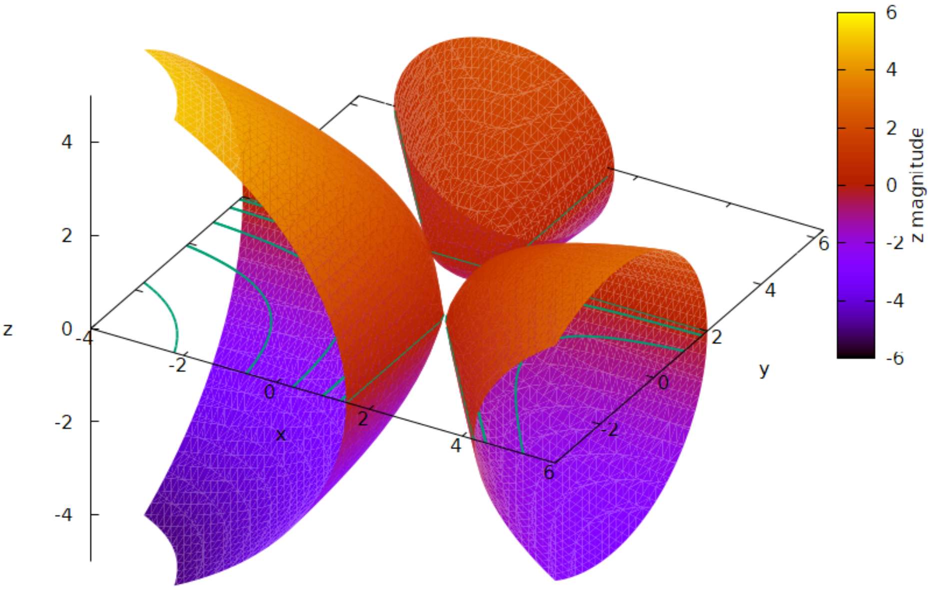

Figure 5. Visual representation of the resulting curvature and contour levels for a 3-4-5 right triangle.

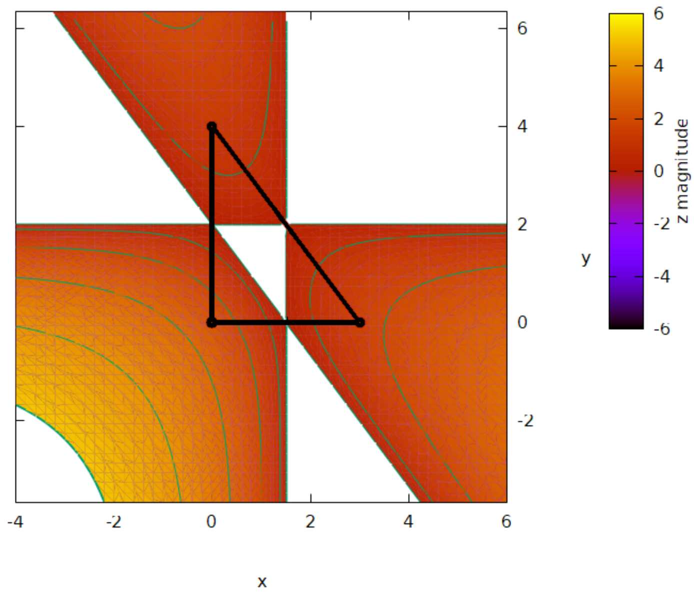

Figure 6. The 3-4-5 right triangle and the xy-plane view of its resulting curvature and contour levels.

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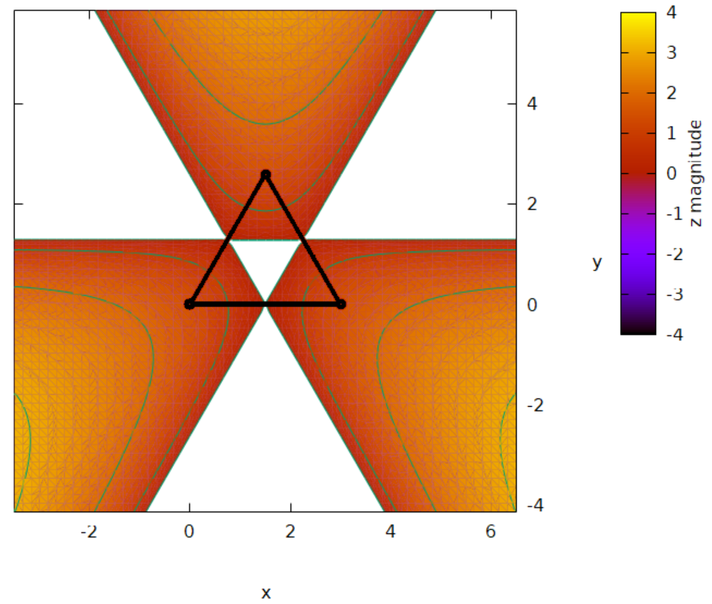

Figure 7. A 3-3-3 isosceles triangle and the xy-plane view of its resulting curvature and contour levels.

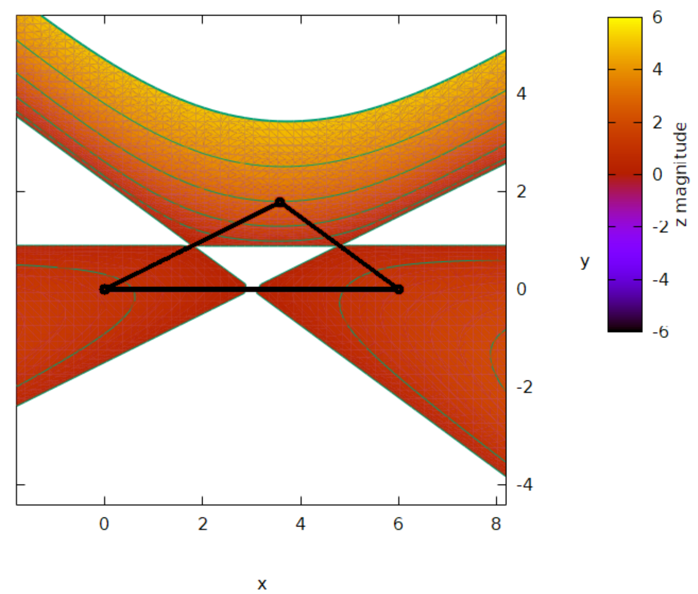

Figure 8. A 3-4-6 scalene triangle and the xy-plane view of its resulting curvature and contour levels.

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