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Nonlinear Differential Equation and SIS Epidemiological Model with a Time Delay Bounded by Two Positive Functions

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Abstract: Delay differential equations were introduced to create more realistic models since many processes depend on past history.Many different mathematical methods, including the Susceptible-Infected-Recovered (SIR) model as well as derivatives thereof, can be used to prognose the spread of many various infectious diseases, including Covid-19. In this paper, we consider an epidemiological Susceptible-Infected-Susceptible (SIS) model in which the individuals cured from the infection return back into the group of people endangered by the infection. We use a nonlinear differential system with time delay as a mathematical formulation of this model. It is examined by the existence of a positive solution which is bounded by two functions $k_1, k_2 \in C([t_0,\infty),(0,\infty))$ with constants $\lambda_1 > 0$, $\lambda_2 > 0$ where $0 < \lambda_1 \leq \lambda_2$. The conditions are put in place for existence of a positive solution that approaches zero in infinity, if $t \to \infty$. Infectious diseases accompany a person throughout their entire life, and in the history of mankind often appeared in frequent epidemics caused high mortality. The occurrence of infectious diseases in the population represents a serious health, social and economic problem. There is a lot of ambiguity concerning the time span and final spread of the epidemic thus making it very demanding for the governing bodies, healthcare institutions, and economic sector to precisely judge the future development.Several examples are shown at the end of this paper.

Keywords: nonlinear differential equation with time delay; SIS model; epidemiological model; positive solution

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1. Introduction

Mathematical modelling spans many fields of study thanks to its versatility. It is useful in biology, economics, or medicine. It is very popular today thanks to big advances in computing technology allowing us to analyze many models and visualize the results in a short time period. One of the very important areas where differential equations can be applied is the epidemiological modelling where they can be used to forecast the infection occurrence or spread. Modelling the infectious diseases is always a very actual topic with new types of infections appearing and old infections reappearing, bacteria becoming resistant to antibiotics and globalization supporting spreading of diseases worldwide [[5]].

Mathematical modelling is a very good tool for understanding the spread of the many many infectious diseases including the most recent COVID-19 pandemic and exploring different scenarios. We are aware though that "one model cannot answer it all" and that more models are necessary to answer all outstanding questions and paint the whole picture. What we aimed to achieve is to connect the mathematical results which we acquired, with a real infectious diseases problem.

The plague in the first half of the 6th century BC killed tens of millions of people and several centuries later the total number of victims grew several times higher. After the first wave of infection receded it periodically reappeared in waves. Other notable pandemics in the history were the Spanish flu,

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HIV/AIDS, SARS, MERS up to the COVID-19 pandemic.
The infectious diseases including the most recent COVID-19 illness, which induced by the severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) has not only caused a large amount of deaths worldwide, but also lead to economic stagnation or even decline in the standard of living. Several published studies have shown the negative financial effect the restricted movement has on the economy, especially in the manufacturing and services sector (such as tourism or retails) [[5],[6],[19],[20]].
We describe the spread of the infectious diseases including the most recent COVID-19 illness by choosing SIS model over the SIR model and we use a nonlinear differential system with time delay as a mathematical formulation of this model.

2. Nonlinear Differential Equation with Time Delay

Chronic diseases like cancer or heart disease today receive more attention, especially in the third-world countries, when compared to infectious diseases which are still the most prominent cause of deaths worldwide. The mechanisms of transfer from an infected individual to susceptible subjects and interactions within the population are very complex and therefore very difficult to completely describe the dynamics of the infection spread without a kind of a formal structure of the mathematical model. Modelling can often be used to compare the behavior of different kinds of diseases in the same group of people, of the same kind of disease in different groups or of the same kind of disease in different time periods. Epidemiological models can also be used to forecast disease occurrence and spread. There has been a revival of interest in the study of differential equations with delays recently, which has been motivated by its many applications[[1],[4],[9],[10],[13],[14][15][16]].

We first consider a nonlinear differential equation with time delay [[18]]

$$\dot{x}(t) = p(t)x(t) - q(t)x(t)x(\tau(t)), \; t \geq t_0, \tag{1}$$

where $p \in C([t_0, \infty), R), q, \tau \in C([t_0, \infty), [0, \infty)), q(t) \not\equiv 0, \tau$ is a non-decreasing function, $\tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

Using this equation we describe the SIS model in section 3 that represents the Covid-19 virus spread. Time delay is the time period between getting infected and the appearance of symptoms.
The solution $x(t)$ of the equation (1) is considered a function $x \in C([t_1 - \tau(t_1), \infty), R)$ for some $t_1 \geq t_0$ and such that the equation (1) is valid for $t \geq t_1$. In the next section we set some of the sufficient conditions which allow the existence of a positive solution of (1) bounded by two positive functions. We will also study the conditions for existence of positive solution which tends to zero in infinity as $t \to \infty$. Next we use a nonlinear differential system with time delay as a mathematical formulation for the SIS model. It is examined by the existence of a positive solution which is bounded by two functions $k_1, k_2 \in C([t_0, \infty), (0, \infty))$ with constants $\lambda_1 > 0$, $\lambda_2 > 0$ where $0 < \lambda_1 \leq \lambda_2$.
In the next part we make use of this lemma:

Lemma 2.1. *[[4],[9]] (Schauder fixed-point theorem). Let Ω be a closed, convex and non-empty subset of the Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping, such that $S\Omega$ is relatively compact subset of X. Then S has at least one fixed point in Ω. That means that $x \in \Omega$ exists for which $Sx = x$.*

2.1. Existence of Positive Solutions

In the following section we set sufficient conditions necessary for the existence of a positive solution of equation (1).

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Theorem 2.1. *Let constants $c > 0$, $0 < \lambda_1 \le \lambda_2$ and bounded functions $k_1, k_2 \in C([t_0, \infty), (0, \infty))$ exist such that*

$$\begin{aligned}
k_1(t) &= k_2(t) = c,\ t_0 \le t \le t_1,\ \tau(t_1) \ge t_0,\\
k_1(t) &\le k_2(t),\ t \ge t_1,\\
\ln \frac{1}{c} k_2(t) &\ge \int_{t_1}^{t} [p(s) - \lambda_1 q(s) k_1(s)]\, ds, t \ge t_1,\\
\ln \frac{1}{c} k_1(t) &\le \int_{t_1}^{t} [p(s) - \lambda_2 q(s) k_2(s)]\, ds, t \ge t_1,\\
\ln \lambda_2 &\ge \int_{\tau(t)}^{t} [-p(s) + \lambda_2 q(s) k_2(s)]\, ds, \tau(t) \ge t_1,\\
\ln \lambda_1 &\le \int_{\tau(t)}^{t} [-p(s) + \lambda_1 q(s) k_1(s)]\, ds, \tau(t) \ge t_1.
\end{aligned}$$

Then the equation (1) has a positive solution bounded by functions $k_1(t)$ a $k_2(t)$.

Proof. [[1],[3]] Let $C([t_0, \infty), R)$ be a set of all continuous bounded functions with a norm $||x|| = \sup_{t \ge t_0} |x(t)|$. Then the $C([t_0, \infty), R)$ is a Banach space. We define a closed, bounded and convex subset Ω of the space $C([t_0, \infty), R)$ as follows:

$$\begin{aligned}
\Omega = \{k_1(t) &\le x(t) \le k_2(t), \quad t \ge t_1,\\
k_1(t) &= k_2(t) = x(t) = c, \quad t_0 \le t \le t_1,\\
x(\tau(t)) &\le \lambda_2 x(t), \quad t \ge t_1,\\
x(\tau(t)) &\ge \lambda_1 x(t), \quad t \ge t_1\}.
\end{aligned}$$

We define a mapping $S : \Omega \to C([t_0, \infty), R)$ this way:

$$(Sx)(t) = \begin{cases} c\ \exp\left(\int_{t_1}^{t} [p(s) - q(s) x(\tau(s))]\, ds \right), t \ge t_1,\\ c, \quad t_0 \le t \le t_1. \end{cases}$$

We will show that for an arbitrary $x \in \Omega$ we have $Sx \in \Omega$. For each $x \in \Omega$ and $t \ge t_1$ we get

$$\begin{aligned}
(Sx)(t) &= c \exp\left(\int_{t_1}^{t} [p(s) - q(s) x(\tau(s))]\, ds \right)\\
&\le c \exp\left(\int_{t_1}^{t} [p(s) - \lambda_1 q(s) x(s)]\, ds \right)\\
&\le c \exp\left(\int_{t_1}^{t} [p(s) - \lambda_1 q(s) k_1(s)]\, ds \right)\\
&\le k_2(t),
\end{aligned}$$

$$\begin{aligned}
(Sx)(t) &= c \exp\left(\int_{t_1}^{t} [p(s) - q(s) x(\tau(s))]\, ds \right)\\
&\ge c \exp\left(\int_{t_1}^{t} [p(s) - \lambda_2 q(s) x(s)]\, ds \right)\\
&\ge c \exp\left(\int_{t_1}^{t} [p(s) - \lambda_2 q(s) k_2(s)]\, ds \right)\\
&\ge k_1(t).
\end{aligned}$$

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For $t_0 \le t \le t_1$ we get $(Sx)(t) = c$, and so $(Sx)(t) \in \Omega$.
Additionally, for $\tau(t) \ge t_1$ we have

$$\begin{aligned}(Sx)(\tau(t)) &= c\,\exp\left(\int_{t_1}^{\tau(t)} [p(s) - q(s)x(\tau(s))]\,ds\right) \\ &= (Sx)(t)\,\exp\left(-\int_{\tau(t)}^{t} [p(s) - q(s)x(\tau(s))]\,ds\right).\end{aligned}$$

With regards to previously inequalities for each $x \in \Omega$ and $\tau(t) \ge t_1$ we get

$$\begin{aligned}(Sx)(\tau(t)) &= (Sx)(t)\ \exp\left(\int_{\tau(t)}^{t} [-p(s) + q(s)x(\tau(s))]\,ds\right) \\ &\le (Sx)(t)\ \exp\left(\int_{\tau(t)}^{t} [-p(s) + \lambda_2 q(s)x(s)]\,ds\right) \\ &\le (Sx)(t)\ \exp\left(\int_{\tau(t)}^{t} [-p(s) + \lambda_2 q(s)k_2(s)]\,ds\right) \\ &\le \lambda_2(Sx)(t), \\ (Sx)(\tau(t)) &\ge (Sx)(t)\ \exp\left(\int_{\tau(t)}^{t} [-p(s) + \lambda_1 q(s)k_1(s)]\,ds\right) \\ &\ge \lambda_1(Sx)(t).\end{aligned}$$

For $t_0 \le \tau(t) \le t_1$ we get $(Sx)(\tau(t)) = c$, and so $(Sx)(\tau(t)) \in \Omega$. This way we have proven that $Sx \in \Omega$ for an arbitrary $x \in \Omega$.
We will now show that S is continuous. Let $x_i = x_i(t) \in \Omega$ be such, that $x_i(t) \to x(t)$ for $i \to \infty$. Since Ω is closed then $x = x(t) \in \Omega$.
For $t \ge t_1$ we have

$$\begin{aligned}&|(Sx_i)(t) - (Sx)(t)| = \\ = &\left|c\left[\exp\left(\int_{t_1}^{t}[p(s) - q(s)x_i(\tau(s))]\,ds\right) - \exp\left(\int_{t_1}^{t}[p(s) - q(s)x(\tau(s))]\,ds\right)\right]\right|.\end{aligned}$$

Since

$$\left|\exp\left(\int_{t_1}^{t}[p(s) - q(s)x_i(\tau(s))]\,ds\right) - \exp\left(\int_{t_1}^{t}[p(s) - q(s)x(\tau(s))]\,ds\right)\right| \to 0$$

for $i \to \infty$, we conclude that

$$\lim_{i\to\infty} ||(Sx_i)(t) - (Sx)(t)|| = 0.$$

This means that S is a continuous mapping. The set of functions $\{Sx : x \in \Omega\}$ is uniformly bounded in $[t_0, \infty)$. This results from the definition of Ω. This subset is also uniformly continuous in $[t_0, \infty)$. Then according to the Arzel-Ascoli theorem the $S\Omega$ is a relatively compact subset of $C([t_0, \infty), R)$.
According to the lemma 2.1 there exists such $x_0 \in \Omega$ that $Sx_0 = x_0$. We can see that $x_0(t)$ is a positive solution of the equation (1). The proof is now complete.

Now we will present two simple examples that prove the theorem 2.1 validity.

Example 2.1. *Consider a nonlinear differential equation with delayed argument*

$$\dot{x}(t) = -\frac{1}{2}x(t) - \frac{1}{10}x(t)x(t-1),\ \ t \ge t_0.$$

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If we take

$$\begin{aligned} k_1(t) &= e^{1-t},\ k_2(t) = 1,\ t \geq t_1 = 1, \\ k_1(t) &= k_2(t) = c = 1,\ 0 \leq t \leq 1, \\ \lambda_1 &= 1, \\ \lambda_2 &= 5, \end{aligned}$$

then we have

$$\begin{aligned} \ln k_2(t) &= 0 \geq \int_1^t \left(-\frac{1}{2} - \frac{1}{10}e^{1-s}\right) ds = -\frac{t}{2} + \frac{1}{10}e^{1-t} + \frac{2}{5},\ t \geq 1, \\ \ln k_1(t) &= 1 - t \leq -\int_1^t ds = 1 - t,\ t \geq 1, \\ \ln \lambda_2 &= \ln 5 \geq \int_{t-1}^t ds = 1,\ t \geq 2, \\ \ln \lambda_1 &= 0 \leq \int_{t-1}^t \left(\frac{1}{2} + \frac{1}{10}e^{1-s}\right) ds = \frac{1}{2} + \frac{e-1}{10}e^{1-t},\ t \geq 2. \end{aligned}$$

Since all the conditions from theorem 2.1 are satisfied, the equation (2) has a positive solution.

Corollary 2.1. *We assume the existence of such constants $c, \lambda \in (0, \infty)$ and such bounded function $k \in C([t_0, \infty), (0, \infty))$, that*

$$\begin{aligned} k(t) &= c > 0, \quad t_0 \leq t \leq t_1,\ \tau(t_1) \geq t_0, \\ \ln \frac{1}{c}k(t) &= \int_{t_1}^t [p(s) - \lambda q(s)k(s)]\, ds,\ t \geq t_1, \\ \ln \lambda &= \int_{\tau(t)}^t [-p(s) + \lambda q(s)k(s)]\, ds,\ tau(t) \geq t_1. \end{aligned}$$

Then the equation (1) has a positive solution

$$x(t) = c \exp\left(\int_{t_1}^t [p(s) - \lambda q(s)k(s)]\, ds\right),\ \tau(t) \geq t_1.$$

Proof. We set $k_1(t) = k_2(t) = k(t)$, $t \geq t_0$ and apply theorem 2.1.

Example 2.2. *Consider a nonlinear differential equation with delayed argument*

$$\dot{x}(t) = (e^{2\tau} - 1)x(t) - e^t x(t)x(t-\delta), \quad t \geq t_0, \tag{2}$$

where $\delta \in (0, \infty)$.
If we take $k(t) = c$ for $-\delta \leq t \leq t_1 = 0$, $k(t) = ce^{-t}$ for $t \geq 0$ and $\lambda = e^\delta = c$, we then get

$$\begin{aligned} \ln \frac{1}{c}k(t) &= -t = \int_0^t (e^{2\delta} - 1 - e^\delta e^s ce^{-s})\, ds = -t,\ t \geq 0, \\ \ln \lambda &= \delta = \int_{t-\delta}^t (1 - e^{2\delta} + e^\delta e^s ce^{-s})\, ds = \int_{t-\delta}^t ds = \delta, \end{aligned}$$

for $t \geq \delta$.
Since all the conditions of the corollary 2.1 are satisfied, the equation (2) has a positive solution [[1]]

$$x(t) = c\ \exp\left(-\int_0^t ds\right) = ce^{-t},\ t \geq \delta.$$

Corollary 2.2. *Let's assume the existence of such constants $c > 0,\ 0 < \lambda_1 \leq \lambda_2$ and such bounded functions $k_1, k_2 \in C([t_0, \infty), (0, \infty))$, that the conditions of theorem 2.1 are satisfied and*

$$\int_{t_1}^{\infty} [p(t) - \lambda_1 q(t) k_1(t)]\, dt = -\infty.$$

Then the equation (1) has a positive solution which converges to zero.

3. SIS Model

Population is considered to be divided into disjunct classes that change in time t. The susceptible class consists of individuals who can contract the disease but were not infected yet. The infected class consists of individuals spreading the infection to others. Parts of the total population in these classes are designated $S(t)$ and $I(t)$ respectively.[[11]]

Our epidemiological model also takes into consideration the following assumptions [[6]]:

1. The observed population has a constant size N, which is sufficiently large for each of the set classes to be considered a continuous variable. We assume that births and natural deaths are in equal amount and all newborns are susceptible. Individuals from both classes are removed by death at a rate proportional to the class size with a constant μ called *daily death rate*.
2. The population is homogeneously intermixed. The number of daily contacts β is a number of average contacts of one infected individual per day. The adequate contact of the infected is defined as an interaction leading to infecting other subject who is not immune. Thus we define βS as the average number of susceptible subjects infected by the infectious individual per day and βSNI as the average number of susceptible subjects infected by the infectious class with size NI per day. The kind of direct or indirect contact necessary for passing on the virus is subject to a particular type of infection.[[12]]
3. The subjects are considered recovered and taken out of the infected class at a rate in proportion to the number of infected using proportional constant γ named *rate of daily recoveries*.
   The number removed from the infected class either by death or recovery equals $\gamma + \mu$. If the recovery does not lead to immunity then we call the model an SIS due to the fact that the individuals are transferred from the susceptible class S to the infected class I and back to the susceptible class S after recovery.[[11]]

The SIS models in general are applicable for certain bacterial diseases like meningitis, plague, sexual diseases, protozoic diseases like malaria or for virus diseases like common flu or the current SARS-COV-2 coronavirus with documented cases of a repeated infections. The medical institutions are investigating the possible reasons for repeated positive tests including virus reactivation or possible errors in testing. In this paper we consider the SIS model valid for the diseases that do not lead to gaining immunity. Natural births and deaths and model history dependent-time delay argument are part of the system. SIS model is represented by the following differential system with time delay argument and these initial conditions:

$$\begin{aligned}
\dot{S}(t) &= -\beta I(t) S(\tau(t)) + \gamma I(t) + \mu - \mu S(t), \\
\dot{I}(t) &= \beta I(t) S(\tau(t)) - \gamma I(t) - \mu I(t),\ t \geq t_0, \\
S(t) &= \varphi_1(t),\ \tau(t_0) \leq t \leq t_0, \varphi_1 \in C([\tau(t_0), t_0], [0, \infty)),\ \varphi_1(t_0) > 0,\ I(t_0) > 0,
\end{aligned}$$

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$$S(t) + I(t) = 1.$$

Let's assume that $\tau \in C([t_0, \infty), [0, \infty))$ is a nondecreasing function, $\tau(t) \leq t$ and $\lim_{t \to \infty} \tau(t) = \infty$. For $\tau(t) = t$ is the model (3) treated in [4]. It is natural to assume that the frequency of contacts β, death rate μ and recovery rate γ change in time t. Thus β, μ and γ in the model (3) will be replaced by variables $\beta(t)$, $\mu(t)$ and $\gamma(t)$.
Since $S(t)$ can be derived from $I(t)$ by using $S(t) = 1 - I(t)$, where it is sufficient to consider [[11]]

$$\begin{aligned} \dot{I}(t) &= (\beta(t) - \gamma(t) - \mu(t))I(t) - \beta(t)I(t)I(\tau(t)), \ t \geq t_0, \\ I(t) &= \varphi_2(t), \quad \tau(t_0) \leq t \leq t_0, \quad \varphi_2 \in C([\tau(t_0), t_0), [0, \infty)), \ \varphi_2(t_0) > 0. \end{aligned} \tag{3}$$

We now apply the theorem 2.1 and the corollaries 2.1, 2.2 on the equation (3) and we get the next theorem.

Theorem 3.1. *Let constants $c > 0$, $0 < \lambda_1 \leq \lambda_2$ and bounded functions $k_1, k_2 \in C([t_0, \infty), (0, \infty))$ exist such that*

$$\begin{aligned} k_1(t) &= k_2(t) = c, t_0 \leq t \leq t_1, \ \tau(t_1) \geq t_0, \\ k_1(t) &\leq k_2(t), \ t \geq t_1, \\ \ln \frac{1}{c} k_2(t) &\geq \int_{t_1}^{t} [\beta(s) - \gamma(s) - \mu(s) - \lambda_1 \beta(s) k_1(s)] \, ds, \ t \geq t_1, \\ \ln \frac{1}{c} k_1(t) &\leq \int_{t_1}^{t} [\beta(s) - \gamma(s) - \mu(s) - \lambda_2 \beta(s) k_2(s)] \, ds, \ t \geq t_1, \\ \ln \lambda_2 &\geq \int_{\tau(t)}^{t} [-\beta(s) + \gamma(s) + \mu(s) + \lambda_2 \beta(s) k_2(s)] \, ds, \ \tau(t) \geq t_1, \\ \ln \lambda_1 &\leq \int_{\tau(t)}^{t} [-\beta(s) + \gamma(s) + \mu(s) + \lambda_1 \beta(s) k_1(s)] \, ds, \ \tau(t) \geq t_1, \end{aligned}$$

then the equation (3) has a positive solution bounded by functions $k_1(t)$ and $k_2(t)$.

Corollary 3.1. *Let's assume the existence of a constant c, $\lambda \in (0, \infty)$ and a bounded function $k \in C([t_0, \infty), [0, \infty))$ such that $k(t) = c$, $t_0 \leq t \leq t_1$, $\tau(t_1) \geq t_0$,*

$$\begin{aligned} \ln \frac{1}{c} k(t) &= \int_{t_1}^{t} [\beta(s) - \gamma(s) - \mu(s) - \lambda \beta(s) k(s)] \, ds, \ t \geq t_1, \\ \ln \lambda &= \int_{\tau(t)}^{t} [-\beta(s) + \gamma(s) + \mu(s) + \lambda \beta(s) k(s)] \, ds, \ \tau(t) \geq t_1, \end{aligned}$$

then the equation (3) has a positive result.

$$I(t) = c \exp \left(\int_{t_1}^{t} [\beta(s) - \gamma(s) - \mu(s) - \lambda \beta(s) k(s)] \, ds \right), \ \tau(t) \geq t_1.$$

Proof. We set $k_1(t) = k_2(t) = k(t)$, $t \geq t_0, \lambda_1 = \lambda_2 = \lambda$, and apply theorem 3.1 and corollary 2.1, where

$$\begin{aligned} p(t) &= \beta(t) - \gamma(t) - \mu(t), \\ q(t) &= \beta(t), \ \tau(t) \geq t_1. \end{aligned}$$

Example 3.1. *Let's assume a nonlinear differential function with time delay*

$$\dot{I}(t) = (e^{b\delta} \beta(t) e^{-b(t-\delta)} - b)I(t) - \beta(t)I(t)I(t - \delta), \ t \geq 0, \tag{4}$$

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where $b, \delta \in (0, \infty),\ \beta(t) > 0.$
This equation is identic with the equation 10, if

$$b = \gamma + \mu.$$

When we set the $k(t) = c$ *for* $-\delta \leq t \leq t_1 = 0,\ k(t) = ce^{-bt}$ *for* $t \geq t_1 = 0$ *and* $c = \lambda = e^{b\delta}$, *we then get*

$$\begin{aligned} \ln \frac{1}{c} k(t) &= \int_0^t [e^{b\delta} \beta(s) e^{-b(s-\delta)} - b - e^{b\delta} \beta(s) c e^{-bs}]\, ds \\ &= \int_0^t (-b)\, ds = -bt,\ t \geq 0, \\ \ln \lambda &= \int_{t-\delta}^t b\, ds = b\delta,\ t \geq \delta. \end{aligned}$$

Since all conditions of the corollary 3.1 are satisfied, the equation (4) has a positive result

$$I(t) = ce^{-bt}, \quad t \geq \delta.$$

Corollary 3.2. *Let's assume the existence of constants* $c > 0,\ 0 < \lambda_1 \leq \lambda_2$ *and bounded functions* $k_1, k_2 \in C([t_0, \infty), (0, \infty))$ *such that the conditions of theorem 3.1 are satisfied and*

$$\int_{t_1}^{\infty} [\beta(t) - \gamma(t) - \mu(t) - \lambda_1 \beta(t) k_1(t)]\, dt = -\infty.$$

Then the equation (3) has a positive solution converging to zero.

Proof. We apply theorem 3.1 and corollary 2.2, where

$$\begin{aligned} p(t) &= \beta(t) - \gamma(t) - \mu(t), \\ q(t) &= \beta(t),\ \tau(t) \geq t_1. \end{aligned}$$

4. Conclusions

Differential delay equations successfully model the actual delay cycles in the physical nature of the model to be solved. The spread of infection becomes uncontrollable without mitigation measures such as quarantine or social distancing. It is recommended to purify the air through air purifying filters inside offices, enclosed spaces and hospitals which will help reduce the percentage of viruses; leaving the virus trapped in the filter and reduces the condition to people who work in these spaces. COVID-19 has caught the world unprepared and the fact that it is highly contagious causes unpredictable complications. We see its impact everywhere, whether it is hospital overcrowding, education or tourism. Unpredictable future development of the pandemic is even more worrying. In this precarious situation, modeling infectious diseases is a very hot topic. Many different theoretically valid mathematical models that have been experimentally validated in the past are used predict the development of the epidemic. We have considered the SIS epidemiological model in our work. We used a nonlinear differential system with time delay as a mathematical formulation of this model. We searched for the existence of a positive solution bounded by two functions k_1, k_2 with constants $\lambda_1 > 0,\ \lambda_2 > 0$ where $0 < \lambda_1 \leq \lambda_2$.

We suppose that $\beta(t), \gamma(t), \mu(t)$ are nonnegative functions for $t \geq t_1$.
If

$$\beta(t) \leq \gamma(t) + \mu(t), \quad t \geq t_1,$$

than

$$\begin{aligned}\beta(t) - \lambda_1\beta(t)k_1(t) < \beta(t) &\leq \gamma(t) + \mu(t),\\ \beta(t) - \lambda_1\beta(t)k_1(t) &< \gamma(t) + \mu(t),\\ \beta(t) - \gamma(t) - \mu(t) - \lambda_1\beta(t)k_1(t) &< 0,\ t \geq t_1.\end{aligned}$$

Thus we have

$$\int_{t_1}^{\infty} [\beta(t) - \gamma(t) - \mu(t) - \lambda_1\beta(t)k_1(t)]\, dt = -\infty,$$

or

$$\int_{t_1}^{\infty} [\beta(t) - \gamma(t) - \mu(t) - \lambda_1\beta(t)k_1(t)]\, dt = -k, \quad k \in (0, \infty).$$

So if

$$\beta(t) \leq \gamma(t) + \mu(t), \quad t \geq t_1,$$

a number of average contacts for one infected individual per day is less or equal to the number removed from the infected class by recovery or death per day, then the number of infected individuals converges to zero, or is less or equal to ce^{-k}, (Corollary 3.2).

The model we used especially in relation with the spread of COVID-19 confirms that the measures put in place to slow down the spread of COVID-19, such as covering the upper airways using a respirator in public areas, public transport, taxis or other means of transport, compulsory quarantine (domestic isolation) or contact limitation are effective.

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