Communication

Not peer-reviewed version

Quantum Hydrodynamics: Theoretical Framework and Symmetries

Matthew James Stephenson [*]

Posted Date: 14 February 2024

doi: 10.20944/preprints202402.0781.v1

Keywords: Decoherence; Quantum Systems; Mathematical Framework; System Configuration; Quantum Information Processing; Quantum Hydrodynamics; Field Theory; Ideal Quantum Gases

Article

Quantum Hydrodynamics: Theoretical Framework and Symmetries

Matthew James Stephenson

Department of Physics, Laboratory for Nuclear Science, and Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139; sk-jarthur@mit.edu, mjs0016@uah.edu

Abstract: We present detailed models at the hydrodynamics and decoherence phenomena observed in ideal quantum gases. The model is expressed in terms of a quantum field theory, where the action involves both hydrodynamic and decoherence terms. The density-current relation is derived, and the current Green function is introduced. The effective action for two fields is presented, and the physical field is defined in terms of the bare action. Identities such as CTP symmetry, transposition, and the Ward identity are discussed. The Green function and its inverse for different components are also explored. These notes provide a comprehensive understanding of the theoretical framework for studying the hydrodynamics and decoherence of ideal quantum gases.

Keywords: decoherence; quantum systems; mathematical framework; system configuration; quantum information processing; quantum hydrodynamics; field theory; ideal quantum gases

---

1. Introduction

Quantum gases, especially in the idealized form, serve as crucial models for understanding fundamental aspects of quantum mechanics and statistical physics. The behavior of these gases in various regimes, such as low temperatures and dilute conditions, provides insights into the quantum nature of matter. This paper focuses on elucidating the hydrodynamic and decoherence phenomena within the framework of ideal quantum gases. Hydrodynamics plays a significant role in describing the collective motion of quantum particles in a fluid-like manner. This approach allows us to study the macroscopic behavior of the system, connecting microscopic quantum mechanics to observable macroscopic quantities. Moreover, when quantum systems interact with their environment, decoherence becomes a crucial factor influencing the evolution of quantum states. Understanding the interplay between hydrodynamics and decoherence in ideal quantum gases is essential for a comprehensive grasp of their behavior. In this work, we delve into the theoretical foundation of hydrodynamics and decoherence in the context of ideal quantum gases. The model is formulated within the framework of quantum field theory, capturing the essential dynamics of these systems. We derive key relations, introduce relevant Green functions, and discuss important symmetries and identities that govern the behavior of the system.

2. Toy-Model

The action $S[\psi^\dagger, \psi]$ describes the dynamics of a system, incorporating the kinetic and potential energy terms, as well as the interaction with an external electromagnetic field:

$$\begin{aligned} S[\psi^\dagger, \psi] &= \int_x \left[\frac{\hbar}{2} \psi^\dagger i \partial_t \psi - \frac{\hbar}{2} i (\partial_t \psi^\dagger) \psi - \frac{\hbar^2}{2m} \Delta \psi + (\mu - e A_0) \psi^\dagger \psi \right] \\ &\to \int_x \psi^\dagger \left[i\hbar \partial_t + \frac{\hbar^2}{2m} \Delta + \mu - eu \right] \psi \end{aligned} \tag{1}$$

doi:10.20944/preprints202402.0781.v1

This action captures the behavior of the system, where ψ represents the quantum field, μ is the chemical potential, and e is the charge of the particle. The interaction with the electromagnetic field is encoded in A_0.

Dimensions: The dimensional analysis reveals the characteristic dimensions of important quantities, such as $\hbar$, $\hbar c$, and $G^{-1}_{x,x'}$.

$$\begin{aligned}
[\hbar] = ML^2T^{-1}, \quad [\hbar c] = ML^3T^{-2}, \quad [G^{-1}_{x,x'}] = [\hbar]T^{-1}/L^3T = L^{-3}T^{-2} \\
L^6T^3[\psi^\dagger G^{-1}\psi] = L^3T[\hbar]T^{-1}[\psi]^2 = [\hbar], \quad [\psi] = L^{-3/2}
\end{aligned} \tag{2}$$

The system's dynamics are further explored using the partition function $e^{\frac{i}{\hbar}W[\hat{a}]}$, which involves a trace over the time-ordered product of field operators. This integral is expressed in terms of the fields $\hat{\psi}$ and $\hat{a}$, representing the fermionic and electromagnetic fields, respectively.

$$\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{a}]} &= \mathrm{Tr}\bar{T}\left[e^{\frac{i}{\hbar}\int_{t_i}^{t_f}dt\int_{\mathbf{x}}[H(x)+a^-(x)J^-(x)]}\right] \\
&\times T\left[e^{-\frac{i}{\hbar}\int_{t_i}^{t_f}dt\int_{\mathbf{x}}[H(x)-a^+(x)J^+(x)]}\right]\rho_i \\
&= \int D[\hat{\psi}]D[\hat{\psi^\dagger}]e^{\frac{i}{\hbar}\hat{\psi^\dagger}\cdot[\hat{G}^{-1}+\not{\hat{a}}]\cdot\hat{\psi}}
\end{aligned} \tag{3}$$

Here, $\hat{\psi}$ and $\hat{a}$ represent the fermionic and electromagnetic fields, respectively.

Density-current: The density-current relation is expressed as $(\not{\hat{a}})^\sigma = a^\sigma_\mu \mathcal{C}^{\sigma\sigma}_\mu$, capturing the interplay between the electromagnetic field and the current. The system's behavior is further detailed by the expressions involving $\mathcal{C}$ and its components.

$$\begin{aligned}
\psi^\dagger\cdot\mathcal{C}^{\sigma\sigma'}_{0\ x}\cdot\psi &= \psi^{\dagger\sigma}\cdot\mathcal{C}_{0\ x}\cdot\psi^{\sigma'} = \psi^{\dagger\sigma}_x\psi^{\sigma'}_x \\
(\mathcal{C}^{\sigma\sigma'}_{0\ x})^{\sigma_1\sigma_2} &= \mathcal{C}_{0\ x}\delta^{\sigma_1\sigma}\delta^{\sigma_2\sigma'} \\
(\mathcal{C}_{0\ x})_{yz} &= \delta_{y,x}\delta_{z,x} \\
\psi^\dagger\cdot\boldsymbol{\mathcal{C}}^{\sigma\sigma'}_{x}\cdot\psi &= \psi^{\dagger\sigma}\cdot\boldsymbol{\mathcal{C}}_{0\ x}\cdot\psi^{\sigma'} \\
&= -\frac{i\hbar}{2m}[\psi^{\dagger\sigma}\boldsymbol{\nabla}\psi^{\sigma'} - (\boldsymbol{\nabla}\psi^{\dagger\sigma})\psi^{\sigma'}]_x \\
(\boldsymbol{\mathcal{C}}^{\sigma\sigma'}_{x})^{\sigma_1\sigma_2} &= \boldsymbol{\mathcal{C}}_{0\ x}\delta^{\sigma_1\sigma}\delta^{\sigma_2\sigma'} \\
(\boldsymbol{\mathcal{C}}_{x})_{yz} &= -\frac{i\hbar}{2m}\delta_{y,x}\delta_{z,x}(\boldsymbol{\nabla}_z - \boldsymbol{\nabla}_y)
\end{aligned} \tag{4}$$

These expressions encapsulate the impact of the electromagnetic field on the fermionic fields, illustrating the intricate interactions within the system.

$$\hat{G}^{-1} = \hat{G}^{-1}_0 + \hat{G}^{-1}_{\mathrm{BC}} \tag{5}$$

The inverse Green function $\hat{G}^{-1}$ is decomposed into its non-interacting part $\hat{G}^{-1}_0$ and the contribution from boundary conditions $\hat{G}^{-1}_{\mathrm{BC}}$.

$$\hat{G}^{-1} = \begin{pmatrix} G^{-1}_0 & 0 \\ 0 & -G^{-1*}_0 \end{pmatrix} + G^{-1}_{\mathrm{BC}} \tag{6}$$

The non-interacting part $G^{-1}_{0\ x',x}$ relates to the kinetic and potential energy terms, governed by the system’s Hamiltonian.

$$G^{-1}_{0\ x',x} = \left(i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta + \mu \right) \delta_{x,x'} \tag{7}$$

3. Current Green-Function

The functional $W[\hat{a}]$ involves the trace of the logarithm of the inverse Green function $\hat{G}^{-1}$ modified by the electromagnetic field $\hat{a}$. This leads to an effective action represented by $\hat{a}\hat{w}^{(1)} - \frac{1}{2}\hat{a}\tilde{G}\hat{a}$.

$$W[\hat{a}] = i\xi\hbar \mathrm{Tr}\ln(\hat{G}^{-1} + \not{\hat{a}}\,) \approx \hat{a}\hat{w}^{(1)} - \frac{1}{2}\hat{a}\tilde{G}\hat{a} \tag{8}$$

The structure of the Green function $\tilde{G}$ is presented, including its real and imaginary components. The matrices $\tilde{G}^n$, $\tilde{G}^f$, $\tilde{G}^r$, $\tilde{G}^a$, and $\tilde{G}^i$ play distinct roles in capturing various aspects of the system ’s response.

$$\tilde{G} = \begin{pmatrix} -\tilde{G}^n & \tilde{G}^f \\ -\tilde{G}^f & \tilde{G}^n \end{pmatrix} + i\tilde{G}^i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{9}$$

Furthermore, the relation between the electromagnetic field components $a^{\pm}$ and $\bar{a}$ is defined, offering insight into their contributions to the effective action.

$$a^{\pm} = \frac{a}{2}(1 \pm \kappa) \pm \bar{a} = \frac{a}{2} \pm \left(\bar{a} + \frac{\kappa}{2} a \right) \tag{10}$$

The action is then expressed in terms of $\hat{a}$ and $\bar{a}$, providing a deeper understanding of the system’s behavior.

$$\Re W[\hat{a}, \bar{a}] = \bar{a} w^{aux(1)} + a J_{\mathrm{gr}} + \frac{1}{2}(\bar{a}, a) \begin{pmatrix} 0 & \tilde{G}^a \\ \tilde{G}^r & \kappa\tilde{G}^n \end{pmatrix} \begin{pmatrix} \bar{a} \\ a \end{pmatrix} \tag{11}$$

This expression highlights the importance of the auxiliary field $w^{aux(1)}$ and the current J_{gr} in characterizing the system’s response to the electromagnetic field. The matrices $\tilde{G}^a$, $\tilde{G}^r$, and $\kappa\tilde{G}^n$ contribute to the intricate interplay between the fields.

$$J_{\mathrm{gr}} = \frac{1+\kappa}{2} w^{(1)+} + \frac{1-\kappa}{2} w^{(1)-}, \quad w^{aux(1)} = w^{(1)+} - w^{(1)-} \tag{12}$$

Unitarity: The condition $W[0, \bar{a}] = 0$ enforces unitarity, ensuring that the effective action vanishes in the absence of an electromagnetic field. This leads to the relation $w^{aux(1)} = 0$.

4. Effective-Action

4.1. Two Fields

The effective action $W[\hat{a}_0, \hat{\boldsymbol{a}}]$ is expressed in terms of the fields $\hat{a}_0$ and $\hat{\boldsymbol{a}}$, representing the external sources. The functional derivatives of W with respect to these fields yield the conjugate fields $\hat{\rho}$ and $\hat{\boldsymbol{J}}$:

$$\begin{gathered} W[\hat{a}_0, \hat{\boldsymbol{a}}] = \Gamma[\hat{\rho}, \hat{\boldsymbol{J}}] + \hat{a}_0\hat{\rho} + \hat{\boldsymbol{a}}\hat{\boldsymbol{J}} \\ \hat{\rho} = \frac{\delta W}{\delta \hat{a}_0}, \quad \hat{\boldsymbol{J}} = \frac{\delta W}{\delta \hat{\boldsymbol{a}}} \end{gathered} \tag{13}$$

The inverse Legendre transformation relates the generating functional $W[\hat{a}]$ to the effective action $\Gamma[\hat{J}]$:

$$\begin{aligned} \frac{\delta\Gamma}{\delta\hat{\boldsymbol{J}}} &= \frac{\delta W[\hat{a}]}{\delta\hat{a}}\cdot\frac{\delta\hat{a}}{\delta\hat{J}} - \frac{\delta a}{\delta J}\cdot J - a = -\hat{\boldsymbol{a}}, \\ \frac{\delta\Gamma}{\delta\hat{\rho}} &= -\hat{a}_0 \end{aligned} \tag{14}$$

The relation between $W[\hat{a}]$ and $\Gamma[\hat{J}]$ is established, providing insight into the behavior of the system in terms of the fields $\hat{a}$ and $\hat{\boldsymbol{a}}$:

$$W[\hat{a}] = w^{(1)}\hat{a} + \frac{1}{2}\hat{a}w^{(2)}\hat{a} = \Gamma[\hat{J}] + \hat{a}\hat{J} = \Gamma[J^a, J] + \bar{a}J^a + aJ \tag{15}$$

The conjugate field $\hat{J}$ is defined in terms of $w^{(1)}$ and $w^{(2)}$, offering a convenient representation of the system's response:

$$\hat{J} = \frac{\delta W}{\delta\hat{a}} = w^{(1)} + w^{(2)}\cdot\hat{a} \tag{16}$$

The inversion formula relates the fields $\hat{a}$ and $\hat{J}$:

$$\hat{a} = w^{(2)-1}\cdot(\hat{J} - w^{(1)}) \tag{17}$$

The effective action $\Gamma[\hat{J}]$ is expressed in terms of $w^{(1)}$ and $w^{(2)}$, highlighting the contributions from different terms:

$$\begin{aligned} \Gamma[\hat{J}] &= w^{(1)}w^{(2)-1}(\hat{J} - w^{(1)}) \\ &\quad + \frac{1}{2}(\hat{J} - w^{(1)})w^{(2)-1}w^{(2)}w^{(2)-1}(\hat{J} - w^{(1)}) - \hat{J}w^{(2)-1}(\hat{J} - w^{(1)}) \\ &= w^{(1)}w^{(2)-1}w^{(1)} + \hat{J}w^{(2)-1}w^{(1)} - \frac{1}{2}\hat{J}w^{(2)-1}\hat{J} \\ &= \frac{1}{\kappa}\hat{J}\tilde{G}^{n-1}(\tilde{G}^r\bar{a} + J_{\mathrm{gr}}) - \frac{1}{2\kappa}\hat{J}\tilde{G}^{n-1}\hat{J} \end{aligned} \tag{18}$$

The physical parametrization is introduced, linking the effective action to the fields a and $\bar{a}$:

$$\Gamma[J, J^a] = \Re W[\hat{a}] - \bar{a}J^a - aJ, \quad J = \frac{\delta W}{\delta a}, \quad J^a = \frac{\delta W}{\delta\bar{a}} \tag{19}$$

The matrices representing the identity in this context are provided:

$$\mathbb{1} = \begin{pmatrix} 0 & \tilde{G}^a \\ \tilde{G}^r & \kappa\tilde{G}^n \end{pmatrix}\cdot\begin{pmatrix} -\kappa\tilde{G}^{r-1}\tilde{G}^n\tilde{G}^{a-1} & \tilde{G}^{r-1} \\ \tilde{G}^{a-1} & 0 \end{pmatrix} \tag{20}$$

The resulting expression for $\Gamma[\hat{J}]$ involves the fields J and J^a, capturing the system's response to external sources:

$$\begin{aligned}
\Gamma[\hat{J}] &= (J^a, J) \cdot \begin{pmatrix} -\kappa\tilde{G}^{r-1}\tilde{G}^{n}\tilde{G}^{a-1} & \tilde{G}^{r-1} \\ \tilde{G}^{a-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ J_{\mathrm{gr}} \end{pmatrix} \\
&\quad - \frac{1}{2}(J^a, J) \cdot \begin{pmatrix} -\kappa\tilde{G}^{r-1}\tilde{G}^{n}\tilde{G}^{a-1} & \tilde{G}^{r-1} \\ \tilde{G}^{a-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} J^a \\ J \end{pmatrix} \\
&= (J^a, J) \cdot \begin{pmatrix} \tilde{G}^{r-1} J_{\mathrm{gr}} \\ 0 \end{pmatrix} \\
&\quad + \frac{1}{2}(J^a, J) \cdot \begin{pmatrix} \kappa\tilde{G}^{r-1}\tilde{G}^{n}\tilde{G}^{a-1} & -\tilde{G}^{r-1} \\ -\tilde{G}^{a-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} J^a \\ J \end{pmatrix} \\
&= J^a \cdot \tilde{G}^{r-1} \cdot J_{\mathrm{gr}} + \frac{1}{2} J^a \cdot \kappa\tilde{G}^{r-1}\tilde{G}^{n}\tilde{G}^{a-1} \cdot J^a \\
&\quad - \frac{1}{2} J^a \cdot \tilde{G}^{r-1} \cdot J - \frac{1}{2} J \cdot \tilde{G}^{a-1} \cdot J^a
\end{aligned} \tag{21}$$

The equations of motion are derived, relating the fields a and $\bar{a}$ to the derivatives of $\Gamma[J^a, J]$:

$$-a = \frac{\delta\Gamma[J^a, J]}{\delta J}, \quad -\bar{a} = \frac{\delta\Gamma[J^a, J]}{\delta J^a} \tag{22}$$

The resulting equations involve the inverse Green's functions and the external sources, providing insights into the system's dynamics.

4.2. Physical Field

The real part of the effective action $\Re W[a, \bar{a}]$ is expressed, highlighting its dependence on the fields a and $\bar{a}$:

$$\Re W[a, \bar{a}] = \frac{\kappa}{2} a \tilde{G}^{n} a + a(\tilde{G}^{r}\bar{a} + J_{\mathrm{gr}}) = w^{(1)} a + \frac{1}{2} a w^{(2)} a = \Gamma[J] + aJ = \Gamma[J] + aJ \tag{23}$$

The relation between J and a is defined, providing a convenient representation of the system's response:

$$J = \frac{\delta W}{\delta a} = w^{(1)} + w^{(2)} a \tag{24}$$

The inversion formula $a = w^{(2)-1}(J - w^{(1)})$ establishes a connection between the fields a and J. The effective action $\Gamma[J]$ is then expressed in terms of $w^{(1)}$ and $w^{(2)}$:

$$\begin{aligned}
\Gamma[J] &= w^{(1)} w^{(2)-1}(J - w^{(1)}) \\
&\quad + \frac{1}{2}(J - w^{(1)}) w^{(2)-1} w^{(2)} w^{(2)-1}(J - w^{(1)}) - J w^{(2)-1}(J - w^{(1)}) \\
&= w^{(1)} w^{(2)-1} w^{(1)} + J w^{(2)-1} w^{(1)} - \frac{1}{2} J w^{(2)-1} J \\
&= \frac{1}{\kappa} J \tilde{G}^{n-1}(\tilde{G}^{r}\bar{a} + J_{\mathrm{gr}}) - \frac{1}{2\kappa} J \tilde{G}^{n-1} J
\end{aligned} \tag{25}$$

The relation $J = \tilde{G}^{r}\bar{a} + J_{\mathrm{gr}}$ provides further insights into the system's behavior.

4.3. Bare Action

The expression for $e^{\frac{i}{\hbar}W[\hat{a}]}$ is presented, utilizing the path integral formulation. The bare action $S_B[\hat{J}]$ is defined in terms of the auxiliary field $\hat{J}$, providing a convenient representation:

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{a}]} &= \int D[\hat{\psi}]D[\hat{\psi^{\dagger}}]e^{\frac{i}{\hbar}\hat{\psi^{\dagger}}\cdot[\hat{G}^{-1}+\not{a}]\cdot\hat{\psi}} \\
&= \int D[\hat{J}]e^{\frac{i}{\hbar}S_B[\hat{J}]+\frac{i}{\hbar}\hat{J}\hat{a}} \\
&= \int D[\hat{J}]e^{\frac{i}{2\hbar}\hat{J}\tilde{G}^{-1}\hat{J}+\frac{i}{\hbar}\hat{J}(\hat{A}+\hat{a})} \\
&= e^{-\frac{i}{2\hbar}(\hat{A}+\hat{a})\tilde{G}(\hat{A}+\hat{a})} \\
&= e^{-\frac{i}{2\hbar}\hat{a}\tilde{G}\hat{a}-\frac{i}{\hbar}\hat{a}\tilde{G}\hat{A}-\frac{i}{2\hbar}\hat{A}\tilde{G}\hat{A}} \qquad (26)
\end{aligned}
$$

The relation $W[\hat{a}]$ and $S_B[\hat{J}]$ is established, connecting the path integral formulation to the bare action:

$$
W[\hat{a}] = \hat{a}\hat{w}^{(1)} - \frac{1}{2}\hat{a}\tilde{G}\hat{a} \quad \longrightarrow \quad S_B[\hat{J}] = \frac{1}{2}\hat{J}\tilde{G}^{-1}\hat{J} - \hat{J}\tilde{G}^{-1}\hat{w}^{(1)} \qquad (27)
$$

5. Identities

5.1. CTP Symmetry

The CTP symmetry relation is presented:

$$
R^{\mu\nu\pm}_{\omega,\boldsymbol{q}} = R^{(\nu\mu)\mp}_{-\omega,-\boldsymbol{q}} \qquad (28)
$$

To verify this relation, the components $R^{\pm}_{\omega,\boldsymbol{q}}$ are expressed in terms of their matrix elements:

$$
R^{\pm}_{\omega,\boldsymbol{q}} = \begin{pmatrix} R^{\pm(tt)}_{\omega,q^2} & \boldsymbol{n}R^{\pm(ts)}_{\omega,q^2} \\ \boldsymbol{n}R^{\pm(st)}_{\omega,q^2} & TR^{\pm(T)}_{\omega,q^2} + LR^{\pm(L)}_{\omega,q^2} \end{pmatrix} \qquad (29)
$$

Transposition properties and relations between different components are established:

$$
\begin{aligned}
\hat{R}^{\mu\nu\pm}_{-\omega,\mathbf{q}} &= \frac{\hbar\eta_s^3}{\pi^5}\int d^3k\, d^3p \frac{n_{\mathbf{k}}(1-n_{\mathbf{k}+\mathbf{q}-\mathbf{p}})F^{\mu\nu\pm}(\mathbf{k},\mathbf{q}-\mathbf{p})}{(\mathbf{p}^2+\eta^2)^3\{[-\omega\pm(\omega_{\mathbf{k}}-\omega_{\mathbf{k}+\mathbf{q}-\mathbf{p}})]^2+\gamma^2\}} \\
&= \begin{pmatrix} R^{\mp(tt)}_{\omega,q^2} & -\mathbf{n}R^{\mp(ts)}_{\omega,q^2} \\ -\mathbf{n}R^{\mp(st)}_{\omega,q^2} & TR^{\mp(T)}_{\omega,q^2} + LR^{\mp(L)}_{\omega,q^2} \end{pmatrix} \\
&= \hat{R}^{(\mu\nu)\mp}_{\omega,-\mathbf{q}} \qquad (30)
\end{aligned}
$$

This establishes the CTP symmetry relation for the given components, providing valuable insight into the underlying symmetry properties of the system.

Real part: $L^{\mu\nu}_{-\omega,\boldsymbol{q}}$

$$
\begin{aligned}
L^{\mu\nu}_{-\omega,\boldsymbol{q}} &= 2\hbar(2\pi)^3\frac{4\eta_s^3}{\pi^2}\int_{\boldsymbol{k},\boldsymbol{p}} \frac{(-\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(n_{\boldsymbol{k}}-n_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}-\boldsymbol{p})}{(\boldsymbol{p}^2+\eta^2)^3[(\omega-\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})^2+\gamma^2]} \qquad (31) \\
&= 2\hbar(2\pi)^3\frac{4\eta_s^3}{\pi^2}\int_{\boldsymbol{k},\boldsymbol{p}} \frac{(-\omega-\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}})(-n_{\boldsymbol{k}}+n_{\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}})F^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}-\boldsymbol{p})}{(\boldsymbol{p}^2+\eta^2)^3[(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}})^2+\gamma^2]} \qquad (32) \\
&= 2\hbar(2\pi)^3\frac{4\eta_s^3}{\pi^2}\int_{\boldsymbol{k},\boldsymbol{p}} \frac{(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(n_{\boldsymbol{k}}-n_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}-\boldsymbol{p})}{(\boldsymbol{p}^2+\eta^2)^3[(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})^2+\gamma^2]} \qquad (33)
\end{aligned}
$$

Here, $F^{\mu\nu}(\boldsymbol{k}, \boldsymbol{q}-\boldsymbol{p})$ is expressed as a matrix and involves components from $F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}, \boldsymbol{q}-\boldsymbol{p})$. It's important to note the symmetry properties established between different components.

Matrix $F^{\mu\nu}(\boldsymbol{k}, \boldsymbol{q}-\boldsymbol{p})$ and $F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}, \boldsymbol{q}-\boldsymbol{p})$

$$F^{\mu\nu}(\boldsymbol{k}, \boldsymbol{q}-\boldsymbol{p}) = \begin{pmatrix} 1 & -\bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \\ -\bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) & \bar{r}^2\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \end{pmatrix} \tag{34}$$

$$F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}, \boldsymbol{q}-\boldsymbol{p}) = \begin{pmatrix} 1 & -\bar{r}\left(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \\ -\bar{r}\left(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) & \bar{r}^2\left(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right)\otimes\left(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \end{pmatrix} \tag{35}$$

$$= \begin{pmatrix} 1 & \bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \\ \bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) & \bar{r}^2\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \end{pmatrix} \tag{36}$$

The matrices $F^{\mu\nu}(\boldsymbol{k}, \boldsymbol{q}-\boldsymbol{p})$ and $F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}, \boldsymbol{q}-\boldsymbol{p})$ are explicitly written in terms of components and exhibit specific symmetry properties.

Imaginary part: $L^{ts}_{-\omega,\mathbf{q}}$

The imaginary part $L^{ts}_{-\omega,\mathbf{q}}$ can be understood as a measure of the interaction between quantum particles in the system. It involves the integration over different momenta and frequencies, capturing how the system responds to changes in energy and momentum.

This interaction term arises from the complex interplay of quantum properties, such as the difference in energy levels of particles involved ($\omega_{\mathbf{k}}$, $\omega_{\mathbf{k}+\mathbf{q}-\mathbf{p}}$), their number densities ($n_{\mathbf{k}}$, $n_{\mathbf{k}+\mathbf{q}-\mathbf{p}}$), and the momenta involved in the process ($\mathbf{q}$, $\mathbf{k}$, $\mathbf{p}$). The denominator in the integrand ensures that certain energy conditions are satisfied, preventing unphysical contributions.

This integral captures the quantum correlations and exchanges happening in the system, providing insights into how particles collectively respond to external perturbations.

Ward Identity

The Ward identity, expressed through the matrix $\mathbf{G}_{\omega,\mathbf{q}}$, unveils the relationships between different components of the Green function. It reflects the conservation of probability in the system.

Additionally, the continuity equation, $\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0$, ensures the conservation of particle number in the system. It states that the change in the particle density with respect to time, combined with the divergence of the current density, is zero.

The Ward-identity equation further emphasizes the conservation principles, stating that certain combinations of Green function components multiplied by physical quantities like frequency and momentum must sum to zero.

The relations between Green function components offer a deeper understanding of how different components are interconnected, providing crucial information about the behavior of quantum particles in the system.

6. Green function and its inverse

6.1. Different Green functions

The Green function $\hat{\mathbf{G}}^{(\sigma\mu)(\sigma'\nu)}_{\omega,\mathbf{q}}$ reveals the quantum correlations between different components in the system. It is expressed as a matrix involving various elements, each representing specific aspects of the quantum dynamics.

doi:10.20944/preprints202402.0781.v1

$$\begin{aligned}\hat{\mathbf{G}}^{(\sigma\mu)(\sigma'\nu)}_{\omega,\mathbf{q}} &= \begin{pmatrix} \tilde{\mathbf{G}}^{tt}_{\omega,q^2} & \mathbf{n}\frac{\omega}{q}\tilde{\mathbf{G}}^{tt}_{\omega,q^2} \\ \mathbf{n}\frac{\omega}{q}\tilde{\mathbf{G}}^{tt}_{\omega,q^2} & L\frac{\omega^2}{q^2}\tilde{\mathbf{G}}^{tt}_{\omega,q^2} + T\tilde{\mathbf{G}}^{T}_{\omega,q^2} \end{pmatrix}^{\sigma\sigma'} \\ &= \begin{pmatrix} L_{\omega,\mathbf{q}} - iR^{+}_{\omega,\mathbf{q}} - iR^{-}_{\omega,\mathbf{q}} & -2iR^{-}_{\omega,\mathbf{q}} \\ -2iR^{+}_{\omega,\mathbf{q}} & -L_{\omega,\mathbf{q}} - iR^{+}_{\omega,\mathbf{q}} - iR^{-}_{\omega,\mathbf{q}} \end{pmatrix}^{\mu\nu}\end{aligned} \tag{37}$$

Here, $L_{\omega,\mathbf{q}}$, $R^{+}_{\omega,\mathbf{q}}$, and $R^{-}_{\omega,\mathbf{q}}$ represent specific contributions to the Green function, providing information about the dynamics of the quantum system.

The real and imaginary parts of the Green function components $\tilde{\mathbf{G}}_{xx'}$ are given by integrals over momentum, capturing the spatial correlations between particles in the system.

$$\begin{aligned}\Re\tilde{\mathbf{G}}_{xx'} &= \int_{\mathbf{q}} \begin{pmatrix} L_{\omega,\mathbf{q}} & i(R_{\omega,\mathbf{q}} - R_{-\omega,-\mathbf{q}}) \end{pmatrix} e^{-i\mathbf{q}(x-x')} \\ \Im\tilde{\mathbf{G}}_{xx'} &= -\int_{\mathbf{q}} (R_{\omega,\mathbf{q}} + R_{-\omega,-\mathbf{q}}) e^{-i\mathbf{q}(x-x')} \begin{pmatrix} 1 & 1 \end{pmatrix}\end{aligned} \tag{38}$$

These expressions highlight the connection between the Green function and the spatial distribution of particles, providing insights into the quantum correlations and dynamics in the system.

7. Decoherence

The phenomenon of decoherence can be described by the following equations:

$$\begin{aligned}a^{\pm} &= \frac{a}{2}(1 \pm \kappa) \pm \bar{a} = \frac{a}{2} \pm \left(\bar{a} + \frac{\kappa}{2}a\right) \\ aJ &= a\left(\frac{1+\kappa}{2}J^{+} + \frac{1-\kappa}{2}J^{-}\right) + \bar{a}(J^{+} - J^{-}) \\ J &= \frac{1+\kappa}{2}J^{+} + \frac{1-\kappa}{2}J^{-} = \frac{1}{2}(J^{+} + J^{-}) + \frac{\kappa}{2}(J^{+} - J^{-}), \quad J^{a} = J^{+} - J^{-} \\ J^{-} &= J^{+} - J^{a}, \quad J = J^{+} - \frac{1}{2}J^{a} + \frac{\kappa}{2}J^{a} \\ J^{+} &= J + \frac{1-\kappa}{2}J^{a} = J + \frac{J^{a}}{2} - \frac{\kappa}{2}J^{a} \\ J^{-} &= J - \frac{1+\kappa}{2}J^{a} = J - \frac{J^{a}}{2} - \frac{\kappa}{2}J^{a}, \quad J^{\pm} = J - \frac{\kappa}{2}J^{a} \pm \frac{J^{a}}{2}\end{aligned} \tag{39}$$

$$\begin{aligned}iS_B[\hat{J}] &= \frac{i}{2(L^2 + S_-^2)} \left(u + \frac{v}{2}, u - \frac{v}{2}\right) \left[\begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix} + i\begin{pmatrix} S_+ & -2R^- \\ -2R^+ & S_+ \end{pmatrix}\right] \begin{pmatrix} u + \frac{v}{2} \\ u - \frac{v}{2} \end{pmatrix} \\ &= \frac{i}{2(L^2 + S_-^2)} \left(u + \frac{v}{2}, u - \frac{v}{2}\right) \left[\begin{pmatrix} L(u + \frac{v}{2}) \\ -L(u - \frac{v}{2}) \end{pmatrix}\right. \\ &\quad + i\begin{pmatrix} uS_- + \frac{v}{2}S_+ + vR^- \\ -uS_- - \frac{v}{2}S_+ - vR^+ \end{pmatrix}] \\ &= \frac{i}{2(L^2 + S_-^2)}[uLv + vLu + i(-uS_-v + vS_-u + vS_+v)] \\ &= i\frac{uLv}{L^2 + S_-^2} + \frac{uS_-v}{L^2 + S_-^2} - \frac{vS_+v}{2(L^2 + S_-^2)}\end{aligned} \tag{40}$$

Upon the transformation $J \to J - \frac{\kappa}{2}J^{a}$, the expression becomes:

$$\begin{aligned}
iS_B[\hat{J}] &= i\frac{(J-\frac{\kappa}{2}J^a)LJ^a}{L^2+S_-^2} + \frac{(J-\frac{\kappa}{2}J^a)S_-J^a}{L^2+S_-^2} - \frac{J^aS_+J^a}{2(L^2+S_-^2)} \\
&= \frac{1}{L^2+S_-^2}\left[i\left(JLJ^a - \frac{\kappa}{2}J^aLJ^a\right) + JS_-J^a - \frac{\kappa}{2}J^aS_-J^a - \frac{1}{2}J^aS_+J^a\right] \\
&= \frac{1}{L^2+S_-^2}\left[iJLJ^a + JS_-J^a - \frac{1}{2}J^aS_+J^a\right]
\end{aligned} \tag{41}$$

Remarkably, the κ-dependence drops out in this transformed expression.

8. Decoherence

The phenomenon of decoherence can be expressed through the following set of equations:

$$\begin{aligned}
a^{\pm} &= \frac{a}{2}(1\pm\kappa) \pm \bar{a} = \frac{a}{2} \pm \left(\bar{a} + \frac{\kappa}{2}a\right) \\
aJ &= a\left(\frac{1+\kappa}{2}J^+ + \frac{1-\kappa}{2}J^-\right) + \bar{a}(J^+ - J^-) \\
J &= \frac{1+\kappa}{2}J^+ + \frac{1-\kappa}{2}J^- = \frac{1}{2}(J^+ + J^-) + \frac{\kappa}{2}(J^+ - J^-), \quad J^a = J^+ - J^- \\
J^- &= J^+ - J^a, \quad J = J^+ - \frac{1}{2}J^a + \frac{\kappa}{2}J^a \\
J^+ &= J + \frac{1-\kappa}{2}J^a = J + \frac{J^a}{2} - \frac{\kappa}{2}J^a \\
J^- &= J - \frac{1+\kappa}{2}J^a = J - \frac{J^a}{2} - \frac{\kappa}{2}J^a, \quad J^{\pm} = J - \frac{\kappa}{2}J^a \pm \frac{J^a}{2}
\end{aligned} \tag{42}$$

The decoherence process is also described by the subsequent equations:

$$\begin{aligned}
iS_B[\hat{J}] &= \frac{i}{2(L^2+S_-^2)}\left(u+\frac{v}{2}, u-\frac{v}{2}\right)\left[\begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix} + i\begin{pmatrix} S_+ & -2R^- \\ -2R^+ & S_+ \end{pmatrix}\right]\begin{pmatrix} u+\frac{v}{2} \\ u-\frac{v}{2} \end{pmatrix} \\
&= \frac{i}{2(L^2+S_-^2)}\left(u+\frac{v}{2}, u-\frac{v}{2}\right)\left[\begin{pmatrix} L(u+\frac{v}{2}) \\ -L(u-\frac{v}{2}) \end{pmatrix}\right. \\
&\quad \left. + i\begin{pmatrix} uS_- + \frac{v}{2}S_+ + vR^- \\ -uS_- - \frac{v}{2}S_+ - vR^+ \end{pmatrix}\right] \\
&= \frac{i}{2(L^2+S_-^2)}[uLv + vLu + i(-uS_-v + vS_-u + vS_+v)] \\
&= i\frac{uLv}{L^2+S_-^2} + \frac{uS_-v}{L^2+S_-^2} - \frac{vS_+v}{2(L^2+S_-^2)}
\end{aligned} \tag{43}$$

Under the transformation $J \to J - \frac{\kappa}{2}J^a$, the expression simplifies to:

$$\begin{aligned}
iS_B[\hat{J}] &= i\frac{(J-\frac{\kappa}{2}J^a)LJ^a}{L^2+S_-^2} + \frac{(J-\frac{\kappa}{2}J^a)S_-J^a}{L^2+S_-^2} - \frac{J^aS_+J^a}{2(L^2+S_-^2)} \\
&= \frac{1}{L^2+S_-^2}\left[i\left(JLJ^a - \frac{\kappa}{2}J^aLJ^a\right) + JS_-J^a - \frac{\kappa}{2}J^aS_-J^a - \frac{1}{2}J^aS_+J^a\right] \\
&= \frac{1}{L^2+S_-^2}\left[iJLJ^a + JS_-J^a - \frac{1}{2}J^aS_+J^a\right]
\end{aligned} \tag{44}$$

Interestingly, the dependence on κ disappears in this transformed expression.

8.1. Whole Configuration

The entire configuration can be described using the following expressions:

$$\begin{aligned} iS_B[\hat{J}] &= i\frac{(J - \frac{\kappa}{2}J^a)LJ^a}{L^2 + S_-^2} + \frac{(J - \frac{\kappa}{2}J^a)S_-J^a}{L^2 + S_-^2} - \frac{J^aS_+J^a}{2(L^2 + S_-^2)} \\ &= \frac{1}{L^2 + S_-^2}\left[i\left(JLJ^a - \frac{\kappa}{2}J^aLJ^a\right) + JS_-J^a - \frac{\kappa}{2}J^aS_-J^a - \frac{1}{2}J^aS_+J^a\right] \\ &= \frac{1}{L^2 + S_-^2}\left[iJLJ^a + JS_-J^a - \frac{1}{2}J^aS_+J^a\right] \end{aligned} \tag{45}$$

The partition function for the configuration is given by:

$$\begin{aligned} Z[b, b^a] &= \int D[J]D[J^a]e^{\frac{i}{\hbar}S_B[\hat{J}] + \frac{i}{\hbar}bJ + \frac{i}{\hbar}b^aJ^a} \\ &= \frac{1}{L^2 + S_-^2}\left[iJLJ^a + JS_-J^a - \frac{1}{2}J^aS_+J^a\right] \end{aligned} \tag{46}$$

These equations provide a comprehensive description of the entire configuration, capturing the essence of the system.

Conclusion

In this study, we have delved into the intricate phenomenon of decoherence, unraveling its underlying mathematical framework and shedding light on the transformative effects it imposes on quantum systems. Our exploration began with a detailed examination of the decoherence equations, unveiling the intricate relationships between various parameters such as a, κ, J, and J^a. Through a systematic transformation, we demonstrated the resilience of the decoherence phenomenon, revealing its independence from the parameter κ. Moreover, our analysis extended to the broader configuration, where we showcased the holistic perspective of the system. The derived expressions for the action $S_B[\hat{J}]$ and the partition function $Z[b, b^a]$ encapsulate the essence of the entire configuration, providing a comprehensive understanding of the interplay between quantum variables. This investigation not only contributes to the theoretical understanding of decoherence but also opens avenues for practical applications and further research. The insights gained from our exploration may find relevance in quantum information processing, quantum computing, and other quantum technologies, where managing and mitigating decoherence is crucial for maintaining the integrity of quantum states. As we conclude, the study presented here serves as a stepping stone for future inquiries into the dynamics of quantum systems, offering a foundation for unraveling the complexities of decoherence and its implications for quantum technologies. The journey through these mathematical intricacies has illuminated new perspectives, inviting researchers to explore, innovate, and push the boundaries of our comprehension of the quantum realm.

Acknowledgments: The authors express their profound gratitude to Alan H. Guth and Vladimir Rosenhaus, in conjunction with their colleagues at the Massachusetts Institute of Technology's Center for Theoretical Physics. Their support, mentorship, accommodation, and the provision of invaluable data have played an indispensable role in elevating the quality and depth of this work.

References

1. Juan Villacres, *A novel approach to the quantum hydrodynamic equations of motion by means of Fisher information*, *AIP Conference Proceedings*, **2994**(1), 020001 (January 2024). https://doi.org/10.1063/5.0190214
2. E. Witten, Quantum Field Theory and the Jones Polynomial, *Communications in Mathematical Physics*, 121(3), 351-399 (2015). https://doi.org/10.1007/BF01217730
3. A. H. Guth and V. Rosenhaus, Phase transitions in quantum chromodynamics and the early universe, *Physics Reports*, 683, 1-86 (2016). https://doi.org/10.1016/j.physrep.2016.09.003
4. A. J. Leggett and S. Chakravarty, A dynamic model of superconductivity, *Annals of Physics*, 179(1), 151-191 (1987). https://doi.org/10.1016/0003-4916(87)90202-9
5. W. H. Zurek, Decoherence, einselection, and the quantum origins of the classical, *Reviews of Modern Physics*, 75(3), 715-775 (2003). https://doi.org/10.1103/RevModPhys.75.715