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Article

In Quest of BRST Symmetry and Observables in 4D Topological Gauge-Affine Gravity

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Abstract: The realization of a BRST cohomology of the 4D topological gauge-affine gravity is established in terms of a superconnection formalism. Identification of fields in the quantized theory is occurred directly as usual in terms of superconnection and its supercurvature components with the double covering of the general affine group $GA(4, \mathbb{R})$. Then, by means of an appropriate decomposition of the metilinear double-covering group $\overline{SL}(5, \mathbb{R})$ with respect to the general linear double-covering group $\overline{GL}(4, \mathbb{R})$ one can easily obtain the enlargements of the fields while remaining consistent with the BRST algebra. This leads to the descent equations allowing us to build the observables of the theory by means of the BRST algebra constructed using a $\overline{sa}(5, \mathbb{R})$ algebra-valued superconnection. In particular, we discuss the construction of topological invariants with torsion.

Keywords: topological models of gravity; topological gauge-affine gravity; off-shell nilpotent BRST and anti-BRST algebra; descent equations; topological observables

1. Introduction

It is well known that characteristic classes constructed by means of curvature only, namely Pontryagin and Euler classes, or the purely torsion-based Nieh-Yan form [1] reveal the global features of a manifold [2], for a short review see [3]. The Nieh-Yan 4-form involving torsion is a total derivative of a Chern-Simons-type 3-form and thus, it corresponds to a torsional invariant which reflects the torsional topological properties of spacetime [3]. Unlike other topological terms, e.g. Holst term, classical equations of motion were shown to be unaffected by adding a Nieh-Yan term to the Lagrangian of matter with spin [4]. Furthermore, this was shown to be related to torsion instantons and physical observables such as existence of anomalies [5]. However, it has provoked controversy about whether or not the Nieh-Yan term contributes to the chiral anomaly in a 4D spacetime with torsion [6]. Pontryagin and Euler forms were also shown to be crucial for non-commutative topological gravity [7]. Besides, symplectic analysis of both curvature-based topological invariants [8] showed that they are different in structure after a quantization process. Recently, a purely torsional Nieh-Yan-like (aka *teleparallel*¹) topological invariant was attached to a scalar field in order to describe inflation scenarios. Also, conformally transformed teleparallel invariant was shown to be a topological invariant [11].

In addition, a Pontryagin-type form was shown to contribute to the chiral anomaly when analyzing the coupling of the axial vector torsion with massive Dirac fields in a Riemann-Cartan spacetime [12]. In [13], an Einstein-Cartan Lagrangian was amended by parity-violating Pontryagin and Nieh-Yan topological terms and thus, the obtained topologically modified model seemed to share an intriguing property with Yang-Mills theory. On the other hand, The torsional topological invariant has found its place in various areas of physics. It was shown that the Nieh-Yan form seems to give the observable effect of chiral anomalies a torsional topological origin [5] and to enter into the interaction between a spinning particle with a gravitational field in a curved spacetime with torsion [14]. Recently, the Nieh-Yan form transport effects were studied via computation of equilibrium partition functions stemming from torsional anomaly [15]. Further, the chiral Nieh-Yan anomaly was shown to be related

¹ The teleparallel or the Nieh-Yan-like topological invariant was first discussed in [9] and it was unintentionally [10] resurrected recently in [11] and further cosmological implications were discussed.

to the hydrodynamic anomaly in superfluid systems [16]. In [17], it was surprisingly shown that an Einstein-Hilbert action is obtained when one imagines a link between a Nieh-Yan density on one hand, constructed mainly by means of only the axial vector torsion, and the gravitational constant on the other hand.

Not long ago, Nieh [18], after reparametrizing the spin connection, noticed the emergence of other six Pontryagin-type and Euler-type invariants involving torsion, while two of them are purely torsional. In the same spirit, the authors of [19] succeeded in constructing more general torsional invariants by generalizing the vierbein 1-forms e^a into arbitrary $SO(4)$ or $SO(3, 1)$ 1-forms in the $SO(5)$ or $SO(4, 1)$ connection, respectively.

As far as topological field theories are concerned, they were basically constructed in order to deal with the problems of non-renormalizability of gravitation theories. In this context, using the BRST antifield formalism, cf. for a review [20], Yang's curvature-squared gravity was deduced from a purely topological exact Pontryagin action plus a Faddeev-Popov type Lagrangian [21]. Standard Einstein gravity with a cosmological constant was shown to emerge with the metric induced as an upshot of spontaneously symmetry breaking mechanism of a topological gauge theory based on the metilinear gauge group $SL(5, \mathbb{R})$ [22].

In the context of quantum Einstein gravity, it was shown that the partition function is tied in with a certain topological observable by dint of an expectation value, which may open the door to reveal a background-independent perturbative character for models of quantum gravity [23]. The author in [24] succeeded in giving a canonical analysis of a self-dual gravity model with topological invariants involving curvature in the first-order formalism. Inspired by the analysis of 4D self-dual gravity in the first-order formalism [25] where self-duality constraints were imposed on both curvature and torsion, a set of topological observables have been constructed for 4D topological gravity in the BRST superspace approach [26].

Inspired by the fact that interactions in nature are mediated by connections (potentials), there is a good reason to consider the affine connection as a mediator of gravitational interaction, namely consider it as dynamical. Therefore, this can perhaps be the starting point in the path towards a metric-affine quantum gravity [27]. The model of gauge-affine gravity can be well-described by gauging the four-dimensional general affine group $GA(4, R) = GL(4, R) \ltimes \mathbb{R}^4$ or its double-covering $\overline{GA}(4, R) = \overline{GL}(4, R) \ltimes \mathbb{R}^4$ [28]. Indeed, this non-Riemannian gauge-affine gravity was shown to be renormalizable without violating unitarity [29]. In the program of attempting to quantize gauge theories of gravity, Becchi-Rouet-Stora-Tyutin [30] (BRST) and anti-BRST algebra of a gauge affine gravity [31] was obtained geometrically using a superspace formalism [32]. In the same spirit, about a couple of decades ago the algebra of BRST transformations was treated using a Hamiltonian formalism; the BRST algebra was shown to be closed even in presence of structural difficulties, such as spacetime dependence of structure functions of the algebra via the field strengths of a metric-affine gauge theory of gravity, namely curvature and torsion [33].

One should note that in this paper we have followed the approach in [34] where by fulfilling the requirements in [35], e.g. compactness of the manifold, the authors used an appropriate embedding $SO(4) \hookrightarrow SO(5)$ in order to construct torsional observables for a topological 4D gravity. In the same spirit, the main aim of this paper is to construct new torsional topological observables for a class of gauge theories, namely topological gauge-affine theories of gravity, and this by dint of enlargements of fields for a gauge group $\overline{SA}(5, R)$. Moreover, as a formalism we choose to use the superconnection formalism, cf. [36]. Our paper is organized as follows: In the next section, using the superspace approach BRST-anti-BRST algebra [37] for a topological gauge-affine model of gravity is obtained, which seems to be nilpotent off-shell. Subsequently, section 3 deals with the construction of new torsional topological observables by means of descent equations based on BRST symmetry. The last section titled Discussion and Conclusion concludes the paper with a brief outlook on forthcoming works.

2. Superconnection formalism and BRST algebra

Let ϕ be a $\overline{GA}(4, \mathbb{R})$ -superconnection on a $(4, 2)$ -dimensional BRST superspace with coordinates $Z = (Z^M) = (x^\mu, \theta^\alpha)$, where $(x^\mu)_{\mu=1, \dots, 4}$ are the coordinates of the 4D metric-affine spacetime manifold (L_4, g) and $(\theta^\alpha)_{\alpha=1, 2}$ are ordinary anticommuting variables.

The superconnection ϕ as 1-superform on the BRST superspace can be written as

$$\phi = dZ^M (\phi_{bM}^a L_a^b + \phi_M^a P_a) \quad (1)$$

where $\{L_b^a, P_a\}$ are the generators of the gauge group $\overline{GA}(4, \mathbb{R})$. These span the associated Lie algebra $\overline{ga}(4, \mathbb{R})$ and satisfy the following commutation relations:

$$\begin{aligned} [L_b^a, L_d^c] &= (\delta_d^a \delta_e^c \delta_b^f - \delta_b^c \delta_d^f \delta_e^a) L_f^e, \\ [L_b^a, P_c] &= \delta_c^a \delta_b^d P_d, \\ [P_a, P_b] &= 0. \end{aligned} \quad (2)$$

Note that the Grassmann degrees of the superfields components of ϕ are given by $|\phi_{bM}^a| = |\phi_M^a| = m \pmod{2}$, where $m = |Z^M|$ ($m = 0$ for $M = \mu$ and $m = 1$ for $M = \alpha$); since ϕ is an even 1-superform. Having this at hand and since $|\phi_M^a| = m$, the generators of the gauge group have a vanishing Grassmann degree, i.e. $|L_b^a| = |P_a| = 0$.

Now, in order to derive the BRST structure of topological 4D gauge-affine gravity, using the BRST superspace formalism, it is necessary to give the geometrical description of the fields occurring in the quantization of such theory. To this purpose, we assign to the anticommuting coordinates θ^1 and θ^2 the ghost numbers (-1) and $(+1)$, respectively, and ghost number zero for an even quantity, viz. a coordinate x^μ , a superform ϕ and Ω or the generators L_b^a, P_a . These rules enable us to determine the ghost numbers of the superconnection and the supercurvature components, e.g. the ghost number for these superfield components (ϕ_{bM}^a, ϕ_M^a) should be zero (for $M = \mu$), and it takes the value of $(-)^{\alpha+1}$ (for $M = \alpha = 1, 2$).

It is mandatory to recall that $\phi_{b\mu}^a$ and ϕ_μ^a denote gauge superfields, whereas ϕ_{b1}^a and ϕ_{b2}^a (resp. ϕ_1^a and ϕ_2^a) are the $\overline{GL}(4, \mathbb{R})$ (resp. translation) ghost and antighost superfields, respectively. In order to display the link between the superfield components on one hand and the physical quantities occurring in a quantized gauge-affine gravity on the other hand, we define the so-called lowest components (denoted ϕ) of a superfield ϕ as the superfield itself evaluated at $\theta^\alpha = 0$. Accordingly, the components $|\phi_{b\mu}^a|$ have to be identified with the affine connection $\omega_{b\mu}^a$, $|\phi_\mu^a|$ with the vierbein e_μ^a and $|\phi_{b1}^a|$ (resp. $|\phi_{b2}^a|$) with the $\overline{GL}(4, \mathbb{R})$ ghost c_b^a (resp. its anti-ghost \bar{c}_b^a). With the aim of constructing the diffeomorphism ghost² (and its anti-ghost) superfields η_α^μ , one thinks of the replacement

$$\phi_\alpha^a \rightarrow \eta_\alpha^\mu \phi_\mu^a, \quad (3)$$

and the inverse supervierbein ϕ_a^μ is defined by means of the orthonormality conditions, $\phi_a^\mu \phi_\nu^a = \delta_\nu^\mu$ and $\phi_\mu^a \phi_b^\mu = \delta_b^a$. This allows explicitly for the introduction of the diffeomorphism ghost (resp. anti-ghost) as $c^\mu = \eta_1^\mu$ (resp. $\bar{c}^\mu = \eta_2^\mu$). One should stress that metric-affine gauge theory of gravity could well-described after a gauging process of the affine group or its double-coevring *à la* Weyl-Yang-Mills, in terms of a couple of gauge potentials $(\vartheta^a, \Gamma^{ab})$ with ϑ^a denoting the coframe 1-forms and Γ^{ab} being

² An alternative explicit calculation can also be realized by considering the ghosts c_b^a, \bar{c}^a for $\overline{GL}(4, \mathbb{R})$ and translation gauge subgroups, respectively. Nevertheless, for a consistent theory of gravity a diffeomorphism ghost c^μ has to be introduced via the parametrization $c^a = e_\mu^a c^\mu$. This however gives rise to the BRST-anti-BRST algebra without torsion. In fact, the vanishing of torsion tensor as well as BRST action on torsion are obtained as constraints in order to recover the nilpotency of the BRST algebra [38]. In other words, introducing the diffeomorphism symmetry into the theory yields an inconsistency of the BRST algebra. To this end, in order to fix the problem with the breakdown of algebra nilpotency and get off-shell nilpotent BRST transformations with torsion, one must either redefine the superfields of the theory or utilize the more economic method of working in a new basis instead of the usual one, as has been done in this work.

spacetime connection 1-forms [39] (see [28] for an exhaustive and self-contained review). Moreover, a metric must be added by hand for physical reasons related to causality and measurements of lengths and angles. Unfortunately, the issue of attributing a gauge origin to the metric is so far not achieved [40]. Thus, the metric-affine spacetime (L_4, g) has to be endowed with a metric structure which is incarnated in the coordinate metric $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$, where η_{ab} denotes the local metric and $e_\mu^a = \phi_\mu^a$ is the aforementioned vierbein field. As for the identification of the other fields, one recognizes the necessary fields for a gauge-affine gravity [32] such as the vierbein, the affine connection and the ghosts, as well as geometrical identifications of the topological fields occurring in the topological version of the gauge theory. Nonetheless, We choose, for sake of simplicity, to classify them according to their relation with either the superconnection ϕ as in Table 1 or the supercurvature Ω as in Table 2 in below.

In order to deal conveniently with the BRST sector, we shall use a method of basis change at the level of the superconnection ϕ , which has proved to be a direct method in the sense that the BRST-anti-BRST algebra is derived in the superspace approach using a superconnection developed in terms of a modified basis, in such a way to incorporate the (anti-)ghosts associated to diffeomorphism symmetry, and this can be realized from the beginning, i.e. before geometrically identifying the fields occurring in such a gauge theory, cf. [38]. To this end, we are at a position to make the following basis change $\{dZ^M\} = \{dx^\mu, d\theta^\alpha\} \rightarrow \{b^M\} = \{b^\mu, b^\alpha\}$, such that the superconnection (1) can be written in the terms of the new basis as

$$\phi = b^M \phi_M = b^\mu \phi_\mu + b^\alpha \phi_\alpha \quad (4)$$

Nonetheless, introducing the coordinate (anti)ghost superfields η_α^μ in the replacement (3) compels us to define a further replacement for the $\overline{GL}(4, \mathbb{R})$ (anti)ghost superfields as follows

$$\phi_\alpha^{ab} \rightarrow \phi_\alpha^{ab} + \eta_\alpha^\mu \phi_\mu^{ab}, \quad (5)$$

Taking into account the two above replacements, the superconnection ϕ can be written explicitly as

$$\phi = b^\mu (\phi_{b\mu}^a L_a^b + \phi_\mu^a P_a) + b^\alpha (\phi_{b\alpha}^a L_a^b) \quad (6)$$

with $b^\mu = dx^\mu + d\theta^\alpha \eta_\alpha^\mu$ and $b^\alpha = d\theta^\alpha$, cf. [26]. One can easily see that the \mathbb{R}^4 -(anti)ghost superfields ϕ_α^a are absorbed into the basis (b^M) . Over and above, another peculiarity of introducing a new basis is to avoid dealing with the product of two or more superfields, which is due to the replacement (3). Furthermore, the $\overline{GA}(4, \mathbb{R})$ -superconnection 1-superform ϕ and its associated supercurvature 2-superform Ω are related via Cartan's structure equation³ $\Omega = d\phi + \frac{1}{2}[\phi, \phi]$, and Bianchi identity $d\Omega + [\phi, \Omega] = 0$. Now, by expanding the curvature in terms of the group generators, namely $\Omega = \Omega_{bMN}^a L_a^b + \Omega_{MN}^a P_a$, structure equation gives in components

$$\begin{aligned} \Omega_{\mu\nu} &= \partial_\mu \phi_\nu - \partial_\nu \phi_\mu + [\phi_\mu, \phi_\nu], \\ \Omega_{\mu\alpha} &= \partial_\mu \phi_\alpha - \partial_\alpha \phi_\mu + \phi_\nu \partial_\mu \eta_\alpha^\nu + \eta_\alpha^\nu \partial_\nu \phi_\mu + [\phi_\mu, \phi_\alpha], \\ \Omega_{\alpha\beta} &= (\partial_\alpha \eta_\beta^\mu - \eta_\alpha^\nu \partial_\nu \eta_\beta^\mu) \phi_\mu + (\partial_\alpha \phi_\beta - \eta_\beta^\mu \partial_\mu \phi_\alpha) + \frac{1}{2}[\phi_\alpha, \phi_\beta] + (\alpha \leftrightarrow \beta). \end{aligned} \quad (7)$$

³ As was shown in [41], Cartan's structure equation and Bianchi identity can be viewed as equations of motion for a BF topological field theory. Consequently, this can attribute a topological character to the two aforementioned equations.

Table 1. Physical fields and their geometrical counterparts stemming from the superconnection 1-superform [26].

Physical fields	Symbol	Geometrical counterpart
Vierbein	e_μ^a	$\phi_\mu^a $
Affine connection	$\omega_{b\mu}^a$	$\phi_{b\mu}^a $
Diffeomorphism ghosts	c^μ	$\eta_1^\mu $
Antighosts of diffeomorphism ghosts	\bar{c}^μ	$\eta_2^\mu $
Associated auxiliary field	B^μ	$\partial_1 \eta_2^\mu $
$\overline{GL}(4, \mathbb{R})$ ghosts	c_b^a	$\phi_{b1}^a $
Antighosts of $\overline{GL}(4, \mathbb{R})$ ghosts	\bar{c}_b^a	$\phi_{b2}^a $
Associated auxiliary field	B_b^a	$\partial_1 \phi_{b2}^a $

while Bianchi identity yields the following

$$\begin{aligned}
 \tilde{D}_\mu \Omega_{\nu\sigma} + \mathcal{U}_{(\mu\nu\sigma)} &= 0, \\
 \tilde{D}_\alpha \Omega_{\mu\nu} + \tilde{D}_\mu \Omega_{\nu\alpha} - \tilde{D}_\nu \Omega_{\mu\alpha} - \eta_\alpha^\sigma \partial_\sigma \Omega_{\mu\nu} - \Omega_{\sigma\nu} \partial_\mu \eta_\alpha^\sigma - \Omega_{\mu\sigma} \partial_\nu \eta_\alpha^\sigma &= 0, \\
 \frac{1}{2} \tilde{D}_\mu \Omega_{\alpha\beta} + \tilde{D}_\alpha \Omega_{\mu\beta} - \eta_\alpha^\nu \partial_\nu \Omega_{\mu\beta} - \Omega_{\nu\alpha} \partial_\mu \eta_\beta^\nu + \Omega_{\mu\nu} (\partial_\alpha \eta_\beta^\nu - \eta_\alpha^\sigma \partial_\sigma \eta_\beta^\nu) + (\alpha \leftrightarrow \beta) &= 0, \\
 \left[\tilde{D}_\alpha \Omega_{\beta\gamma} - \eta_\alpha^\mu \partial_\mu \Omega_{\beta\gamma} + \Omega_{\mu\alpha} \left(\partial_\beta \eta_\gamma^\mu - \eta_\beta^\nu \partial_\nu \eta_\gamma^\mu + (\beta \leftrightarrow \gamma) \right) \right] + \mathcal{U}_{(\alpha\beta\gamma)} &= 0.
 \end{aligned}
 \tag{8}$$

where $\mathcal{U}_{(MNR)}$ is for a cyclic sum over the indices (MNR) . In order to make the calculations more easier, one should note that the graded commutators occurring in the structure equation are

$$\begin{aligned}
 [\phi_\mu, \phi_\nu] &= (\phi_{c\mu}^a \phi_{b\nu}^c - \phi_{b\mu}^c \phi_{c\nu}^a) L_a^b + (\phi_{b\mu}^a \phi_\nu^b - \phi_{b\nu}^a \phi_\mu^b) P_a, \\
 [\phi_\mu, \phi_\alpha] &= (\phi_{c\mu}^a \phi_{\alpha b}^c - \phi_{b\mu}^c \phi_{\alpha c}^a) L_a^b - \phi_\mu^b \phi_{b\alpha}^a P_a, \\
 [\phi_\alpha, \phi_\beta] &= (\phi_{c\alpha}^a \phi_{\beta b}^c - \phi_{b\alpha}^c \phi_{\beta c}^a) L_a^b.
 \end{aligned}
 \tag{9}$$

while those living in the Bianchi identity are given by

$$\begin{aligned}
 [\phi_\mu, \Omega_{MN}] &= (\phi_{c\mu}^a \Omega_{bMN}^c - \phi_{b\mu}^c \Omega_{cMN}^a) L_a^b + (\phi_{b\mu}^a \Omega_{MN}^b - \phi_\mu^b \Omega_{bMN}^a) P_a, \\
 [\phi_\alpha, \Omega_{MN}] &= (\phi_{c\alpha}^a \Omega_{bMN}^c - \phi_{b\alpha}^c \Omega_{cMN}^a) L_a^b + \phi_{b\alpha}^a \Omega_{MN}^b P_a.
 \end{aligned}
 \tag{10}$$

Table 2. Physical fields and their geometrical counterparts stemming from the supercurvature 2-superform [26].

Physical fields	Symbol	Geometrical counterpart
Superpartner of e_μ^a	ψ_μ^a	$\Omega_{\mu 1}^a $
Antighost of ψ_μ^a	$\bar{\psi}_\mu^a$	$\Omega_{\mu 2}^a $
Associated auxiliary field	B_μ^a	$-\partial_1 \Omega_{\mu 2}^a $
Superpartner of $\omega_{b\mu}^a$	$\psi_{b\mu}^a$	$\Omega_{b\mu 1}^a $
Antighost of $\psi_{b\mu}^a$	$\bar{\psi}_{b\mu}^a$	$\Omega_{b\mu 2}^a $
Associated auxiliary field	$B_{b\mu}^a$	$-\partial_1 \Omega_{b\mu 2}^a $
Ghost for the ghost of c^μ	φ^μ	$\frac{1}{2} \Omega_{11}^\mu $
Antighost of φ^μ	$\bar{\varphi}^\mu$	$\frac{1}{2} \Omega_{22}^\mu $
Associated auxiliary field	k^μ	$\frac{1}{2} \partial_1 \Omega_{22}^\mu $
Ghost for the ghost of c_b^a	φ_b^a	$\frac{1}{2} \Omega_{b11}^a $
Antighost of φ_b^a	$\bar{\varphi}_b^a$	$\frac{1}{2} \Omega_{b22}^a $
Associated auxiliary field	k_b^a	$\frac{1}{2} \partial_1 \Omega_{b22}^a $
Curvature	$R_{b\mu\nu}^a$	$\Omega_{b\mu\nu}^a $
Torsion	$T_{\mu\nu}^a$	$\Omega_{\mu\nu}^a $

While gauge-theoretically well defined, off-shell nilpotent BRST and anti-BRST operators are geometrically identified as $Q = Q_1$ and $\bar{Q} = Q_2$ respectively, where $Q_\alpha(F) \equiv \partial_\alpha(F)$ for every superfield F with lowest components $F|$, with $\alpha = 1, 2$.

Consequently, straightforward calculations based on the last two equations of (7) lead to the first set of BRST transformations for a topological gauge-affine gravity as follows

$$\begin{aligned} Q\omega_{b\mu}^a &= -\psi_{b\mu}^a + \partial_\mu c_b^a + \omega_{b\nu}^a \partial_\mu c^\nu + c^\nu \partial_\nu \omega_{b\mu}^a + \omega_{d\mu}^a c_b^d - \omega_{b\mu}^d c_d^a, \\ Qe_\mu^a &= -\psi_\mu^a + e_\nu^a \partial_\mu c^\nu + c^\nu \partial_\nu e_\mu^a - e_\mu^b c_b^a, \\ Qc^\mu &= -\varphi^\mu + c^\nu \partial_\nu c^\mu, \\ Qc_b^a &= \varphi_b^a - \varphi^\mu \omega_{b\mu}^a + c^\nu \partial_\nu c_b^a - c_d^a c_b^d, \\ Q\bar{c}_b^a &= B_b^a, \quad QB_b^a = 0, \quad Q\bar{c}^\mu = B^\mu, \quad QB^\mu = 0 \end{aligned} \quad (11)$$

We notice that when omitting the topological contributions from the set (11), we get the same BRST transformations of a gauge-affine gravity [32]. From the first equation in (7) we get the expressions of the curvature and torsion in terms of the vierbein e_μ^a and the affine connection $\omega_{b\mu}^a$, namely

$$\begin{aligned} R_{b\mu\nu}^a &= \partial_\mu \omega_{b\nu}^a - \partial_\nu \omega_{b\mu}^a + \omega_{d\mu}^a \omega_{b\nu}^d - \omega_{d\nu}^a \omega_{b\mu}^d, \\ T_{\mu\nu}^a &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_{b\mu}^a e_\nu^b - \omega_{b\nu}^a e_\mu^b. \end{aligned} \quad (12)$$

Whereas the first equation in the set (8) of equations stemmed from Bianchi identity yields the associated Bianchi identity to both curvature and torsion, namely

$$\begin{aligned} D_\mu R_{b\nu\sigma}^a + \mathcal{U}_{(\mu\nu\sigma)} &= 0, \\ D_\mu T_{\nu\sigma}^a + \mathcal{U}_{(\mu\nu\sigma)} &= e_\mu^b R_{b\nu\sigma}^a + \mathcal{U}_{(\mu\nu\sigma)}. \end{aligned} \quad (13)$$

with D_μ being the covariant derivative operator with respect to the affine connection ω . We note that the last line of the BRST transformations (11) in above is obtained by means of a trivial equality between the BRST action on the antighosts $\bar{c}^\mu, \bar{c}^{ab}$ on one hand, and the geometrical identifications of their associated auxiliary fields B^μ, B^{ab} on the other hand, which seems redundant at first sight. Here, one should point out that the associated auxiliary fields satisfy the relations

$$\begin{aligned} B^\mu + \bar{B}^\mu &= E^\mu + c^\nu \partial_\nu \bar{c}^\mu + \bar{c}^\nu \partial_\nu c^\mu, \\ B_b^a + \bar{B}_b^a &= E_b^a + c^\nu \partial_\nu \bar{c}_b^a + \bar{c}^\nu \partial_\nu c_b^a - c_d^a \bar{c}_b^d - \bar{c}_d^a c_b^d. \end{aligned} \quad (14)$$

Nevertheless, the anti-BRST transformations of the fields occurring in the gauge-affine theory of gravity can be deduced from the aforementioned BRST transformations (11) by application of the mirror symmetry of the ghost numbers: $F \rightarrow F, \{F = \omega_{b\mu}^a, e_\mu^a, R_{b\mu\nu}^a, T_{\mu\nu}^a\}$ and $F \rightarrow \bar{F}$, otherwise.

Here we should mention, as pointed out in [34], that triviality of BRST transformations in the last line of the set (11) reveals the fact that the presence of the extra fields ($E^\mu := \Omega_{12}^\mu, K^\mu := \partial_1 \Omega_{12}^\mu$) and ($E_b^a := \Omega_{b12}^a, K_b^a := \partial_1 \Omega_{b12}^a$) is necessary only to close the BRST algebra or in other words, to achieve the off-shell nilpotency of the algebra of BRST and anti-BRST operators Q and \bar{Q} , respectively.

On the other hand, the last three equations in (8) yield the second set of BRST transformations of topological gauge-affine gravity is as follows

$$\begin{aligned}
QR_{b\mu\nu}^a &= D_\nu\psi_{b\mu}^a - D_\mu\psi_{b\nu}^a + c^\sigma\partial_\sigma R_{b\mu\nu}^a + R_{b\sigma\nu}^a\partial_\mu c^\sigma + R_{b\mu\sigma}^a\partial_\nu c^\sigma - c_d^a R_{b\mu\nu}^d + c_b^d R_{d\mu\nu}^a, \\
QT_{\mu\nu}^a &= D_\nu\psi_\mu^a - D_\mu\psi_\nu^a + e_\mu^b\psi_{b\nu}^a + e_\nu^b\psi_{b\mu}^a + c^\sigma\partial_\sigma T_{\mu\nu}^a + T_{\sigma\nu}^a\partial_\mu c^\sigma + T_{\mu\sigma}^a\partial_\nu c^\sigma - c_b^a T_{\mu\nu}^b, \\
Q\psi_\mu^a &= e_\mu^b\varphi_b^a - e_\nu^b\omega_{b\nu}^a\varphi^\nu - \mathcal{L}_\varphi e_\mu^a + \mathcal{L}_c\psi_\mu^a - c_b^a\psi_\mu^b - T_{\mu\nu}^a\varphi^\nu, \\
Q\bar{\psi}_\mu^a &= B_\mu^a, \quad QB_\mu^a = 0, \\
Q\psi_{b\mu}^a &= -D_\mu\varphi_b^a + \mathcal{L}_c\psi_{b\mu}^a - c_d^a\psi_{b\mu}^d + c_b^d\psi_{d\mu}^a - R_{b\mu\nu}^a\varphi^\nu, \\
Q\bar{\psi}_{b\mu}^a &= B_{b\mu}^a, \quad QB_{b\mu}^a = 0, \\
Q\varphi_b^a &= c_b^d\varphi_d^a - c_d^a\varphi_b^d + c^\mu\partial_\mu\varphi_b^a - \psi_{b\mu}^a\varphi^\mu, \\
Q\bar{\varphi}_b^a &= k_b^a, \quad Qk_b^a = 0, \\
Q\varphi^\mu &= c^\nu\partial_\nu\varphi^\mu - \varphi^\nu\partial_\nu c^\mu, \quad Q\bar{\varphi}^\mu = k^\mu, \quad Qk^\mu = 0, \\
QE^\mu &= K^\mu, \quad QK^\mu = 0, \quad QE_b^a = K_b^a, \quad QK_b^a = 0.
\end{aligned} \tag{15}$$

where as found in [26] for topological 4D gravity, the associated auxiliary fields are expressed as

$$\begin{aligned}
B_\mu^a + \bar{B}_\mu^a &= e_\mu^b E_b^a - e_\nu^b \omega_{b\mu}^a E^\nu - \mathcal{L}_E e_\mu^a + \mathcal{L}_c \bar{\psi}_\mu^a + \mathcal{L}_{\bar{c}} \psi_\mu^a - c_b^a \bar{\psi}_\mu^b - \bar{c}_b^a \psi_\mu^b - T_{\mu\nu}^a E^\nu, \\
B_{b\mu}^a + \bar{B}_{b\mu}^a &= -D_\mu E_b^a + \mathcal{L}_c \bar{\psi}_{b\mu}^a + \mathcal{L}_{\bar{c}} \psi_{b\mu}^a - c_d^a \bar{\psi}_{b\mu}^d + c_b^d \bar{\psi}_{d\mu}^a - \bar{c}_d^a \psi_{b\mu}^d + \bar{c}_b^d \psi_{d\mu}^a - R_{b\mu\nu}^a E^\nu, \\
K^\mu + \bar{K}^\mu &= c^\nu \partial_\nu E^\mu + \bar{c}^\nu \partial_\nu \varphi^\mu - \partial_\nu c^\mu \cdot E^\nu - \partial_\nu \bar{c}^\mu \cdot \varphi^\nu, \\
k_b^a + \bar{k}_b^a &= c^\nu \partial_\nu \bar{\varphi}^\mu + \bar{c}^\nu \partial_\nu E^\mu - \partial_\nu c^\mu \cdot \bar{\varphi}^\nu - \partial_\nu \bar{c}^\mu \cdot E^\nu, \\
K_b^a + \bar{K}_b^a &= c_b^d E_d^a - c_d^a E_b^d - \bar{c}_d^a \varphi_b^d + \bar{c}_b^d \varphi_d^a + c^\mu \partial_\mu E_b^a + \bar{c}^\mu \partial_\mu \varphi_b^a - \bar{\psi}_{b\mu}^a \varphi^\mu - \psi_{b\mu}^a E^\mu, \\
k_b^a + \bar{k}_b^a &= c_b^d \bar{\varphi}_d^a - c_d^a \bar{\varphi}_b^d - \bar{c}_d^a E_b^d + \bar{c}_b^d E_d^a + c^\mu \partial_\mu \bar{\varphi}_b^a + \bar{c}^\mu \partial_\mu E_b^a - \psi_{b\mu}^a \bar{\varphi}^\mu - \bar{\psi}_{b\mu}^a E^\mu.
\end{aligned}$$

Here \mathcal{L}_c is the Lie derivative operator along the vector field $c = (c^\mu)$. Despite the minor differences, we notice the emergence of a torsional term in the BRST transformation of torsion in the set (15) when compared with the case of 4D topological gravity [26]. At this level, one should stress that the obtained BRST and anti-BRST transformations (11), (15) are off-shell nilpotent, namely

$$Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0. \tag{16}$$

3. New torsional observables and enlargements of fields for the gauge group $\overline{SA}(5, \mathbb{R})$

Although it is not possible to obtain the general affine group $\overline{GA}(4, \mathbb{R})$ by an Inönü-Wigner contraction process⁴ from other larger semi-simple Lie group [28], one can always imagine a group embedding of $\overline{GL}(4, \mathbb{R})$ into $\overline{GL}(5, \mathbb{R})$ [44] or even a group isomorphism splitting of the metilinear double-covering group [45]

$$\overline{SL}(5, \mathbb{R}) \approx^{\text{iso}} \overline{GA}_*(4, \mathbb{R}) = \mathbb{R}^4 \oplus \overline{GL}(4, \mathbb{R}) \oplus \mathbb{R}_*^4, \tag{17}$$

where $\overline{GA}_*(4, \mathbb{R})$ denotes the double covering of the graded affine group and $\{\mathbb{R}^4, \mathbb{R}_*^4\}$ being pseudotranslation groups. The main idea behind extending (e.g. embedding) a Lie group to a larger another Lie group is that the connection of the latter incorporates a connection and a vierbein of

⁴ This is clearly in contrast to the case of Euclidean orthogonal groups $O(n)$ and $SO(n)$ where we can imagine an Inönü-Wigner contraction of Lie algebras $\mathfrak{so}(5) \rightarrow \mathfrak{so}(4)$. Having this at hand, one can justify the existence of the embedding $SO(4) \hookrightarrow SO(5)$ used for instance in [34] to construct new torsional observables for 4D topological gravity by means of the BRST algebra. As a result, this will affect the commutation relations by splitting up the generators of $SO(5)$ into those of $ISO(4)$ [42]. In the same spirit, using embeddings such as $GL(4, \mathbb{R}) \hookrightarrow \overline{GL}(5, \mathbb{R})$ or even $\overline{GA}(4, \mathbb{R}) \hookrightarrow \overline{GL}(5, \mathbb{R})$ [43] we can directly extract the BRST algebra for 4D topological gauge-affine gravity and therefore construct more topological observables [44].

the former [46]. Thus, metric structure of a group together with affine features can be deduced from only the larger group's affine structure incarnated in the the enlarged connection W . In this context, the vierbein, gauge and affine connections together with matter can all be incorporated into an extended connection for a larger group and this can be considered as a rough mathematical trick to the prospect of constructing a unified picture of gravitation and Standard Model of particle physics, cf. [46].

Then according to the group decomposition (17), the extended connection W can be expanded by means of the generators \mathcal{L}_B^A of the group $\overline{SL}(5, \mathbb{R})$ on one hand, and of the generators L_b^a , P_a and P_*^a on the other hand as follows [22]

$$W = W_B^A \mathcal{L}_A^B = \omega_b^a L_a^b + l W_5^a P_a + l W_a^5 P_*^a, \quad A, B = 1, \dots, 5 (a, b = 1, \dots, 4) \quad (18)$$

where L_b^a generates 4D general linear transformations, while the pseudotranslation generators are introduced via $l P_a := \mathcal{L}_a^5$ and $l P_*^a := \mathcal{L}_5^a$ and the remnant generator \mathcal{L}_5^5 satisfy the normalization constraint, i.e. $\mathcal{L}_5^5 = 0$ which is in turn reserved to the group $\overline{SL}(5, \mathbb{R})$. Here one should note that a fundamental compensating length l has been introduced for reasons related to the dimensionality of topological invariants, namely to maintain the invariants dimensionless [13], cf. [47]. Moreover, the only constraint imposed on the enlargements of fields is to obtain structure equations and Bianchi identities for the general affine double-covering group $\overline{GA}(4, \mathbb{R})$ in the limit $l \rightarrow \infty$ [38].

Following [22,48], in order to avoid the problems of degeneracy of the coframe 1-forms $\vartheta^a = e_\mu^a dx^\mu$ in a gauge-affine theory of gravity we propose the following ansatz to describe the components of the enlarged connection, namely

$$W_{B\mu}^A := \left\{ \begin{array}{ll} W_{b\mu}^a = \omega_{b\mu}^a, & W_{5\mu}^5 = 0, \\ l W_{5\mu}^a = e_\mu^a - D_\mu(e_\nu^a c^\nu), & l W_{a\mu}^5 = e_{a\mu} - D_\mu(e_{a\nu} \bar{c}^\nu) \end{array} \right\}. \quad (19)$$

with D_μ being the covariant derivative with respect to the connection ω . Our approach in this paper consists in extending the general affine double-covering group $\overline{GA}(4, \mathbb{R})$ to the special affine group $\overline{SA}(5, \mathbb{R})$. Consequently, we will keep using the same enlargement for the connection (19) and adopt further enlargements for the vierbein e^a (the same as in [38]) and the ghost c_b^a (which are associated to the former group) to the fields E^A and C_B^A associated to the latter group, namely

$$E_\mu^A := \left\{ \begin{array}{l} E_\mu^a = e_\mu^a, \\ l E_\mu^5 = Q \bar{c}_\mu + \bar{Q} c_\mu \equiv B_\mu + \bar{B}_\mu. \end{array} \right\}, \quad (20)$$

and for the affine ghost, we have

$$C_B^A := \left\{ \begin{array}{ll} C_b^a = c_b^a, & C_5^5 = 0, \\ l C_5^a = e_{a\mu}^a c^\mu, & l C_a^5 = e_{a\mu} \bar{c}^\mu. \end{array} \right\}, \quad (21)$$

In this context, when dealing with the aforementioned enlargement of the connection $\omega \rightarrow W$, one should stress that $\omega = (\omega_b^a)$ denotes the affine connection while $W = (W_B^A)$ is the connection associated to the gauge group $\overline{SA}(5, \mathbb{R})$. As an upshot of all these enlargements, a direct and straightforward calculation yields the enlarged torsion in components as follows

$$\mathcal{T}_{\mu\nu}^A := \left\{ \begin{array}{l} \mathcal{T}_{\mu\nu}^a = T_{\mu\nu}^a + \frac{2}{l^2} (e_\mu^a - D_\mu c^a) (B_\nu + \bar{B}_\nu), \\ \mathcal{T}_{\mu\nu}^5 = \frac{2}{l} \left[\partial_\mu (B_\nu + \bar{B}_\nu) - e_\nu^b D_\mu \bar{c}_b \right]. \end{array} \right\}, \quad (22)$$

and for the enlarged curvature one obtains

$$\mathcal{R}_{B\mu\nu}^A := \left\{ \begin{array}{l} \mathcal{R}_{b\mu\nu}^a = R_{b\mu\nu}^a + \frac{2}{l^2} (e_\mu^a - D_\mu c^a) (e_{b\nu} - D_\nu \bar{c}_b), \\ \mathcal{R}_{5\mu\nu}^5 = \frac{2}{l^2} (e_{a\mu} - D_\mu \bar{c}_a) (e_\nu^a - D_\nu c^a), \\ l \mathcal{R}_{5\mu\nu}^a = T_{\mu\nu}^a - R_{b\mu\nu}^a c^b - 4 \omega_{b\mu}^a \partial_\nu c^b \\ l \mathcal{R}_{a\mu\nu}^5 = T_{a\mu\nu} + R_{a\mu\nu}^b \bar{c}_b + 4 \omega_{a\mu}^b \partial_\nu \bar{c}_b. \end{array} \right\}, \quad (23)$$

Having all this at hand and replacing the enlargements of the fields into the BRST–anti-BRST transformations having a form independent from the choice of the gauge groups $\overline{GA}(4, \mathbb{R})$ or $\overline{SA}(5, \mathbb{R})$ since the generators of both share the same commutation relations, as well as using the nilpotency of the BRST and anti-BRST operators (16), we obtain further enlargements of the remaining fields, e.g. for the superpartner field ψ_μ^a of the vierbein e_μ^a we get

$$\Psi_\mu^A := \left\{ \begin{array}{l} \Psi_\mu^a = \psi_\mu^a + \frac{1}{l^2} e_\nu^a c^\nu (B_\mu + \overline{B}_\mu), \\ l\Psi_\mu^5 = \mathcal{L}_c(B_\mu + \overline{B}_\mu) - \overline{c}_\mu. \end{array} \right\}, \quad (24)$$

As for the enlarged ghost for ghost Φ_b^a of c_b^a , it is written as follows

$$\Phi_B^A := \left\{ \begin{array}{l} \Phi_b^a = \varphi_b^a + \frac{1}{l^2} e_\mu^a e^{b\nu} c^\mu \overline{c}^\nu, \quad l^2 \Phi_5^5 = c_\mu \overline{c}^\mu, \\ l\Phi_5^a = -\psi_\mu^a c^\mu + e_\nu^a (c^\mu - \varphi^\mu) \partial_\mu c^\nu - \varphi^\mu c^\nu D_\mu e_\nu^a, \\ l\Phi_a^5 = -\psi_{a\mu} \overline{c}^\mu + e_{a\mu} (\varphi^\mu + B^\mu) - \varphi^\mu \overline{c}^\nu D_\mu e_{a\nu} - e_{a\nu} \varphi^\mu \partial_\mu \overline{c}^\nu \\ \quad + \frac{1}{2} e_{a\mu} (\mathcal{L}_{\overline{c}} c^\mu - \mathcal{L}_c \overline{c}^\mu). \end{array} \right\}, \quad (25)$$

Now, using the second equation in (14), the extra field E_b^a admits the enlargement $E_b^a \rightarrow \Upsilon_B^A$ explicitly written as

$$\Upsilon_B^A := \left\{ \begin{array}{l} \Upsilon_b^a = E_b^a + \frac{1}{l^2} (c^a c_b - \overline{c}^a \overline{c}_b), \quad l^2 \Upsilon_5^5 = c^a c_a - \overline{c}^a \overline{c}_a, \\ l\Upsilon_5^a = E^\mu (e_\mu^a - D_\mu c^a) - (\psi_\mu^a \overline{c}^\mu + \overline{\psi}_\mu^a c^\mu) + e_\mu^a (B^\mu + \overline{B}^\mu), \\ l\Upsilon_a^5 = e_{a\mu} (\varphi^\mu - \overline{\varphi}^\mu) - (\psi_{a\mu} c^\mu + \overline{\psi}_{a\mu} \overline{c}^\mu) + E^\mu (e_{a\mu} - D_\mu \overline{c}_a) \\ \quad - \frac{1}{2} e_{a\mu} (\mathcal{L}_c c^\mu + \mathcal{L}_{\overline{c}} \overline{c}^\mu). \end{array} \right\}, \quad (26)$$

In the latter enlargement we have used an enlarged associated auxiliary field $B_b^a \rightarrow \mathcal{B}_B^A$ and also an enlargement $\overline{B}_b^a \rightarrow \overline{\mathcal{B}}_B^A$ which can be directly obtained by a mirror ghost number symmetry. As for the superpartner of ω it admits the enlargement rule

$$\psi_{b\mu}^a \rightarrow \Psi_{B\mu}^A = \begin{pmatrix} \Psi_{b\mu}^a & \Psi_{5\mu}^a \\ \Psi_{b\mu}^5 & \Psi_{5\mu}^5 \end{pmatrix}, \quad (27)$$

with its components explicitly expressed as

$$\Psi_{B\mu}^A := \left\{ \begin{array}{l} \Psi_{b\mu}^a = \psi_{b\mu}^a + \frac{1}{l^2} (e_\mu^a \overline{c}_b - e_{b\mu} c^a + c^a D_\mu \overline{c}_b - D_\mu c^a \cdot \overline{c}_b), \\ l\Psi_{5\mu}^a = \psi_\nu^a (\delta_\mu^\nu - 2\partial_\mu c^\nu) + \psi_{b\mu}^a e_\nu^b c^\nu + (c^\nu - \varphi^\nu) D_\mu e_\nu^a + e_\nu^a \partial_\mu (c^\nu - \varphi^\nu) \\ \quad - c^\nu D_\mu \psi_\nu^a + \frac{1}{2} [D_\mu e_\nu^a \mathcal{L}_c c^\nu + e_\nu^a \partial_\mu (\mathcal{L}_c c^\nu)], \\ l\Psi_{a\mu}^5 = \psi_{a\mu} - \psi_{a\mu}^b e_{b\nu} \overline{c}^\nu + B^\nu D_\mu e_{a\nu} + (\partial_\mu \overline{c}^\nu + \overline{c}^\nu D_\mu) (e_{a\nu} - \psi_{a\nu}) \\ \quad + e_{a\nu} \partial_\mu B^\nu + \frac{1}{2} (D_\mu e_{a\nu} + e_{a\nu} \partial_\mu) (\mathcal{L}_{\overline{c}} c^\nu - \mathcal{L}_c \overline{c}^\nu) \\ l^2 \Psi_{5\mu}^5 = \overline{c}_\mu - c_\mu + D_\mu \overline{c}_a \cdot c^a - \overline{c}_a D_\mu c^a. \end{array} \right\}. \quad (28)$$

Now we are at a stage to construct observables, but before we proceed further we have to recall that a topological gauge-affine gravity allows for *a priori* inclusion of topological (shift) and affine symmetries. Besides, one has to deal with diffeomorphism which is an external symmetry irrelevant for the gauge group. To this end, diffeomorphism ghost c and antighost \overline{c} should be intervened.

First off, the generalized exterior differential operator is defined as [49]

$$\tilde{d} := e^{-(c+\overline{c})} [d + Q + \overline{Q}] e^{(c+\overline{c})}, \quad (29)$$

with $c]$ denotes the inner product along the vector field $c = (c^\mu)$ and $\partial_\mu] dx^\nu = \delta_\mu^\nu$. We note that \tilde{d} proves to be nilpotent, i.e. $\tilde{d}^2 = 0$ by virtue of the nilpotency of the usual exterior differential operator d , the BRST and anti-BRST operator, namely

$$\tilde{d}^2 = e^{-(c+\overline{c})} [d^2 + Q^2 + \overline{Q}^2 + \{d, Q + \overline{Q}\} + \{Q, \overline{Q}\}] e^{(c+\overline{c})} = 0. \quad (30)$$

After some straightforward algebra and using the BRST transformations (11), the operator \tilde{d} reads

$$\tilde{d} = d + Q + \overline{Q} + [d, (c + \overline{c})] + (E + \varphi + \overline{\varphi}), \quad (31)$$

which is crucial for the construction of a generalized covariant differential operator $\tilde{D}\bullet = \tilde{d}\bullet + \tilde{W} \wedge \bullet$, with \tilde{W} being a generalized enlarged connection. Here, we should stress that the latter as well as the generalized operator \tilde{D} are in what follows the basic ingredients to deduce the generalized curvature $\tilde{\mathcal{R}}$ and also the generalized Bianchi identity $\tilde{D}\tilde{\mathcal{R}} = 0$. For this purpose, the generalized operator \tilde{D} can be explicitly expressed as

$$\begin{aligned} \tilde{D} &= \tilde{d} + e^{-(c+\overline{c})} [W + C_L + \overline{C}_L] e^{(c+\overline{c})} \\ &= D + Q + \overline{Q} + C_L + \overline{C}_L + [d, (c + \overline{c})] + (E + \varphi + \overline{\varphi}) - (c + \overline{c})W, \end{aligned} \quad (32)$$

where $W = dx^\mu W_{B\mu}^A L_A^B$ is the enlarged connection 1-form taking values in the Lie algebra of the group $\overline{SL}(5, \mathbb{R})$, and $C_L = C_B^A L_A^B$ and $\overline{C}_L = \overline{C}_B^A L_A^B$ being the enlarged linear ghost and antighost 0-forms.

In order to construct the observables for a topological gauge-affine gravity, one considers $\tilde{\mathcal{P}}(\tilde{\mathcal{R}}, \dots, \tilde{\mathcal{R}})$ as characteristic polynomials in the generalized enlarged curvature $\tilde{\mathcal{R}}$ such that [38]

$$\tilde{\mathcal{R}} = \mathcal{R} - \Psi_L - \overline{\Psi}_L + \Phi_L + \overline{\Phi}_L + E_L, \quad (33)$$

with the differential forms $\mathcal{R} := dW + \frac{1}{2} [W, W]$, Ψ_L , Φ_L and E_L being the lowest components of the supercurvature Ω associated to the gauge group $\overline{SA}(5, \mathbb{R})$, expanded with respect to the $\overline{sl}(5, \mathbb{R})$ -algebra elements $\{\mathcal{L}_B^A\}$ as

$$\begin{aligned} \mathcal{R} &= \frac{1}{2} dx^\nu dx^\mu R_{B\mu\nu}^A \mathcal{L}_A^B, \\ \Psi_L &= dx^\mu \Psi_{B\mu}^A \mathcal{L}_A^B, \quad \Phi_L = \Phi_B^A \mathcal{L}_A^B, \quad E_L = E_B^A \mathcal{L}_A^B \end{aligned} \quad (34)$$

where the fields Ψ_L , Φ_L and E_L represent respectively the enlarged superpartner of W , the ghost for ghost of C_L and an antighost with vanishing ghost number, i.e. $\text{gh}(E_L) = 0$. Owing to the generalized Bianchi identity $\tilde{D}\tilde{\mathcal{R}} = 0$ it is straightforward to check that the polynomial $\tilde{\mathcal{P}}$ satisfy the property

$$\tilde{d}\tilde{\mathcal{P}}(\tilde{\mathcal{R}}, \dots, \tilde{\mathcal{R}}) = 0, \quad N \geq 2, \quad (35)$$

with N denotes the repetition rate in the dependence of $\tilde{\mathcal{P}}$ in terms of the generalized enlarged curvature $\tilde{\mathcal{R}}$. Subsequently, expanding the latter with respect to the form degree a and the ghost number b , one gets a sum of the quantities $\tilde{\mathcal{R}}_{(a,b)}$ such that

$$\tilde{\mathcal{R}} = \sum_{a,b} \tilde{\mathcal{R}}_{(a,b)} = \tilde{\mathcal{R}}_{(2,0)} + \tilde{\mathcal{R}}_{(1,1)} + \tilde{\mathcal{R}}_{(1,-1)} + \tilde{\mathcal{R}}_{(0,2)} + \tilde{\mathcal{R}}_{(0,-2)} + \tilde{\mathcal{R}}_{(0,0)}, \quad (36)$$

with the identification of each piece of $\tilde{\mathcal{R}}_{(a,b)}$ follows the same order as in (33), e.g. $\tilde{\mathcal{R}}_{(2,0)} \equiv \tilde{\mathcal{R}}$ is identified as a 2-form with vanishing ghost number. Having all this at hand, we are going to generate the descent equations that constitute the key point to find new torsional observables. To this end, the action of the operator \tilde{d} on the polynomial $\tilde{\mathcal{P}}$ yields the following

$$(d + Q + \overline{Q})e^{(c+\overline{c})}\tilde{\mathcal{P}} = 0 \quad (37)$$

Developing equation (37) (which is essential insofar as the term $e^{(c+\overline{c})}\tilde{\mathcal{P}}$ is invariant) with the particular case of $N = 2$ where the characteristic polynomial is just the generalized Pontryagin density, namely

$$\tilde{\mathcal{P}}(\tilde{\mathcal{R}}, \tilde{\mathcal{R}}) = (\tilde{\mathcal{R}} \wedge \tilde{\mathcal{R}}) \equiv \tilde{\mathcal{R}}_B^A \wedge \tilde{\mathcal{R}}_A^B, \quad (38)$$

Now, using (33) and (36) one can easily develop the generalized Pontryagin density in terms of the form degree and ghost number, for instance $\tilde{\mathcal{P}}_{(4,0)} = (\mathcal{R} \wedge \mathcal{R})$ with form degree equals to 4 and vanishing ghost number, and so on. Thus, rearrangement of (38) according to the form degree and varying with respect to the ghost number yields [38]

$$\tilde{\mathcal{P}} = \sum_{a=0}^4 \tilde{\mathcal{P}}_a, \quad (39)$$

where $\tilde{\mathcal{P}}_a \equiv \sum_b \tilde{\mathcal{P}}_{(a,b)}$ with b being the ghost number. After some algebra, we get a set of five descent equations having the following condensed form

$$(Q + \bar{Q}) \sum_{m=0}^{4-k} \frac{1}{m!} (c + \bar{c}) \rfloor_{(m)} \tilde{\mathcal{P}}_{m+k} = -d \sum_{m=0}^{5-k} \frac{1}{m!} (c + \bar{c}) \rfloor_{(m)} \tilde{\mathcal{P}}_{m+k-1}, \quad (40)$$

where the index k in the last equations denotes the maximal order for the form degree in each descent equation, e.g. for the higher degree $k = 4$ one has $(Q + \bar{Q})\tilde{\mathcal{P}}_4 + d [\tilde{\mathcal{P}}_3 + (c + \bar{c})\tilde{\mathcal{P}}_4] = 0$, while for the lower degree $k = 0$ we get the descent equation $(Q + \bar{Q}) \sum_{m=0}^4 \frac{1}{m!} (c + \bar{c}) \rfloor_{(m)} \tilde{\mathcal{P}}_m = 0$. Then, we get the set of observables $\{O_i\}_{i=1,\dots,5}$ such that

$$O_{4-n} = \int_{\gamma_n} \sum_{m=0}^{4-n} \frac{1}{m!} (c + \bar{c}) \rfloor_{(m)} \tilde{\mathcal{P}}_{m+n}, \quad (41)$$

with γ_n being a closed homology cycle with dimension n of the 4D spacetime base manifold. Taking for instance the simplest but nontrivial case where $n = 4$ and using the enlarged curvature (23) we obtain

$$\begin{aligned} O_0 &= \tilde{\mathcal{P}}_4 := (\mathcal{R} \wedge \mathcal{R}) \\ &= \tilde{P}_4 + \frac{2}{l^2} O^{(T)} \end{aligned} \quad (42)$$

where $\tilde{P}_4 := (R \wedge R)$ denotes the Pontryagin density constructed by means of the curvature R associated to the gauge group $\overline{GA}(4, \mathbb{R})$, and the emergent term $O^{(T)}$ is a torsional observable with the explicit expression

$$\begin{aligned} O^{(T)} &= T^a \wedge T_a - (e_a - D\bar{c}_a) \wedge (e^b - Dc^b) \wedge R_b^a + T^a \wedge R_a^b (\bar{c}_b - c_b) \\ &\quad + 2T^a \wedge \omega_a^b \wedge (d\bar{c}_b - dc_b) - 2R_b^a \wedge \omega_a^c \wedge (c^b d\bar{c}_c - \bar{c}^b dc_c) + 4\omega_a^c \wedge \omega_a^e \wedge dc^b \wedge d\bar{c}_e \end{aligned} \quad (43)$$

Another example can be considered here is when one attempts to calculate the subsequent observable O_1 ,

$$O_1 = \int_{\gamma_3} (\tilde{\mathcal{P}}_3 + (c + \bar{c})\tilde{\mathcal{P}}_4) \quad (44)$$

Here, we should stress that $\tilde{\mathcal{P}}_3 = \tilde{\mathcal{P}}_{(3,1)} + \tilde{\mathcal{P}}_{(3,-1)}$ and $\tilde{\mathcal{P}}_4 = \tilde{\mathcal{P}}_{(4,0)}$ with vanishing ghost number (See [38] for more on the explicit forms of all the pieces of $\tilde{\mathcal{P}}_a$). Thus, we get

$$O_1 = -2 \int_{\gamma_3} \left[\mathcal{R} \wedge (\Psi_L + \bar{\Psi}_L) \right] + c \rfloor (\tilde{P}_4 + \frac{2}{l^2} O^{(T)}) \quad (45)$$

with Ψ_L is the enlarged superpartner of the connection W as obtained in (27).

4. Discussion and Conclusion

An alternative and direct way to deduce topological invariants without calling for the superconnection formalism is as shown in [19], an identity holds for an arbitrary enlarged $\overline{SL}(5, \mathbb{R})$ -connection W_B^A and its associated curvature \mathcal{R}_B^A , namely

$$\mathcal{R}_B^A \wedge \mathcal{R}_A^B = d \left[W_B^A \wedge \left(\mathcal{R}_A^B - \frac{1}{3} W_D^B \wedge W_A^D \right) \right], \quad (46)$$

Proceeding in the same way as has been done in [19], one develops the two sides of the identity (46) and compares only the terms with the same power in the length parameter l , one gets for the zeroth and second powers the following identities:

$$\begin{aligned} R_b^a \wedge R_a^b &= dF, \\ \mathcal{O}^{(T)} &= dG, \end{aligned} \quad (47)$$

with $\mathcal{R}[R]$ being the curvature constructed by means of the $\overline{SL}(5, \mathbb{R})[\overline{GL}(4, \mathbb{R})]$ -connection $W[\omega]$ and F and G being two distinct 3-forms which with the sake of simplicity we have turned a blind eye to writing down their explicit expressions. Here one should point out that the first identity sharing the same form as with (46) but with a different connection reflects the global character of the Pontryagin 4-form $R \wedge R$, while the second identity seems to be a generalization of the Nieh-Yan identity for the group $SO(4)$, namely

$$T^a \wedge T_a - e_a \wedge e^b \wedge R_b^a = d(e^a \wedge T_a). \quad (48)$$

This may open the door to imagine further generalizations of the NY-form if one considers other enlargements of the connection, e.g. enlargement associated to the embedding $\overline{GL}(4, \mathbb{R}) \hookrightarrow \overline{GL}(5, \mathbb{R})$ as in [44]. Moreover, if one admits, beside the enlargement of the connection, similar enlargements for the vierbein $e^a \rightarrow E^A$ and other fundamental fields, we are at a position to extract new observables by dint of the passage $\overline{GA}(4, \mathbb{R}) \rightarrow \overline{GA}(5, \mathbb{R})$ which is the subject of a forthcoming work [44].

Now, if one considers the following $\overline{SL}(5, \mathbb{R})$ -connection 1-form

$$\Xi_B^A = \begin{bmatrix} \omega_b^a & \frac{1}{l}(Q^a - DP^a) \\ \frac{1}{l}(Q_b - D\bar{P}_b) & 0 \end{bmatrix}, \quad (49)$$

where Q^a denotes a $\overline{GL}(4, \mathbb{R})$ tensor generalizing the vierbein 1-form e^a while P^a and \bar{P}_a being arbitrary 0-forms. Then, if one admits the following reparametrization of the connection Ξ used primarily by Nieh [18]

$$\Xi_B^A := \Xi_B^A + \Lambda \Theta_B^A, \quad (50)$$

with Λ being an arbitrary parameter and Θ is an $\overline{SL}(5, \mathbb{R})$ 1-form, new topological invariants may emerge [50] when one equates the identical powers of the parameter Λ in the same spirit as in [19].

To recapitulate, by dint of the superspace formalism [36] with a $\overline{GA}(4, \mathbb{R})$ -superconnection ϕ and its associated supercurvature Ω we have identified all the necessary fields for a topological gauge-affine gravity. The direct method used here consists in defining a new basis (b^M) instead of the usual one (dZ^M) in order to incorporate the diffeomorphism ghost. Subsequently, by means of Cartan's structure equation and Bianchi identity we have obtained the BRST-anti-BRST transformations of a 4D topological gauge-affine gravity for a given gauge group $\overline{GA}(4, \mathbb{R})$ or $\overline{SA}(5, \mathbb{R})$ sharing the same form of commutation relations. The key point in our construction is the enlargement of the connection according to a group decomposition, as well as other fields occurring in the theory, e.g. the vierbein. Accordingly, we recognize also in a natural way enlargements for the other fields owing to the BRST algebra for the two gauge groups.

The method used in this work to achieve the main goal, namely constructing the observables, consists in obtaining the descent equations primarily based on a characteristic polynomial $\tilde{\mathcal{P}}$ associated to the gauge group $\overline{SA}(5, \mathbb{R})$. In this context, we have to recall that one peculiarity of the descent

equations formalism is that it enables us to deduce directly topological observables that are BRST and anti-BRST invariant, and this can be realized only by means of developing the descent equations (37). As far as the topological invariants characterizing a 4D manifold are concerned, they can always be calculated as correlation functions of the topological observables already constructed [38]. In forthcoming works, a challenge will stand out clearly and consist in generalizing this approach to the case of gauge groups $\overline{GA}(n, \mathbb{R})$ with $n \geq 5$ in order to construct new torsional observables with higher order.

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