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Article

On the Conjecture over Dimensions of Associated Lie Algebra to the Isolated Singularities

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Abstract: Lie algebra plays an important role in the study of singularity theory and other field of sciences. Finding numerous invariants linked with isolated singularities is always a main interest in the classification theory of isolated singularities. Any Lie algebra that characterizes simple singularity produces a natural question. The study of properties such as to find the dimensions of newly defined algebra is a remarkable work. Hussain, Yau and Zuo [24] have been found a new class of Lie algebra $\mathcal{L}_k(V)$, $k \geq 1$ i.e., $\text{Der}(M_k(V), M_k(V))$ and purposed a conjecture over its dimension $\delta_k(V)$ for $k \geq 0$. Later they proved it true for k up to $k = 1, 2, 3, 4, 5$. In this work, the main concern is whether it's true for a higher value of k . According to this, we calculate first, the dimension of Lie algebra $\mathcal{L}_k(V)$ for $k = 6$ and then compute the upper estimate conjecture of fewnomial isolated singularities. Along with, we also justify the inequality conjecture: $\delta_{k+1}(V) < \delta_k(V)$ for $k = 6$. Our calculated results are innovative and a new addition to the study of singularity theory.

Keywords: simple singularity; hypersurface singularity; derivation Lie algebra; inequality conjecture.

1. Introduction

A mathematical structure known as a Lie algebra is made up of vector spaces and a particular binary operation known as the Lie bracket which used to measure the "twist" between two infinitesimal transformations geometrically. Lie algebras are essential to many branches of mathematics and science. In quantum mechanics and particle physics, they are widely used, and in physical phenomena, they arise as symmetry groups. In this regard, Brieskorn established the relationship between Lie algebras and simple singularities. Recent studies have investigated relations between finite-dimensional solvable Lie algebras and isolated hypersurface singularities in complex analysis. In different areas of science and mathematics, singularities have arisen naturally. That's why singularity theory has become the interlinking path between the various applications of mathematics with its conclusive parts. As an example, it coordinates regular polyhedra theory and simple Lie algebra with the optimal caustics investigations, and relates knot theory to wave fronts of hyperbolic PDE, while it also connects commutative algebra to the theory of solids shape. In most problems of singularity theory, the core aim is to determine the dependence of various physical phenomena while deal with the geometric objects.

The germs of holomorphic functions are widely accepted to be at the origin of \mathbb{C}^n and \mathcal{O}_n . Naturally, the \mathcal{O}_n can be used to identify the algebra of n indeterminate power series. Yau takes into account the Lie algebras derived from the moduli algebra $A(V) := \mathcal{O}_n / (g, \frac{\partial g}{\partial y_1}, \dots, \frac{\partial g}{\partial y_n})$, where $L(V)$ define as $\text{Der}(A(V), A(V))$ and V denote the isolated hypersurface singularity. Lie algebra $L(V)$ is a famous solvable finite-dimensional algebra ([1–3]). Yau algebra of V is used in singularity theory to distinguish $L(V)$ from the other types of Lie algebra ([4,5]). The complex analytical set of isolated hypersurface singularities and the finite set of solvable dimensional Lie algebras (nilpotent) have many

new natural connections that Hussain, Yau, Zuo, and their research associates have discovered in recent years ([6–12]). They have presented three distinct methods for connecting isolated hypersurface singularities to Lie algebra.

These associations are helpful to understand the solvable Lie algebra (nilpotent), from a geometric point of view [8]. A lot of work has been done by Yau and his research collaborates, from 1980s [8,12–23].

Let a holomorphic function $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ associated to the isolated hypersurface singularity $(V, 0)$ and its multiplicity denoted as $\text{mult}(g)$. The $\mathcal{L}_k(V)$ [24] defined as:

Let ideal $\mathbb{J}_k(g)$ generated by $\langle \frac{\partial^k g}{\partial y_{i_1} \cdots \partial y_{i_k}} \mid 1 \leq i_1, \dots, i_k \leq n \rangle$ and $\text{mult}(g) = m, 1 \leq k \leq m$. Then $M_k(V) := \mathcal{O}_n / (g + \mathbb{J}_1(g) + \cdots + \mathbb{J}_k(g))$ define the k -th local algebra and $\mathcal{L}_k(V)$ denoted the derivations Lie algebras and its dimension denoted as $\delta_k(V)$. It is further note that the $\mathcal{L}_k(V)$ is the generalization of Yau algebra. Further detail can be check from ([12,24]).

In [24], the sharp upper estimate and inequality conjectures introduced in following pattern:

Conjecture 1. [24] Let $\delta_k(\{y_1^{\alpha_1} + \cdots + y_n^{\alpha_n} = 0\}) = h_k(\alpha_1, \dots, \alpha_n), 0 \leq k \leq n$ and $(V, 0) = \{(y_1, y_2, \dots, y_n) \in \mathbb{C}^n : g(y_1, y_2, \dots, y_n) = 0\}, (n \geq 2)$ be an isolated singularity with weight type $(w_1, w_2, \dots, w_n; 1)$. Then $\delta_k(V) \leq h_k(1/w_1, \dots, 1/w_n)$.

Conjecture 2. [24] using the above notations, suppose $(V, 0)$ defined by $g \in \mathcal{O}_n, n \geq 2$. Then

$$\delta_{(k+1)}(V) < \delta_k(V), k \geq 1.$$

The Conjecture 1 have been proved for binomial and trinomial singularities when $k = 1, 2, 3, 4$ ([11,17,20,24,25]) and the Conjecture 2 also have been proved for $k = 1, 2, 3$ ([24,25]).

The main results of this paper is to prove the Conjecture 1 and 2) for binomial and trinomial singularities for particular value of k . The key findings of this paper following as:

Theorem 3. For $(V, 0) = \{(y_1, y_2, \dots, y_n) \in \mathbb{C}^n : y_1^{\alpha_1} + \cdots + y_n^{\alpha_n} = 0\}, (n \geq 2; \alpha_l \geq 7, 1 \leq j \leq n)$

$$\delta_6(V) = h_6(\alpha_1, \dots, \alpha_n) = \sum_{j=1}^n \frac{\alpha_j - 7}{\alpha_j - 6} \prod_{l=1}^n (\alpha_l - 6).$$

Theorem 4. For binomial singularity $(V, 0)$ defined by $g(y_1, y_2)$ a weighted homogeneous polynomial with $\text{mult}(g) \geq 8$ and weight type $(w_1, w_2; 1)$,

$$\delta_6(V) \leq h_6\left(\frac{1}{w_1}, \frac{1}{w_2}\right) = \sum_{l=1}^2 \frac{\frac{1}{w_l} - 7}{\frac{1}{w_l} - 6} \prod_{j=1}^2 \left(\frac{1}{w_j} - 6\right).$$

Theorem 5. For binomial singularity $(V, 0)$ defined by $g(y_1, y_2)$ a weighted homogeneous polynomial with $\text{mult}(g) \geq 8$ and weight type $(w_1, w_2; 1)$,

$$\delta_6(V) < \delta_5(V).$$

Theorem 6. For feunomial singularity $(V, 0)$ defined by $g(y_1, y_2, y_3)$ a weighted homogeneous polynomial with $\text{mult}(g) \geq 8$ and weight type $(w_1, w_2, w_3; 1)$, Then

$$\delta_6(V) \leq h_6\left(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}\right) = \sum_{l=1}^3 \frac{\frac{1}{w_l} - 7}{\frac{1}{w_l} - 6} \prod_{j=1}^3 \left(\frac{1}{w_j} - 6\right).$$

Theorem 7. For trinomial singularity $(V, 0)$ defined by $g(y_1, y_2, y_3)$ a weighted homogeneous polynomial with $\text{mult}(g) \geq 8$ and weight type $(w_1, w_2, w_3; 1)$. Then

$$\delta_6(V) < \delta_5(V).$$

2. Preliminaries

Proposition 1. Analytically, the series Type A., Type B., and Type C. are the homogeneous fewnomial Isolated singularity g having $\text{mult}(g) \geq 3$

$$\text{Type A. } y_1^{\alpha_1} + y_2^{\alpha_2} + \cdots + y_{n-1}^{\alpha_{n-1}} + y_n^{\alpha_n}, n \geq 1,$$

$$\text{Type B. } y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_3 + \cdots + y_{n-1}^{\alpha_{n-1}} y_n + y_n^{\alpha_n}, n \geq 2,$$

$$\text{Type C. } y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_3 + \cdots + y_{n-1}^{\alpha_{n-1}} y_n + y_n^{\alpha_n} y_1, n \geq 2.$$

Corollary 1. Analytically, the series A, B, and C are the Homogeneous fewnomial Isolated binomial singularities g having $\text{mult}(g) \geq 3$

$$\text{A: } y_1^{\alpha_1} + y_2^{\alpha_2},$$

$$\text{B: } y_1^{\alpha_1} y_2 + y_2^{\alpha_2},$$

$$\text{C: } y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_1.$$

Proposition 2. ([28]) Analytically $g(y_1, y_2, y_3)$, $\text{mult}(g) \geq 3$ is,

$$\text{Type 1. } y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3},$$

$$\text{Type 2. } y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_3 + y_3^{\alpha_3},$$

$$\text{Type 3. } y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_3 + y_3^{\alpha_3} y_1,$$

$$\text{Type 4. } y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3} y_1,$$

$$\text{Type 5. } y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_1 + y_3^{\alpha_3}.$$

3. Proof of theorems

Before going to the proof of main theorems, first we will prove some propositions.

Proposition 3. For $(V, 0)$ defined by $g = y_1^{\alpha_1} + y_2^{\alpha_2} + \cdots + y_n^{\alpha_n}$ ($\alpha_l \geq 8, l = 1, 2, \dots, n$) with weight type $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \dots, \frac{1}{\alpha_n}; 1)$,

$$\delta_6(V) = \sum_{j=1}^n \frac{\alpha_j - 7}{\alpha_j - 6} \prod_{l=1}^n (\alpha_l - 6).$$

Proof. $\prod_{l=1}^n (\alpha_l - 6)$ be the dimension of moduli algebra $M_6(V)$ and monomial basis are of the form

$$\{y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n}, 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, \dots, 0 \leq l_n \leq \alpha_n - 7\},$$

with following relations:

$$y_1^{\alpha_1 - 6} = 0, y_2^{\alpha_2 - 6} = 0, y_3^{\alpha_3 - 6} = 0, \dots, y_n^{\alpha_n - 6} = 0. \quad (1)$$

This implies

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7} \sum_{l_2=0}^{\alpha_2-7} \cdots \sum_{l_n=0}^{\alpha_n-7} \alpha_{l_1, l_2, \dots, l_n}^j y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n}, j = 1, 2, \dots, n.$$

Using 1 we may determine a derivation of $M_6(V)$ as

$$\begin{aligned} \alpha_{0,l_2,l_3,\dots,l_n}^1 &= 0; 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \dots, 0 \leq l_n \leq \alpha_n - 7; \\ \alpha_{l_1,0,l_3,\dots,l_n}^2 &= 0; 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \dots, 0 \leq l_n \leq \alpha_n - 7; \\ \alpha_{l_1,l_2,0,\dots,l_n}^3 &= 0; 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, \dots, 0 \leq l_n \leq \alpha_n - 7; \\ &\vdots \\ \alpha_{l_1,l_2,l_3,\dots,l_{n-1},0}^n &= 0; 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, \dots, 0 \leq l_{n-1} \leq \alpha_{n-1} - 7. \end{aligned}$$

The derivation basis are of the form

$$\begin{aligned} y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n} \partial_1, & 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \dots, 0 \leq l_n \leq \alpha_n - 7; \\ y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n} \partial_2, & 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \dots, 0 \leq l_n \leq \alpha_n - 7; \\ y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n} \partial_3, & 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 7, 0 \leq l_4 \leq \alpha_4 - 7, \\ & 0 \leq l_5 \leq \alpha_5 - 7, 0 \leq l_6 \leq \alpha_6 - 7, \dots, 0 \leq l_n \leq \alpha_n - 7; \\ &\vdots \\ y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n} \partial_n, & 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \dots, 1 \leq l_n \leq \alpha_n - 7. \end{aligned}$$

This implies

$$\delta_6(V) = \sum_{j=1}^n \frac{\alpha_j - 7}{\alpha_j - 6} \prod_{l=1}^n (\alpha_l - 6).$$

□

Remark 1. For weighted isolated homogeneous fewnomial singularity $(V, 0)$ of type A defined by $g = y_1^{\alpha_1} + y_2^{\alpha_2}$ ($\alpha_1 \geq 8, \alpha_2 \geq 8$) with weight type $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}; 1)$ the Proposition 3 implies

$$\delta_6(V) = 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 84.$$

Proposition 4. For isolated binomial singularity $(V, 0)$ of type B defined by $g = y_1^{\alpha_1}y_2 + y_2^{\alpha_2}$ ($\alpha_1 \geq 7, \alpha_2 \geq 8$) with weight type $(\frac{\alpha_2-1}{\alpha_1\alpha_2}, \frac{1}{\alpha_2}; 1)$,

$$\delta_6(V) = 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 87.$$

For $\text{mult}(g) \geq 8$,

$$2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 87 \leq \frac{2\alpha_1\alpha_2^2}{\alpha_2 - 1} - 13\left(\frac{\alpha_1\alpha_2}{\alpha_2 - 1} + \alpha_2\right) + 84.$$

Proof. $\alpha_1\alpha_2 - 6(\alpha_1 + \alpha_2) + 37$ be the dimension of

$$M_6(V) = \mathbb{C}\{y_1, y_2\} / (\mathcal{G}_{y_1y_1y_1y_1y_1}, \mathcal{G}_{y_2y_2y_2y_2y_2}, \mathcal{G}_{y_1y_2y_2y_2y_2}, \mathcal{G}_{y_1y_1y_2y_2y_2}, \mathcal{G}_{y_1y_1y_1y_2y_2}, \mathcal{G}_{y_1y_1y_1y_1y_2})$$

and monomial basis are of the form

$$\{y_1^{l_1} y_2^{l_2}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; y_1^{\alpha_1 - 6}\}. \quad (2)$$

This implies,

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7} \sum_{l_2=0}^{\alpha_2-7} \alpha_{l_1, l_2}^i y_1^{l_1} y_2^{l_2} + \alpha_{\alpha_1-6, 0}^i y_1^{\alpha_1-6}, \quad i = 1, 2.$$

The basis of $\mathcal{L}_6(V)$ are

$$y_1^{l_1} y_2^{l_2} \partial_1, 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7; y_1^{l_1} y_2^{l_2} \partial_2, 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7;$$

$$y_2^{\alpha_2-7} \partial_1; y_1^{\alpha_1-6} \partial_1; y_1^{\alpha_1-6} \partial_2.$$

We get following formula

$$\delta_6(V) = 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 87.$$

Finally we need to show that

$$2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 87 \leq \frac{2\alpha_1\alpha_2^2}{\alpha_2-1} - 13\left(\frac{\alpha_1\alpha_2}{\alpha_2-1} + \alpha_2\right) + 84. \quad (3)$$

After solving 3 we have $\alpha_1(\alpha_2 - 10) + \alpha_2(\alpha_1 - 6) + 6 \geq 0$. \square

Proposition 5. For isolated binomial singularity $(V, 0)$ of type C defined by $g = y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_1$ ($\alpha_1 \geq 7, \alpha_2 \geq 7$) with weight type $(\frac{\alpha_2-1}{\alpha_1\alpha_2-1}, \frac{\alpha_1-1}{\alpha_1\alpha_2-1}; 1)$,

$$\delta_6(V) = \begin{cases} 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 90; & \alpha_1 \geq 8, \alpha_2 \geq 8 \\ \alpha_2 - 3; & \alpha_1 = 7, \alpha_2 \geq 7. \end{cases}$$

For $\text{mult}(g) \geq 8$,

$$2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 90 \leq \frac{2(\alpha_1\alpha_2 - 1)^2}{(\alpha_1 - 1)(\alpha_2 - 1)} - 13(\alpha_1\alpha_2 - 1) \left(\frac{\alpha_1 + \alpha_2 - 2}{(\alpha_1 - 1)(\alpha_2 - 1)} \right) + 84.$$

Proof. The Moduli algebra has dimension $\alpha_1\alpha_2 - 6(\alpha_1 + \alpha_2) + 38$. The monomial basis of

$$M_5(V) = \mathbb{C}\{y_1, y_2\} / (g_{y_1 y_1 y_1 y_1 y_1}, g_{y_2 y_2 y_2 y_2 y_2}, g_{y_1 y_2 y_2 y_2 y_2}, g_{y_1 y_1 y_2 y_2 y_2}, g_{y_1 y_1 y_1 y_2 y_2})$$

are of the form

$$\{y_1^{l_1} y_2^{l_2}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; y_1^{\alpha_1-6}; y_2^{\alpha_2-6}\}. \quad (4)$$

This implies,

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7} \sum_{l_2=0}^{\alpha_2-7} \alpha_{l_1, l_2}^i y_1^{l_1} y_2^{l_2} + \alpha_{\alpha_1-6, 0}^i y_1^{\alpha_1-6} + \alpha_{0, \alpha_2-6}^i y_2^{\alpha_2-6}, \quad i = 1, 2.$$

The derivation basis are of the form

$$y_1^{l_1} y_2^{l_2} \partial_1, 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7; y_1^{l_1} y_2^{l_2} \partial_2, 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7;$$

$$y_2^{\alpha_2-7} \partial_1; y_2^{\alpha_2-6} \partial_1; y_1^{\alpha_1-6} \partial_1; y_2^{\alpha_2-6} \partial_2; y_1^{\alpha_1-7} \partial_2; y_1^{\alpha_1-6} \partial_2.$$

This implies,

$$\delta_6(V) = 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 90.$$

For $\alpha_1 = 7, \alpha_2 \geq 7$, we get $\mathcal{L}_6(V)$ as

$$y_2^{l_2} \partial_2, 1 \leq l_2 \leq \alpha_2 - 6; y_2^{\alpha_2 - 6} \partial_1; y_1 \partial_1; y_1 \partial_2.$$

Next we will prove that

$$2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 90 \leq \frac{2(\alpha_1\alpha_2 - 1)^2}{(\alpha_1 - 1)(\alpha_2 - 1)} - 13(\alpha_1\alpha_2 - 1) \left(\frac{\alpha_1 + \alpha_2 - 2}{(\alpha_1 - 1)(\alpha_2 - 1)} \right) + 84. \quad (5)$$

After solving 5 we have

$$\alpha_1\alpha_2^2[(\alpha_2 - 5)(\alpha_1 - 5) - \alpha_1(\alpha_2 - 8)] + \alpha_2^3 + 4\alpha_1^2\alpha_2 + 10\alpha_2^2(\alpha_1 - 6) + 6\alpha_1\alpha_2(\alpha_1 - 6) + 3\alpha_1^2(\alpha_2 - 6) + \alpha_1\alpha_2(\alpha_1 - 6) + 15\alpha_1 + 2(\alpha_2 - 6) \geq 0.$$

Similarly, the conjecture 1 hold true for $\alpha_1 = 7, \alpha_2 \geq 7$. \square

Remark 2. For feynomial surface isolated singularity $(V, 0)$ of the type 1 defined by $g = y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3}$ ($\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 8$) with weight type $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}; 1)$. From Proposition 3, we get

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 + 120(\alpha_1 + \alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 756.$$

Proposition 6. For feynomial surface isolated singularity $(V, 0)$ of the type 2 defined by $g = y_1^{\alpha_1}y_2 + y_2^{\alpha_2}y_3 + y_3^{\alpha_3}$ ($\alpha_1 \geq 7, \alpha_2 \geq 7, \alpha_3 \geq 8$) with weight type $(\frac{1-\alpha_3+\alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3}, \frac{\alpha_3-1}{\alpha_2\alpha_3}, \frac{1}{\alpha_3}; 1)$,

$$\delta_6(V) = \begin{cases} 3\alpha_1\alpha_2\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 124(\alpha_1 + \alpha_3) \\ + 120\alpha_2 - 807; & \alpha_1 \geq 7, \alpha_2 \geq 8, \alpha_3 \geq 8 \\ 2\alpha_1\alpha_3 - 9\alpha_1 - 11\alpha_3 + 47; & \alpha_1 \geq 7, \alpha_2 = 7, \alpha_3 \geq 8. \end{cases}$$

For $\alpha_1 \geq 7, \alpha_2 \geq 8, \alpha_3 \geq 8$, we need to prove following inequality:

$$\begin{aligned} 3\alpha_1\alpha_2\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 124(\alpha_1 + \alpha_3) + 120\alpha_2 - 807 &\leq \frac{3\alpha_1\alpha_2^2\alpha_3^3}{(1 - \alpha_3 + \alpha_2\alpha_3)(\alpha_3 - 1)} \\ - 19\left(\frac{\alpha_1\alpha_2^2\alpha_3^2}{(1 - \alpha_3 + \alpha_2\alpha_3)(\alpha_3 - 1)} + \frac{\alpha_1\alpha_2\alpha_3^2}{1 - \alpha_3 + \alpha_2\alpha_3} + \frac{\alpha_2\alpha_3^2}{\alpha_3 - 1}\right) &+ 120\left(\frac{\alpha_1\alpha_2\alpha_3}{1 - \alpha_3 + \alpha_2\alpha_3} \right. \\ \left. + \frac{\alpha_2\alpha_3}{\alpha_3 - 1} + \alpha_3\right) - 756. \end{aligned}$$

Proof. The Moduli algebra has dimension $\alpha_1\alpha_2\alpha_3 - 6(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 37(\alpha_1 + \alpha_3) + 36\alpha_2 - 228$. The monomial basis of $M_6(V)$ are of the form:

$$\{y_1^{l_1}y_2^{l_2}y_3^{l_3}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1 - 6}y_3^{l_3}, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_3^{\alpha_3 - 6}, 0 \leq l_1 \leq \alpha_1 - 7\}.$$

This implies,

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7} \sum_{l_2=0}^{\alpha_2-7} \sum_{l_3=0}^{\alpha_3-7} \alpha_{l_1, l_2, l_3}^i y_1^{l_1} y_2^{l_2} y_3^{l_3} + \sum_{l_1=0}^{\alpha_1-7} \alpha_{l_1, 0, \alpha_3-6}^i y_1^{l_1} y_3^{\alpha_3-6} + \sum_{l_3=0}^{\alpha_3-7} \alpha_{\alpha_1-6, 0, l_3}^i y_3^{l_3} y_1^{\alpha_1-6}, \quad i = 1, 2, 3.$$

The derivation basis are of the form

$$\begin{aligned}
& y_1^{l_1} y_2^{l_2} y_3^{l_3} \partial_1, 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1 - 6} y_3^{l_3} \partial_1, 0 \leq l_3 \leq \alpha_3 - 7, \\
& y_2^{\alpha_2 - 7} y_3^{l_3} \partial_1, 1 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1} y_2^{\alpha_2 - 6} \partial_1, 0 \leq l_1 \leq \alpha_1 - 7, \\
& y_1^{l_1} y_2^{l_2} y_3^{l_3} \partial_2, 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1 - 6} y_3^{l_3} \partial_2, 0 \leq l_3 \leq \alpha_3 - 7, \\
& y_1^{l_1} y_2^{\alpha_2 - 6} \partial_2, 0 \leq l_1 \leq \alpha_1 - 7; y_1^{l_1} y_3^{\alpha_3 - 7} \partial_2, 1 \leq l_1 \leq \alpha_1 - 7, \\
& y_1^{l_1} y_2^{l_2} y_3^{l_3} \partial_3, 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1} y_2^{\alpha_2 - 6} \partial_3, 0 \leq l_1 \leq \alpha_1 - 7, \\
& y_1^{\alpha_1 - 6} y_3^{l_3} \partial_3, 1 \leq l_3 \leq \alpha_3 - 7.
\end{aligned}$$

We get

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 124(\alpha_1 + \alpha_3) + 120\alpha_2 - 807.$$

For $\alpha_1 \geq 7, \alpha_2 = 7, \alpha_3 \geq 8$, we get the following basis:

$$\begin{aligned}
& y_1^{l_1} y_3^{l_3} \partial_1, 1 \leq l_1 \leq \alpha_1 - 6, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1} y_2 \partial_1, 0 \leq l_1 \leq \alpha_1 - 7, \\
& y_1^{l_1} y_2 \partial_2, 0 \leq l_1 \leq \alpha_1 - 7; y_1^{l_1} y_3^{\alpha_3 - 7} \partial_2, 1 \leq l_1 \leq \alpha_1 - 6, \\
& y_1^{l_1} y_3^{l_3} \partial_3, 0 \leq l_1 \leq \alpha_1 - 6, 1 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1} y_2 \partial_3, 0 \leq l_1 \leq \alpha_1 - 7.
\end{aligned}$$

We get

$$\delta_6(V) = 2\alpha_1\alpha_3 - 9\alpha_1 - 11\alpha_3 + 47.$$

For $\alpha_1 \geq 7, \alpha_2 \geq 8, \alpha_3 \geq 8$, we need to prove following inequality:

$$\begin{aligned}
& 3\alpha_1\alpha_2\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 124(\alpha_1 + \alpha_3) + 120\alpha_2 - 807 \leq \frac{3\alpha_1\alpha_2^2\alpha_3^3}{(1 - \alpha_3 + \alpha_2\alpha_3)(\alpha_3 - 1)} \\
& - 19\left(\frac{\alpha_1\alpha_2^2\alpha_3^2}{(1 - \alpha_3 + \alpha_2\alpha_3)(\alpha_3 - 1)} + \frac{\alpha_1\alpha_2\alpha_3^2}{1 - \alpha_3 + \alpha_2\alpha_3} + \frac{\alpha_2\alpha_3^2}{\alpha_3 - 1}\right) + 120\left(\frac{\alpha_1\alpha_2\alpha_3}{1 - \alpha_3 + \alpha_2\alpha_3}\right. \\
& \left. + \frac{\alpha_2\alpha_3}{\alpha_3 - 1} + \alpha_3\right) - 756.
\end{aligned}$$

After simplification we get

$$(\alpha_1 - 5)^3(\alpha_2 - 7)\alpha_3 + (\alpha_2 - 6)\alpha_1\alpha_3((\alpha_3 - 5)(\alpha_1 - 7) + (\alpha_2 - 5)(\alpha_3 - 5)) + \alpha_2(3\alpha_3 - 6)(\alpha_1 - 5) + \alpha_2(\alpha_1 - 4) + 6 \geq 0.$$

Similarly, one can prove that for $\alpha_1 \geq 7, \alpha_2 = 7, \alpha_3 \geq 8$ the conjecture 1 hold true. \square

Proposition 7. For isolated fewnomial surface singularity $(V, 0)$ of type 3 defined by $g = y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_3 + y_3^{\alpha_3} y_1$ ($\alpha_1 \geq 7, \alpha_2 \geq 7, \alpha_3 \geq 7$) with weight type

$$\left(\frac{1 - \alpha_3 + \alpha_2\alpha_3}{1 + \alpha_1\alpha_2\alpha_3}, \frac{1 - \alpha_1 + \alpha_1\alpha_3}{1 + \alpha_1\alpha_2\alpha_3}, \frac{1 - \alpha_2 + \alpha_1\alpha_2}{1 + \alpha_1\alpha_2\alpha_3}; 1\right)$$

$$\delta_6(V) = \begin{cases} 3\alpha_1\alpha_2\alpha_3 + 124(\alpha_1 + \alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 831; & \alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 8 \\ 2\alpha_2\alpha_3 - 11\alpha_2 - 9\alpha_3 + 51; & \alpha_1 = 7, \alpha_2 \geq 8, \alpha_3 \geq 7 \\ 2\alpha_1\alpha_3 - 9\alpha_1 - 11\alpha_3 + 51; & \alpha_1 \geq 7, \alpha_2 = 7, \alpha_3 \geq 7 \\ 2\alpha_1\alpha_2 - 11\alpha_1 - 9\alpha_2 + 51; & \alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 = 7 \end{cases}$$

Assuming that $\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 8$, then we need to prove following inequality:

$$3\alpha_1\alpha_2\alpha_3 + 124(\alpha_1 + \alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 831 \leq \frac{3(1+\alpha_1\alpha_2\alpha_3)^3}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_1+\alpha_1\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)} \\ + 120\left(\frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_3+\alpha_2\alpha_3} + \frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_1+\alpha_1\alpha_3} + \frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_2+\alpha_1\alpha_2}\right) - 19\left(\frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_1+\alpha_1\alpha_3)} + \frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_1+\alpha_1\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)}\right) \\ + \frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)} - 756.$$

Proof. $(\alpha_1\alpha_2\alpha_3 - 6(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 37(\alpha_1 + \alpha_2 + \alpha_3) - 234)$ be the dimension of $M_5(V)$ and monomial basis are of the form

$$\{y_1^{l_1}y_2^{l_2}y_3^{l_3}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6}y_3^{l_3}, 0 \leq l_3 \leq \alpha_3 - 7; \\ y_2^{l_2}y_3^{\alpha_3-6}, 0 \leq l_2 \leq \alpha_2 - 7; y_1^{l_1}y_2^{\alpha_2-6}, 0 \leq l_1 \leq \alpha_1 - 7\}.$$

This implies,

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7} \sum_{l_2=0}^{\alpha_2-7} \sum_{l_3=0}^{\alpha_3-7} \alpha_{l_1,l_2,l_3}^i y_1^{l_1}y_2^{l_2}y_3^{l_3} + \sum_{l_1=0}^{\alpha_1-7} \alpha_{l_1,\alpha_2-6,0}^i y_1^{l_1}y_2^{\alpha_2-6} + \sum_{l_3=0}^{\alpha_3-7} \alpha_{\alpha_1-6,0,l_3}^i y_1^{\alpha_1-6}y_3^{l_3} \\ + \sum_{l_2=0}^{\alpha_2-7} \alpha_{0,l_2,\alpha_3-6}^i y_2^{l_2}y_3^{\alpha_3-6}, \quad i = 1, 2, 3.$$

The derivation basis are of the form

$$y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_1, 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_2^{l_2}y_3^{\alpha_3-6}\partial_1, 0 \leq l_2 \leq \alpha_2 - 7, \\ y_2^{\alpha_2-7}y_3^{l_3}\partial_1, 1 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_2^{\alpha_2-6}\partial_1, 0 \leq l_1 \leq \alpha_1 - 7; y_1^{\alpha_1-6}y_3^{l_3}\partial_1, 0 \leq l_3 \leq \alpha_3 - 7, \\ y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_2, 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6}y_3^{l_3}\partial_2, 0 \leq l_3 \leq \alpha_3 - 7, \\ y_1^{l_1}y_2^{\alpha_2-6}\partial_2, 0 \leq l_1 \leq \alpha_1 - 7; y_1^{l_1}y_3^{\alpha_3-7}\partial_2, 1 \leq l_1 \leq \alpha_1 - 7; y_2^{l_2}y_3^{\alpha_3-6}\partial_2, 0 \leq l_2 \leq \alpha_2 - 7, \\ y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_3, 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_2^{\alpha_2-6}\partial_3, 0 \leq l_1 \leq \alpha_1 - 7, \\ y_1^{\alpha_1-7}y_2^{l_2}\partial_3, 1 \leq l_2 \leq \alpha_2 - 7; y_2^{l_2}y_3^{\alpha_3-6}\partial_3, 0 \leq l_2 \leq \alpha_2 - 7; y_1^{\alpha_1-4}y_3^{l_3}\partial_3, 0 \leq l_3 \leq \alpha_3 - 7.$$

Therefore we have

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 + 124(\alpha_1 + \alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 831.$$

In case of $\alpha_1 = 7, \alpha_2 \geq 8, \alpha_3 \geq 7$, we obtain basis as

$$y_2^{\alpha_2-7}y_3^{l_3}\partial_1, 1 \leq l_3 \leq \alpha_3 - 6; y_1y_3^{l_3}\partial_1, 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6}\partial_1; y_3^{\alpha_3-6}\partial_2, \\ y_1y_3^{l_3}\partial_2, 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6}\partial_2; y_2^{l_2}y_3^{l_3}\partial_2, 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 6, \\ y_2^{l_2}y_3^{l_3}\partial_3, 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 6, y_1y_3^{l_3}\partial_3, 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6}\partial_3.$$

Therefore we have

$$\delta_6(V) = 2\alpha_2\alpha_3 - 11\alpha_2 - 9\alpha_3 + 51.$$

Similarly, we can get bases for $\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 = 7$ and $\alpha_1 \geq 7, \alpha_2 = 7, \alpha_3 \geq 7$.

For $\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 8$, we need to prove following inequality:

$$3\alpha_1\alpha_2\alpha_3 + 124(\alpha_1 + \alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 851 \leq \frac{3(1+\alpha_1\alpha_2\alpha_3)^3}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_1+\alpha_1\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)} \\ + 120\left(\frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_3+\alpha_2\alpha_3} + \frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_1+\alpha_1\alpha_3} + \frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_2+\alpha_1\alpha_2}\right) - 19\left(\frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_1+\alpha_1\alpha_3)} + \frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_1+\alpha_1\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)}\right) \\ + \frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)} - 756.$$

After simplification we get

$$4(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3) + \alpha_1(\alpha_2 - 7) + \alpha_2(\alpha_3 - 7) + \alpha_3(\alpha_1 - 7) + 4\alpha_1^2[\alpha_2(\alpha_3 - 7) + \alpha_3(\alpha_2 - 7)] \\ + 3\alpha_2^2[\alpha_1(\alpha_3 - 6) + \alpha_3(\alpha_1 - 7)] + 5\alpha_3^2[\alpha_1(\alpha_2 - 7) + \alpha_2(\alpha_1 - 6)] + 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + 3(\alpha_1^3\alpha_2 + \alpha_2^3\alpha_3 \\ + \alpha_3^3\alpha_1) + 2\alpha_1^2\alpha_2^2\alpha_3^2 + 5(\alpha_1\alpha_2^2\alpha_3 + \alpha_1\alpha_2\alpha_3^2) + 2\alpha_1^2\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_3[2\alpha_1 - 11] + \alpha_1^3\alpha_2\alpha_3^2(\alpha_3 - 7)(\alpha_2 - 7) \\ + \alpha_1^2\alpha_3^2(\alpha_3 - 7)(\alpha_1\alpha_2 - 7) + \alpha_1^2\alpha_2\alpha_3^2(\alpha_3 + \alpha_2 - 8) + 3\alpha_1\alpha_2\alpha_3^3(\alpha_1 - 7) + \alpha_1^2\alpha_2^3\alpha_3(\alpha_3 - 7)(\alpha_1 - 6) \\ + \alpha_1^2\alpha_2^2(\alpha_1 - 7)(\alpha_2\alpha_3 - 6) + \alpha_1^3\alpha_2\alpha_3(\alpha_2 - 7) + \alpha_1^2\alpha_2^2\alpha_3(\alpha_1 - 6 + (\alpha_3 - 7)) + \alpha_1\alpha_2^2\alpha_3^3(\alpha_2 - 7)(\alpha_1 \\ - 6) + \alpha_2^2\alpha_3^2(\alpha_2 - 7)(\alpha_1\alpha_3 - 7) + 12 \geq 0.$$

Similarly, conjecture 1 we can prove for $\alpha_1, \alpha_3 \geq 7, \alpha_2 = 7; \alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 = 7$ and $\alpha_1 = 7, \alpha_2 \geq 8, \alpha_3 \geq 7$. \square

Proposition 8. For isolated fewnomial surface singularity $(V, 0)$ of type 4 defined by $g = y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3}y_2$ ($\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 7$) with weight type $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{\alpha_2-1}{\alpha_2\alpha_3}; 1)$,

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 + 124\alpha_1 + 120(\alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 781.$$

Assuming that $\text{mult}(g) \geq 8$, we need to prove following inequality

$$3\alpha_1\alpha_2\alpha_3 + 124\alpha_1 + 120(\alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 781 \leq \frac{3\alpha_2^2\alpha_1\alpha_3}{\alpha_2-1} + 120(\alpha_1 + \alpha_2 + \frac{\alpha_2\alpha_3}{\alpha_2-1}) \\ - 19(\alpha_1\alpha_2 + \frac{\alpha_1\alpha_2\alpha_3}{\alpha_2-1} + \frac{\alpha_2^2\alpha_3}{\alpha_2-1}) - 756.$$

Proof. $(\alpha_1\alpha_2\alpha_3 - 6(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 36(\alpha_2 + \alpha_3) + 37\alpha_1 - 222)$ be the dimension of $M_6(V)$ and monomial basis are of the form

$$\{y_1^{l_1}y_2^{l_2}y_3^{l_3}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; 0 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_3^{\alpha_3-6}, 0 \leq l_2 \leq \alpha_1 - 7\}.$$

This implies,

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7} \sum_{l_2=0}^{\alpha_2-7} \sum_{l_3=0}^{\alpha_3-7} \alpha_{l_1, l_2, l_3}^i y_1^{l_1} y_2^{l_2} y_3^{l_3} + \sum_{l_1=0}^{\alpha_1-7} \alpha_{l_1, 0, \alpha_3-6}^i y_1^{l_1} y_3^{\alpha_3-6}, \quad i = 1, 2, 3.$$

The derivation basis are of the form

$$y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_1, \quad 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_3^{\alpha_3-6}\partial_1, \quad 1 \leq l_1 \leq \alpha_1 - 7, \\ y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_2, \quad 1 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_3^{\alpha_3-6}\partial_2, \quad 0 \leq l_1 \leq \alpha_1 - 7, \\ y_2^{l_2}y_3^{l_3}\partial_2, \quad 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, y_1^{l_1}y_2^{\alpha_2-7}\partial_3, \quad 0 \leq l_1 \leq \alpha_1 - 7 \\ y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_3, \quad 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 7, y_1^{l_1}y_3^{\alpha_3-6}\partial_3, \quad 0 \leq l_1 \leq \alpha_1 - 7.$$

Therefore we have

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 + 124\alpha_1 + 120(\alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 781.$$

Next, we also need to show that when $\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 7$, then

$$3\alpha_1\alpha_2\alpha_3 + 124\alpha_1 + 120(\alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 781 \leq \frac{3\alpha_2^2\alpha_1\alpha_3}{\alpha_2-1} + 120(\alpha_1 + \alpha_2 + \frac{\alpha_2\alpha_3}{\alpha_2-1}) \\ - 19(\alpha_1\alpha_2 + \frac{\alpha_1\alpha_2\alpha_3}{\alpha_2-1} + \frac{\alpha_2^2\alpha_3}{\alpha_2-1}) - 756.$$

From above inequality, we get

$$\frac{\alpha_1\alpha_3(2\alpha_2 - 12)}{\alpha_2 - 7} + \alpha_2\alpha_3 + \alpha_3(\alpha_2 - 5) + \frac{6\alpha_3}{\alpha_2 - 6} + \frac{\alpha_1[\alpha_2(\alpha_3 - 6) + 7]}{\alpha_2 - 6} \geq 0.$$

□

Proposition 9. For isolated fewnomial surface singularity $(V, 0)$ of type 5 defined by $g = y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_1 + y_3^{\alpha_3}$ ($\alpha_1 \geq 7, \alpha_2 \geq 7, \alpha_3 \geq 8$) with weight type $(\frac{\alpha_2-1}{\alpha_1\alpha_2-1}, \frac{\alpha_1-1}{\alpha_1\alpha_2-1}, \frac{1}{\alpha_3}; 1)$,

$$\delta_6(V) = \begin{cases} 3\alpha_1\alpha_2\alpha_3 + 120(\alpha_1 + \alpha_2) + 128\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 806; & \alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 8 \\ 2\alpha_2\alpha_3 - 13\alpha_2 - 8\alpha_3 + 53; & \alpha_1 = 7, \alpha_2 \geq 7, \alpha_3 \geq 8 \end{cases}$$

Assuming that $\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 8$, then we need to prove following inequality:

$$3\alpha_1\alpha_2\alpha_3 + 120(\alpha_1 + \alpha_2) + 128\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 806 \leq \frac{3\alpha_3(\alpha_1\alpha_2-1)^2}{(\alpha_2-1)(\alpha_1-1)} + 120\left(\frac{\alpha_1\alpha_2-1}{\alpha_2-1} + \frac{\alpha_1\alpha_2-1}{\alpha_1-1} + \alpha_3\right) - 19\left(\frac{(\alpha_1\alpha_2-1)^2}{(\alpha_2-1)(\alpha_1-1)} + \frac{\alpha_3(\alpha_1\alpha_2-1)}{\alpha_1-1} + \frac{\alpha_3(\alpha_1\alpha_2-1)}{\alpha_2-1}\right) - 756.$$

Proof. $\alpha_1\alpha_2\alpha_3 - 6(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 36(\alpha_1 + \alpha_2) + 38\alpha_3 - 228$ be the number of dimension of $M_6(V)$ and monomial basis are of the form

$$\{y_1^{l_1} y_2^{l_2} y_3^{l_3}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6} y_3^{l_3}, 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6} y_3^{l_3}, 0 \leq l_3 \leq \alpha_3 - 7\},$$

This implies,

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7} \sum_{l_2=0}^{\alpha_2-7} \sum_{l_3=0}^{\alpha_3-7} \alpha_{l_1, l_2, l_3}^i y_1^{l_1} y_2^{l_2} y_3^{l_3} + \sum_{l_3=0}^{\alpha_3-7} \alpha_{\alpha_1-6, 0, l_3}^i y_1^{\alpha_1-6} y_3^{l_3} + \sum_{l_3=0}^{\alpha_3-7} \alpha_{0, \alpha_2-6, l_3}^i y_2^{\alpha_2-6} y_3^{l_3}, \quad i = 1, 2, 3.$$

The derivation basis are of the form

$$\begin{aligned} & y_1^{l_1} y_2^{l_2} y_3^{l_3} \partial_1, \quad 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6} y_3^{l_3} \partial_1, \quad 0 \leq l_3 \leq \alpha_3 - 7, \\ & y_2^{\alpha_2-6} y_3^{l_3} \partial_1, \quad 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-7} y_3^{l_3} \partial_1, \quad 0 \leq l_3 \leq \alpha_3 - 7, \\ & y_1^{l_1} y_2^{l_2} y_3^{l_3} \partial_2, \quad 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6} y_3^{l_3} \partial_2, \quad 0 \leq l_3 \leq \alpha_3 - 7, \\ & y_2^{\alpha_2-6} y_3^{l_3} \partial_2, \quad 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-7} y_3^{l_3} \partial_2, \quad 0 \leq l_3 \leq \alpha_3 - 7, \\ & y_1^{l_1} y_2^{l_2} y_3^{l_3} \partial_3, \quad 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6} y_3^{l_3} \partial_3, \quad 1 \leq l_3 \leq \alpha_3 - 7, \\ & y_2^{\alpha_2-6} y_3^{l_3} \partial_3, \quad 1 \leq l_3 \leq \alpha_3 - 7. \end{aligned}$$

Therefore we have

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 + 120(\alpha_1 + \alpha_2) + 128\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 806.$$

For $\alpha_1 = 7, \alpha_2 \geq 7, \alpha_3 \geq 8$, we obtain the following basis:

$$\begin{aligned} & y_2^{l_2} y_3^{l_3} \partial_2, \quad 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6} y_3^{l_3} \partial_1, \quad 0 \leq l_3 \leq \alpha_3 - 7, \\ & y_1 y_3^{l_3} \partial_1, \quad 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6} y_3^{l_3} \partial_2, \quad 0 \leq l_3 \leq \alpha_3 - 7, \\ & y_2^{l_2} y_3^{l_3} \partial_3, \quad 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 7; y_1 y_3^{l_3} \partial_2, \quad 0 \leq l_3 \leq \alpha_3 - 7, \\ & y_1 y_3^{l_3} \partial_3, \quad 1 \leq l_3 \leq \alpha_3 - 7. \end{aligned}$$

We have

$$\delta_6(V) = 2\alpha_2\alpha_3 - 13\alpha_2 - 8\alpha_3 + 53.$$

Next, we need to show that when $\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 8$, then

$$3\alpha_1\alpha_2\alpha_3 + 120(\alpha_1 + \alpha_2) + 128\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 806 \leq \frac{3\alpha_3(\alpha_1\alpha_2-1)^2}{(\alpha_2-1)(\alpha_1-1)} + 120\left(\frac{\alpha_1\alpha_2-1}{\alpha_2-1} + \frac{\alpha_1\alpha_2-1}{\alpha_1-1}\right) + \alpha_3 - 19\left(\frac{(\alpha_1\alpha_2-1)^2}{(\alpha_2-1)(\alpha_1-1)} + \frac{\alpha_3(\alpha_1\alpha_2-1)}{\alpha_1-1} + \frac{\alpha_3(\alpha_1\alpha_2-1)}{\alpha_2-1}\right) - 756.$$

After simplification, we get

$$\begin{aligned} & \alpha_1(\alpha_1 - 7)(\alpha_2 - 6)(\alpha_3 + (\alpha_1 - 5)\alpha_2(\alpha_2 - 7)\alpha_3) + \alpha_1^2(\alpha_3 - 6)(\alpha_2 - 5) + \alpha_2^2\alpha_1 + 4\alpha_1(\alpha_2 - 6) \\ & + 4\alpha_2(\alpha_1 - 6) + 4\alpha_3(\alpha_1 - 5) + 12\alpha_1\alpha_2 + 14\alpha_1\alpha_3 + 5\alpha_2\alpha_3 + 22\alpha_2 + \alpha_1\alpha_2(\alpha_1 - 6) \\ & + (\alpha_1 - 5)\alpha_2(\alpha_2 - 6)(\alpha_3 - 5) + (\alpha_1 - 6)(\alpha_3 - 7) + 22 \geq 0. \end{aligned}$$

Similarly, for $\alpha_1 = 7, \alpha_2 \geq 7, \alpha_3 \geq 8$ the conjecture 1 also hold true. \square

Proof of Theorem 3.

Proof. The proof of theorem 3 is the consequence of Proposition 3. \square

Proof of Theorem 4.

Proof. The Proposition 4, 5 and Remark 1 implies Theorem 4 as a corollary. \square

Proof of Theorem 5.

Proof. The Propositions 4, 5, 6 and Remark 3 of [24] along with Remark 1 Propositions 4 and 5 implies that the inequality $\delta_6(V) < \delta_5(V)$ holds true for binomial singularities. \square

Proof of Theorem 6.

Proof. The Theorem 6 is the consequence of the Remark 2 and Propositions 6, 7, 8, 9. \square

Proof of Theorem 7.

Proof. The Propositions 6, 7, 8, 9 and Remark 4 of [24] along with Remark 2 and Propositions 6, 7, 8, 9 implies that the inequality $\delta_6(V) < \delta_5(V)$ holds true trinomial singularities. \square

4. Conclusion

Finding the dimension of an algebra is critical for studying its applications. This work is the partial proof of the conjecture over dimensions $\delta_k(V)$ of k -th Yau Algebra. First of all we determine general formula fo the dimension of fewnomial isolated singularities for $\mathcal{L}_6(V)$ in Prepositions 3. Then on the behave of this result determine formulas for weighted binomial isolated singularities of all the three types listed in Corollary 1 in Preposition 2 4 and 5. Then Using general formula for dimension $\delta_6(V)$ of $\mathcal{L}_6(V)$ determine formulas for weighted binomial isolated singularities in Preposition 6, 7, 8, and 9 of all the five types listed in Preposition 2. On the behave of all these findings and dimension of $\mathcal{L}_5(V)$ determind in [29], we proved inequality conjecture: $\delta_6(V) < \delta_5(V)$ for binomial and trinomial isolated singularities in theorem 3 and 3.

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