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Article

Revolutionary Analysis of the Pointwise Stationary Fluid Flow Approximation Model of the Non-stationary $M/D/1$ Queue with Pointwise Stationary Fluid Flow Approximation Applications to the Internet of Things

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Abstract: This paper provides the first-time ever analytic modelling for Pointwise Stationary Fluid Flow Approximation (PSFFA) model of the non-stationary $M/D/1$ queueing system, for a longstanding solution via simulation. This is by proposing a constant ratio β (Ismail's ratio) that relates the time-dependent mean arrival and mean service rates, offering an exact analytical solution. The stability dynamics of the time-varying $M/D/1$ queueing system are then examined numerically in relation to time, β , and the queueing parameters. Some Pointwise Fluid Flow Approximation (PSFFA) applications to IoT are provided. The paper ends with summary and future research directions.

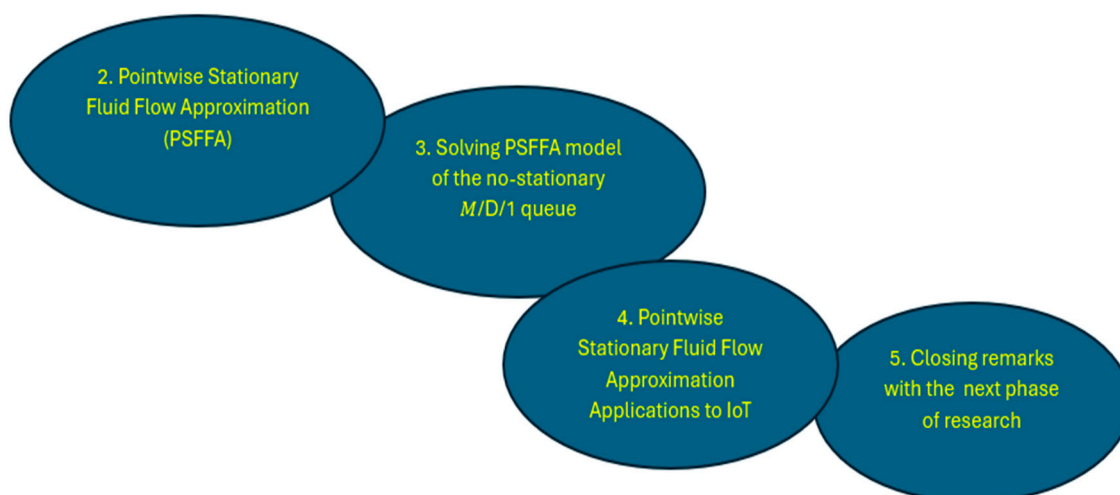
Keywords: time varying $M/D/1$ queue; modelling; simulation; Pointwise Fluid Flow Approximation (PSFFA)

1. Introduction

The field of transient/non-stationary analysis has limited literature, which can be categorized into simulation, transient analysis, analysis, and applications techniques. These categories encompass various approaches to studying systems that change over time, including simulations, analysing transient behaviour, and exploring non-stationary phenomena. In certain cases, mathematical transformations can be used to obtain a closed form expression for analysing non-stationary queueing systems. However, evaluating these expressions can be computationally complex. As a result, there has been a focus on numerically determining the transient behaviour of such systems instead of deriving closed form expressions.

The current exposition contributes to solving for first time ever, the longstanding unsolved problem of obtaining the state variable of the time varying $M/D/1$ queueing system.

The following flowchart shows how this paper is organized.



2. PSFFA

Let $f_{in}(t)$ and $f_{out}(t)$ serve as the temporal flow in, and flow out, respectively. Therefore,

$$\frac{dx(t)}{dt} = \dot{x}(t) = -f_{out}(t) + f_{in}(t), x(t) \text{ as the state variable} \quad (2.1)$$

$f_{out}(t)$ links server utilization, $\rho(t)$ and the time-dependent mean service rate, $\mu(t)$ by:

$$f_{out}(t) = \mu(t)\rho(t) \quad (2.2)$$

For an infinite queue waiting space:

$$f_{in}(t) = \text{Mean arrival rate} = \lambda(t) \quad (2.3)$$

Thus (2.1) rewrites to:

$$\dot{x}(t) = -\mu(t)\rho(t) + \lambda(t), \quad 1 > \rho(t) = \frac{\lambda(t)}{\mu(t)} > 0 \quad (2.4)$$

The stable subcase of (2.4) (i.e., $\dot{x}(t) = 0$), implies:

$$x = G_1(\rho) \quad (2.5)$$

The numerical invertibility of $G_1(\rho)$, yields

$$\rho = G_1^{-1}(x) \quad (2.6)$$

Hence,

$$\dot{x}(t) = -\mu(t)(G_1^{-1}(x(t))) + \lambda(t) \quad (2.7)$$

The $M/D/1$ queueing system is made of Poisson arrival, one exponential (Poisson) server, FIFO (First-In-First-Out). Thus, $M/D/1$ queueing system's G_1 (c.f., [1]) reads:

$$G_1(x) = ((x + 1) - \sqrt{(x + 1)^2 - 1}) \quad (2.8)$$

Therefore, the time varying $M/D/1$ queueing system's -PSFFA model reads:

$$\dot{x} = -\mu((x + 1) - \sqrt{(x + 1)^2 - 1}) + \lambda \quad (2.9)$$

Non-stationary queues' life example [6] is depicted by Figure 1.

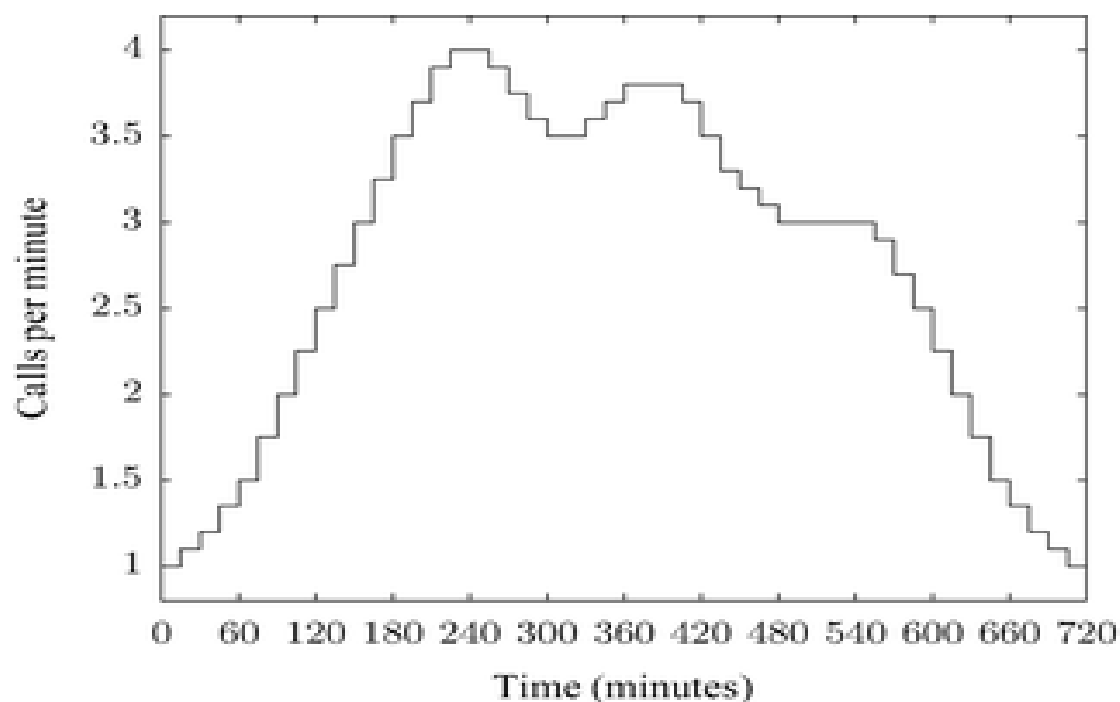


Figure 1.

3. Solving the non-stationary $M/D/1$ queueing system's PSFFA (c.f., (2.9))

Theorem 3.1 Ismail's ration, β solves (2.9), with a closed form expression to read as:

$$(x - \beta(x + 1))^{1/(1-\beta)^2} = \gamma e^{\int \frac{(x+1)}{(1-\beta)} \mu(t) dt}, \beta = \frac{\lambda(t)}{\mu(t)} \quad (3.1)$$

Proof

We have

$$x' = -\mu((x + 1) - \sqrt{(x^2 + 1)}) + \lambda \quad (\text{c.f., (2.9)})$$

Let, then $x = -y \operatorname{coth} y \operatorname{csch} y$. Setting, $\beta = \frac{\lambda(t)}{\mu(t)}$. Thus, we have

$$-y \operatorname{coth} y \operatorname{csch} y = -\mu((1 + \operatorname{csch} y) - \operatorname{coth} y) + \lambda = -\mu[(1 + \operatorname{csch} y) - \operatorname{coth} y] - \beta] \quad (3.2)$$

Therefore, we have

$$\frac{-\operatorname{coth} y \operatorname{csch} y dy}{[(1 + \operatorname{csch} y) - \operatorname{coth} y] - \beta} = -\mu dt$$

$$\frac{\operatorname{cosh} y dy}{[-(1 + \operatorname{sinh} y - \operatorname{cosh} y) + \beta \operatorname{sinh} y] \operatorname{sinh} y} = -\mu dt \quad (3.3)$$

$$\frac{\frac{2}{\beta} (e^{3y} + e^y) dy}{\left[e^{2y} - \frac{e^y}{\beta} + \left(\frac{2}{\beta} - 1 \right) \right] (e^{2y} - 1)} = -\mu dt$$

$$e^{2y} - \frac{e^y}{\beta} + \left(\frac{2}{\beta} - 1 \right) = 0 \Rightarrow e^y = a, b$$

where

$$a = \frac{\left(\frac{1}{\beta} + \sqrt{\frac{1}{\beta^2} - \frac{8}{\beta} + 4} \right)}{2}, b = \frac{\left(\frac{1}{\beta} - \sqrt{\frac{1}{\beta^2} - \frac{8}{\beta} + 4} \right)}{2}$$

Let

$$\frac{\frac{2}{\beta}(e^{3y}+e^y)}{[e^{2y}-\frac{e^y}{\beta}+(\frac{2}{\beta}-1)](e^{2y}-1)} = \frac{A}{(e^y-a)} + \frac{B}{(e^y-b)} + \frac{C}{(e^y-1)} + \frac{D}{(e^y+1)} \quad (3.4)$$

Hence, it is implied that:

$$A + B + C + D = \frac{2}{\beta} \Rightarrow A = \frac{2}{\beta} - (B + C + D)$$

$$\therefore B = \frac{(1-a)(C-D)}{(\frac{2}{\beta}+a)} \quad (3.5)$$

Thus,

$$C = \frac{\frac{4}{\beta}(\frac{2}{\beta}+a) - \left[\left(1 + \frac{2}{\beta} + 2a\right) - (ab - (a+b)) \left(\frac{2}{\beta} + a\right) \right] D}{\left(\frac{2}{\beta}\right) + a + [ab - (a+b)] \left(\frac{2}{\beta} + a\right)}$$

$$bA + aB + abC - abD = 0$$

Therefore, we have

D

$$= \frac{\frac{2b}{\beta} + a \left(ab - \frac{b(1+\frac{2}{\beta})}{(\frac{2}{\beta}+a)} + a \left[\frac{(1-a)}{(\frac{2}{\beta}+a)} \right] \right) \left[\frac{\frac{4}{\beta}(\frac{2}{\beta}+a)}{(\frac{2}{\beta}) + a + [ab - (a+b)] (\frac{2}{\beta} + a)} \right]}{(ab + b \left[\frac{(1+\frac{2}{\beta}+2a)}{(\frac{2}{\beta}+a)} \right] + a \left[\frac{(1-a)}{(\frac{2}{\beta}+a)} \right]) + a \left(ab - \frac{b(1+\frac{2}{\beta})}{(\frac{2}{\beta}+a)} + a \left[\frac{(1-a)}{(\frac{2}{\beta}+a)} \right] \right) \left[\frac{[(1+\frac{2}{\beta}+2a) - (ab - (a+b)) (\frac{2}{\beta} + a)]}{(\frac{2}{\beta}) + a + [ab - (a+b)] (\frac{2}{\beta} + a)} \right]} \quad (3.6)$$

$C =$

$$\frac{\frac{4}{\beta}(\frac{2}{\beta}+a) - \frac{2b}{\beta} \left[(1+\frac{2}{\beta}+2a) - (ab - (a+b)) (\frac{2}{\beta} + a) \right] + \frac{4a}{\beta} (\frac{2}{\beta} + a) \left(ab - \frac{b(1+\frac{2}{\beta})}{(\frac{2}{\beta}+a)} + a \left[\frac{(1-a)}{(\frac{2}{\beta}+a)} \right] \right)}{(ab + b \left[\frac{b(1+\frac{2}{\beta}+2a)}{(\frac{2}{\beta}+a)} + a(1-a) \right] \frac{2}{\beta} + a + [ab - (a+b)] (\frac{2}{\beta} + a) + a \left(ab - \frac{b(1+\frac{2}{\beta})}{(\frac{2}{\beta}+a)} + a \left[\frac{(1-a)}{(\frac{2}{\beta}+a)} \right] \right) \left[(1+\frac{2}{\beta}+2a) - (ab - (a+b)) (\frac{2}{\beta} + a) \right])} \left(\frac{2}{\beta} \right) + a + [ab - (a+b)] \left(\frac{2}{\beta} + a \right) \quad (3.7)$$

$$A = \frac{2}{\beta} - \frac{(1+\frac{2}{\beta})C}{(\frac{2}{\beta}+a)} - \frac{(1+\frac{2}{\beta}+2a)D}{(\frac{2}{\beta}+a)}, \quad B = \frac{(1-a)(C-D)}{(\frac{2}{\beta}+a)} \quad (3.8)$$

This finally solves the complicated mathematical computations to obtain:

$$\left[\frac{Ae^{-y}}{(1-ae^{-y})} + \frac{Be^{-y}}{(1-be^{-y})} + \frac{Ce^{-y}}{(1-e^{-y})} + \frac{De^{-y}}{(1+e^{-y})} \right] dy = -\mu dt$$

Integrating both sides, implies for some non – negative real constant parameter η ,

$$\frac{|(1-ae^{-y})|^A |(1-be^{-y})|^B |(1-e^{-y})|^C}{(1+e^{-y})^D} = \eta e^{-\int \mu dt} \quad (3.9)$$

This transforms to the final required closed form solution:

$$\frac{|(1-ae^{-csch^{-1}(x)})|^A |(1-be^{-csch^{-1}(x)})|^B |(1-e^{-csch^{-1}(x)})|^C}{(1+e^{-csch^{-1}(x)})^D} = \eta e^{-f \mu dt} \quad (3.10)$$

By mathematical analysis, it is well known that:

$$csch^{-1}(x) = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) \quad (3.11)$$

with the domain of real line with zero removed

Thus, one gets

$$\frac{(|(1-\frac{ax}{1+\sqrt{1+x^2}})|)^A (|(1-\frac{bx}{1+\sqrt{1+x^2}})|)^B (|(1-\frac{x}{1+\sqrt{1+x^2}})|)^C}{(1+\frac{x}{1+\sqrt{1+x^2}})^D} = \eta e^{-f \mu dt} \quad (3.12)$$

Corollary 2:

As $x(t) \rightarrow 0$, we have

$$\eta e^{-f \mu dt} = \lim_{x(t) \rightarrow 0} \frac{\left(|1 - \frac{ax}{1 + \sqrt{1 + x^2}}|\right)^A \left(|1 - \frac{bx}{1 + \sqrt{1 + x^2}}|\right)^B \left(|1 - \frac{x}{1 + \sqrt{1 + x^2}}|\right)^C}{\left(1 + \frac{x}{1 + \sqrt{1 + x^2}}\right)^D} = 1$$

Implying

$$\int \mu dt = \ln \eta \quad (3.13)$$

Corollary 3

As $x(t) \rightarrow \infty$, we have

$$\begin{aligned} \eta e^{-f \mu dt} &= \lim_{x(t) \rightarrow \infty} \frac{\left(|1 - \frac{ax}{1 + \sqrt{1 + x^2}}|\right)^A \left(|1 - \frac{bx}{1 + \sqrt{1 + x^2}}|\right)^B \left(|1 - \frac{x}{1 + \sqrt{1 + x^2}}|\right)^C}{\left(1 + \frac{x}{1 + \sqrt{1 + x^2}}\right)^D} \\ &= \lim_{x(t) \rightarrow \infty} \frac{\left(|1 - \frac{a}{\frac{1}{x} + \sqrt{1 + (\frac{1}{x})^2}}|\right)^A \left(|1 - \frac{b}{\frac{1}{x} + \sqrt{1 + (\frac{1}{x})^2}}|\right)^B \left(|1 - \frac{1}{\frac{1}{x} + \sqrt{1 + (\frac{1}{x})^2}}|\right)^C}{\left(1 + \frac{1}{\frac{1}{x} + \sqrt{1 + (\frac{1}{x})^2}}\right)^D} \\ &= \frac{(|1-a|)^A (|1-b|)^B \left(|1 - \frac{1}{\frac{1}{x} + \sqrt{1 + (\frac{1}{x})^2}}|\right)^C}{\left(1 + \frac{1}{\frac{1}{x} + \sqrt{1 + (\frac{1}{x})^2}}\right)^D} = 0 \rightarrow \int \mu dt \rightarrow \infty \end{aligned} \quad (3.14)$$

Numerical experiments

Numerical experiment One

Let $\beta = 2$, then $a = 0.5, b = 0, \eta = 1, \mu(t) = t, A = -7.923076923, B = 1.846153846, C = 6, D = 0.4615384615$

We have

$$\sqrt{2} \left[\frac{(|(1-\frac{0.5x}{1+\sqrt{1+x^2}})|)^{-7.923076923} (|(1-\frac{x}{1+\sqrt{1+x^2}})|)^6}{(1+\frac{x}{1+\sqrt{1+x^2}})^{0.4615384615}} \right] = t$$

Figure 2 shows a new phenomenon to queuing theorist. The possibility that time will converge to a certain value for sufficiently large number in the time varying $M/D/1$ queuing system. This is for an increasing temporal mean service rate.

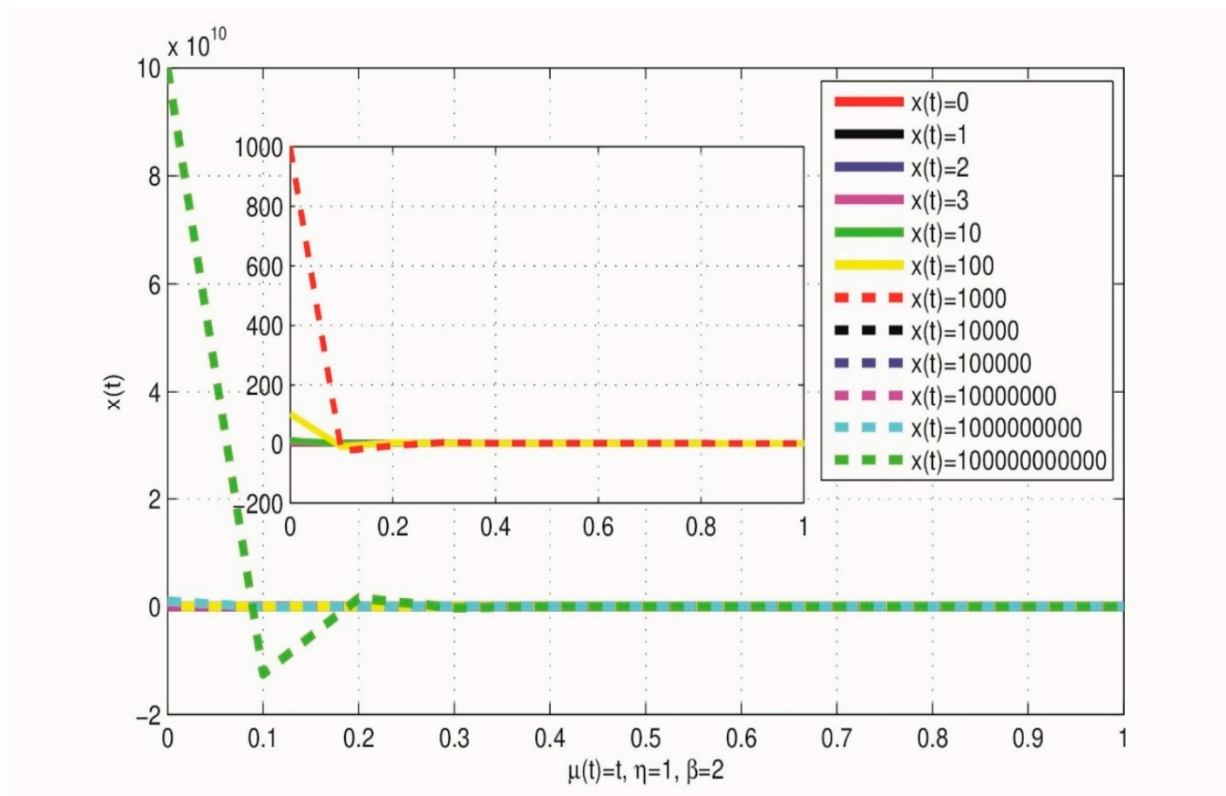


Figure 2.

This shows that as the time varying $M/D/1$ queuing system's state variable becomes sufficiently large, time vanishes.

Numerical experiment Two

Let $\beta = 2$, then $a = 0.5, b = 0, \eta = 1, \mu(t) = \frac{1}{t}$

$A = -7.923076923, B = 1.846153846, C = 6, D = 0.4615384615$

We have

$$\left[\frac{\left(\left| 1 - \frac{0.5x}{1 + \sqrt{1+x^2}} \right| \right)^{7.923076923} \left(\left| 1 - \frac{x}{1 + \sqrt{1+x^2}} \right| \right)^{-6}}{\left(1 + \frac{x}{1 + \sqrt{1+x^2}} \right)^{-0.4615384615}} \right] = t$$

Figure 3 visualizes a new phenomenon to queuing theorist. The possibility that time will converge to a certain value for sufficiently large number in the time varying $M/D/1$ queuing system. This is for a decreasing temporal mean service rate.

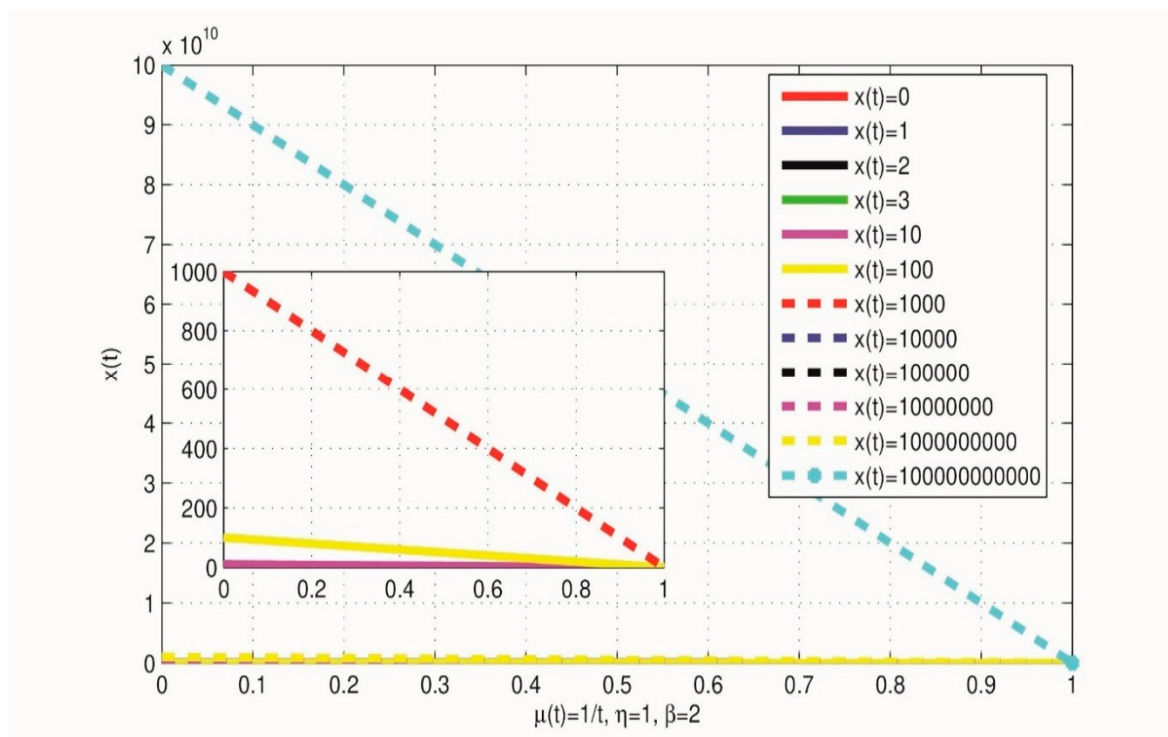


Figure 3.

4. Some PSFFA applications to IoT

By describing how vehicles in a platoon use 802.11p communication to exchange messages and change their movement characteristics at intersections, a time-dependent model for assessing the platooning communications' effectiveness at intersections phase was thoroughly investigated[7]. The model evaluates the effectiveness of platooning communications and addresses potential safety concerns by considering variables including vehicle behaviours, traffic signals, and the changing connectivity among vehicles.

The authors [7] used PSFFA to describe the transmission queue's dynamic behaviour in platooning communications. They also create models that characterise the continuous backoff freeze and four access categories (ACs) of 802.11p as they relate to the time-dependent access procedure.

For 802.11p communication in platooning situations at junctions, the authors created models. [7] consider continual backoff freezing and use the PSFFA to describe the gearbox queue's dynamic behaviour. The access process with its four Access Categories (ACs), they also use a z-domain linear model was demonstrated by [7].

The display of time-dependent packet transmission delay and packet delivery ratio of four Acs can be visualized by Figures 4a, 4b, 4c, 5a, 5b and 5c, respectively.

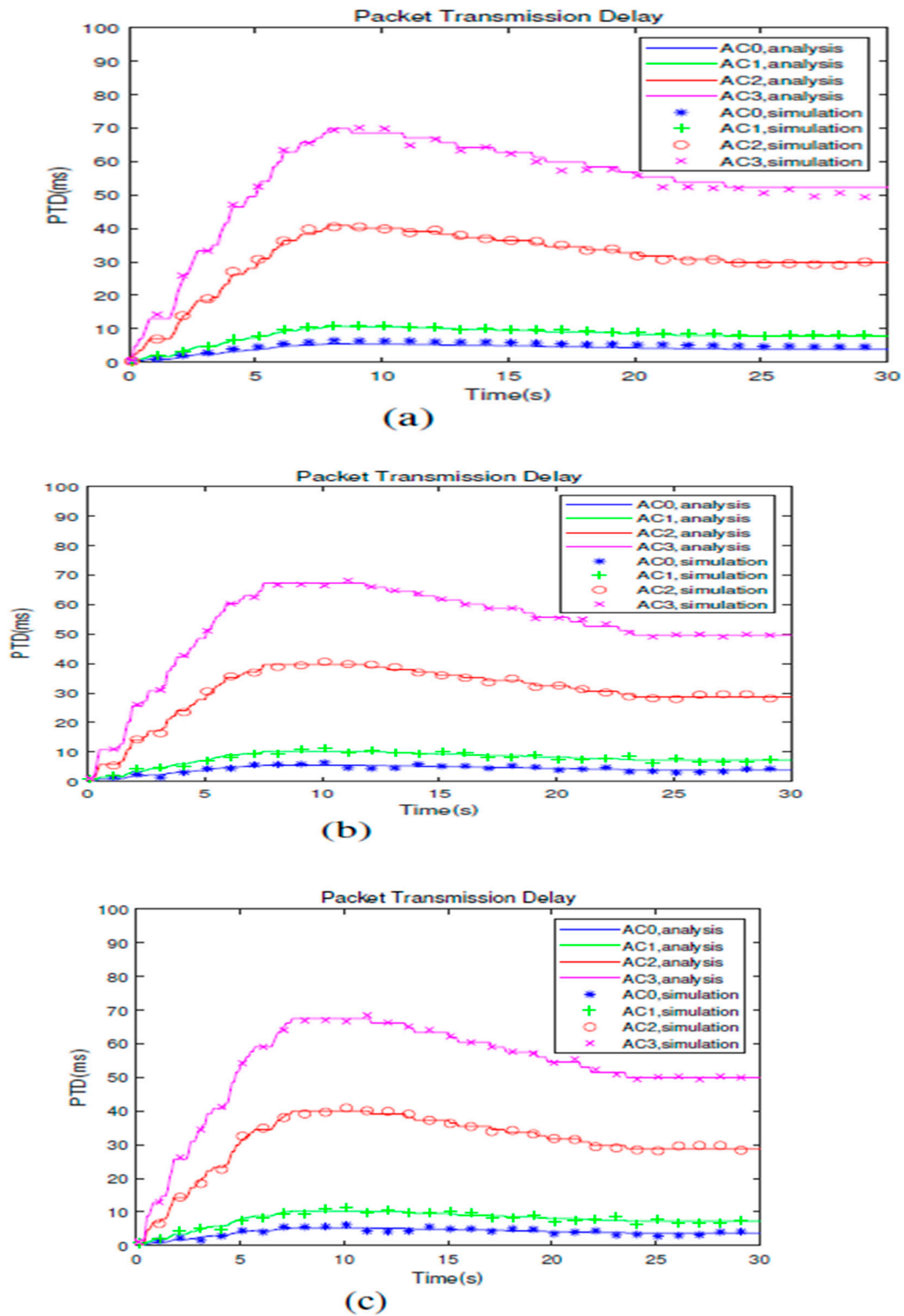
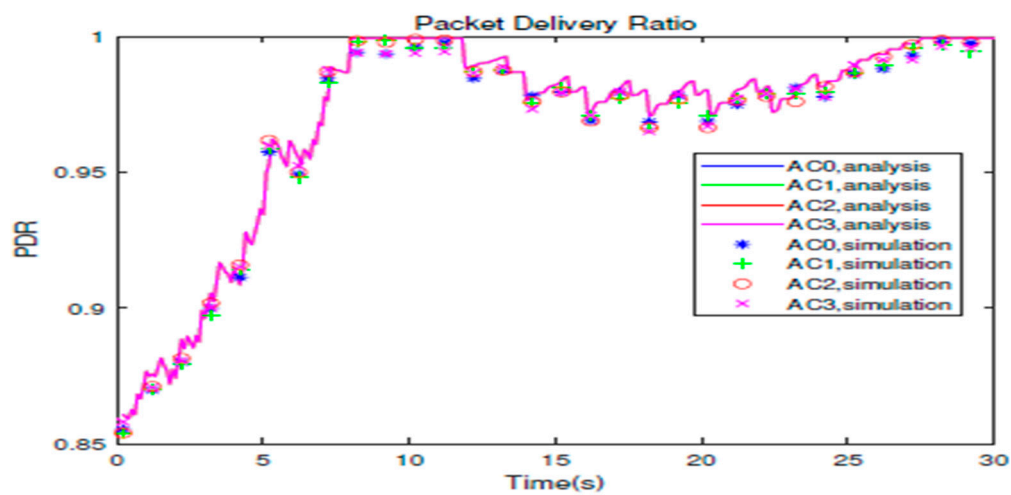
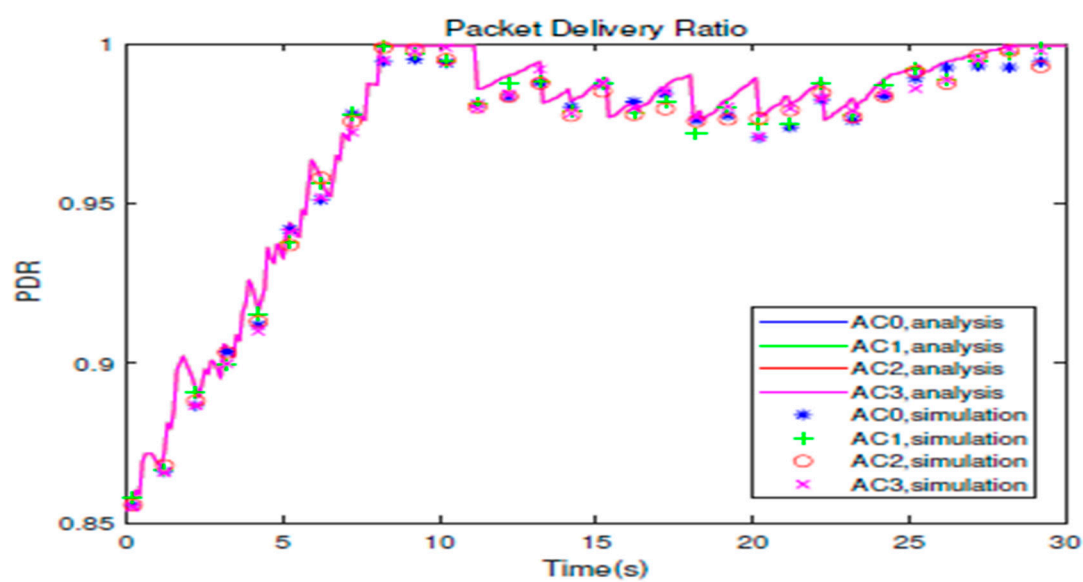


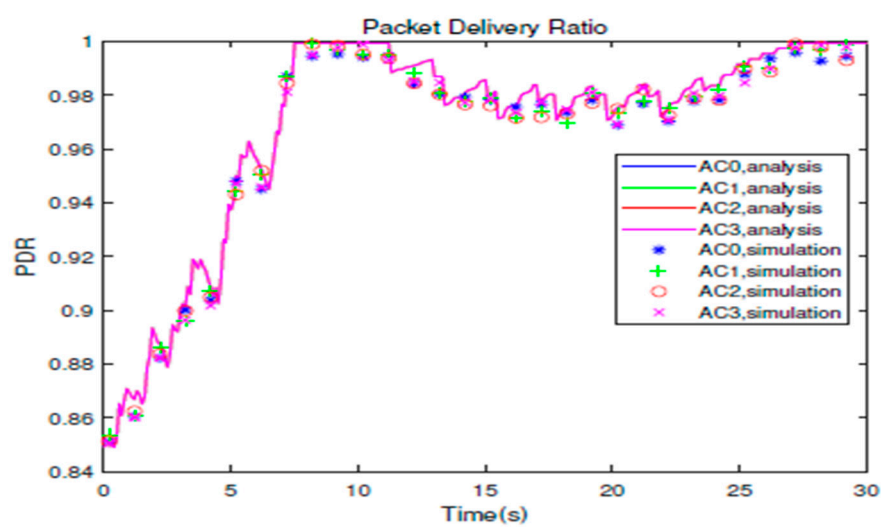
Figure 4. The packet transmission delay that is time dependent. A left turn, a straight shot, or a right turn, respectively[7].



(a)



(b)



(c)

Figure 5. The proportion of time-dependent packet deliveries. A left turn, a straight shot, or a right turn, respectively[7].

The findings of a study[8] into the variables influencing queue utilisation dynamics on routers in telecommunication networks. The investigation shows that the average queue length reaches a steady state value following a transient process lasting from a few to tens of seconds when utilising a dynamic model, specifically PSFFA. It is advised to calculate the average queue length using steady state estimations only after the transient process has finished and a more precise differential model may be used.

To accurately predict the average queue length while analysing the average queue length and Quality of Service[8] in a network, a dynamic model with a nonlinear differential equation must be used. Only when the transient process has ended, and the length of the transient process is controlled by variables like flow rate, router interface capacity, and service discipline, are steady-state estimations useful for determining the average queue length. A better choice of queuing models and smaller packet sizes can further hasten the average queue length's convergence.

5. Closing remarks with next phase of research

In this work, a challenging topic in queueing theory is examined; more precisely, the underlying queue's state variable is determined. The article offers a solution to this issue by utilizing a pointwise stationary fluid flow approximation (PSFFA) technique to formulate the non-stationary $M/D/1$ queueing system. Future work will concentrate on resolving open research issues and investigating applications of non-stationary queues in other scientific areas. The study also examines the effects of time, combined with and queueing parameters on the underlying queue's stability dynamics. More fundamentally, some interesting PSFFA applications to IoT are provided. Future work involves further investigation of the impact of $\sigma(t), 1 > \sigma(t) > 0$ on the stability of $G/M/1$ PSFFA model of the non – stationary $E_k/M/1$ and $IPP/M/1$ queueing systems.

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