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Identification of an Open Polynomial Tricompartmental Catenary System of $(\alpha + \beta)$ Order by a Linearization Method

Kemari Ayoub [1,†], Salim Khelifa [2,†] and Abita Rahmoune [3,*]

1 Laboratoire LPMA–Ecole Normale Superieure de Kouba (Vieux Kouba) - Alger; rmath205@yahoo.com
2 Laboratoire LPMA–Ecole Normale Superieure de Kouba (Vieux Kouba) - Alger; ksusthb@gmail.com
3 Department of mathematics & Informatics, Amar Telidji University, Laghouat, Algeria, Laboratory of Pure and Applied Mathematics, Laghouat, Algeria; abitamaths@gmail.com
* Correspondence: abitamaths@gmail.com
† These authors contributed equally to this work.

Abstract: To identify the exchange coefficients of a nonlinear polynomial tricompartmental general system of $(\alpha + \beta)$, we follow the procedure; firstly, the recommended solution is to introduce an adequate time $t^* > 0$ in a determinable manner. That is, after injecting quantity a into the main compartment, wait a moment for the exchange in the polynomial $(\alpha + \beta)$ order nonlinear general system to settle, and then compare this compartment to compartment 2 at t^*. Secondly, applying the Taylor formula will linearize the system and identify the exchange coefficients. Finally, we will prove that the linearization method is stable.

Keywords: linear compartmental system; nonlinear compartmental system; inverse problem; identification; numerical analysis; ordinary differential equation

MSC: 34C35; 3402; 58F40; 9202; 92B05

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1. Introduction

Nonlinear compartmental systems of polynomial type are encountered particularly in population dynamics. These systems are controlled by the following hypothesis: "The quantity passing from the compartment i to the compartment j is equal to $k_{ij}x_i^{\alpha}x_j^{\beta}$ ($\beta = 0$ if compartment j is outside environment) where $x_i(t)$ denotes the mass quantity of compartment i at time t and k_{ij} the exchange coefficient and α , β are constants characterizing the compartmental system. This so-called hypothesis of order polynomial exchange $(\alpha + \beta)$. In the present study our aim is to identify the coefficients exchange of an open polynomial tricompartmental catenary system, using linearization process. The measurements given by the experimenter will be used in conjunction with a time delay technique to adapt the results obtained for identification in linear compartmental systems.

2. Definitions and Notations

We consider the nonlinear tricompartmental system of polynomial type, namely $(\alpha + \beta)$, shown in Figure 1.

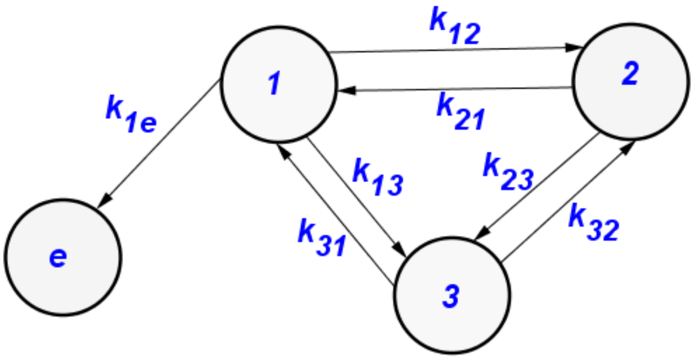

Figure 1. $\left(S_{NL}^{(P)}\right)$:Nonlinear polynomial tricompartmental system

The mass balance principle in each compartment leads to nonlinear differential equations (see [2]). The identification is done by exiting the system with and instantaneous injection of substance quantity a in the first compartment. Thus we can say that the tricompartmental catenary system is governed by the following differential system with initial condition:

$$\begin{cases} x_1'(t) = \left(k_{21}x_2^{\alpha}(t) + k_{31}x_3^{\alpha}(t)\right) x_1^{\beta}(t) - \left(k_{12}x_2^{\beta}(t) + k_{13}x_3^{\beta}(t) + k_{1e}\right) x_1^{\alpha}(t) \\ x_2'(t) = \left(k_{12}x_1^{\alpha}(t) + k_{32}x_3^{\alpha}(t)\right) x_2^{\beta}(t) - \left(k_{21}x_1^{\beta}(t) + k_{23}x_3^{\beta}(t)\right) x_2^{\alpha}(t) \\ x_3'(t) = \left(k_{13}x_1^{\alpha}(t) + k_{23}x_2^{\alpha}(t)\right) x_3^{\beta}(t) - \left(k_{31}x_1^{\beta}(t) + k_{32}x_2^{\beta}(t)\right) x_3^{\alpha}(t) \\ x_1(0) = a \\ x_2(0) = 0 \\ x_3(0) = 0 \end{cases} \tag{1}$$

we note :

$$\begin{array}{ccc} X : [0, +\infty[& \longrightarrow & \mathbb{R}^3 \\ t & \longrightarrow & X^T(t) = (x_1(t), x_2(t), x_3(t)) \end{array}$$

The state function of the tricompartmental catenary system $(S_{\mathrm{NL}}^{(P)})$, is:

$$\begin{array}{ccl} F : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x_1, x_2, x_3) & \longrightarrow & F(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3)) \end{array}$$

such that:

$$\begin{cases} f_1(x_1, x_2, x_3) = \left(k_{21}x_2^{\alpha} + k_{31}x_3^{\alpha}\right) x_1^{\beta} - \left(k_{12}x_2^{\beta} + k_{13}x_3^{\beta} + k_{1e}\right) x_1^{\alpha} \\ f_2(x_1, x_2, x_3) = \left(k_{12}x_1^{\alpha} + k_{32}x_3^{\alpha}\right) x_2^{\beta} - \left(k_{21}x_1^{\beta} + k_{23}x_3^{\beta}\right) x_2^{\alpha} \\ f_3(x_1, x_2, x_3) = \left(k_{13}x_1^{\alpha} + k_{23}x_2^{\alpha}\right) x_3^{\beta} - \left(k_{31}x_1^{\beta} + k_{32}x_2^{\beta}\right) x_3^{\alpha} \end{cases}$$

With these notations we can write the differential system (1) under the vectorial form:

$$\begin{cases} X'^T(X^T(t)) \\ \\ X(0) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \end{cases} \tag{2}$$

3. Preliminary Study

The partial derivatives of the function F being:

$$\begin{cases} \dfrac{\partial f_1}{\partial x_1}(x_1,x_2,x_3) = \left(\beta k_{21} x_2^{\alpha} + \beta k_{31} x_3^{\alpha}\right) x_1^{\beta-1} - \alpha \left(k_{12} x_2^{\beta} + k_{13} x_3^{\beta} + k_{1e}\right) x_1^{\alpha-1} \\ \dfrac{\partial f_2}{\partial x_1}(x_1,x_2,x_3) = \alpha k_{12} x_2^{\beta} x_1^{\alpha-1} - \beta k_{21} x_2^{\alpha} x_1^{\beta-1} \\ \dfrac{\partial f_3}{\partial x_1}(x_1,x_2,x_3) = \alpha k_{13} x_3^{\beta} x_1^{\alpha-1} - \beta k_{31} x_3^{\alpha} x_1^{\beta-1}, \end{cases}$$

$$\begin{cases} \dfrac{\partial f_1}{\partial x_2}(x_1,x_2,x_3) = \alpha k_{21} x_1^{\beta} x_2^{\alpha-1} - \beta k_{12} x_1^{\alpha} x_2^{\beta-1} \\ \dfrac{\partial f_2}{\partial x_2}(x_1,x_2,x_3) = \beta \left(k_{12} x_1^{\alpha} + k_{32} x_3^{\alpha}\right) x_2^{\beta-1} - \alpha \left(k_{21} x_1^{\beta} + k_{23} x_3^{\beta}\right) x_2^{\alpha-1} \\ \dfrac{\partial f_3}{\partial x_2}(x_1,x_2,x_3) = \alpha k_{23} x_3^{\beta} x_2^{\alpha-1} - \beta k_{32} x_3^{\alpha} x_2^{\beta-1}, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial f_1}{\partial x_3}(x_1,x_2,x_3) = \alpha k_{31} x_1^{\beta} x_3^{\alpha-1} - \beta k_{13} x_1^{\alpha} x_3^{\beta-1} \\ \dfrac{\partial f_2}{\partial x_3}(x_1,x_2,x_3) = \alpha k_{32} x_2^{\beta} x_3^{\alpha-1} - \beta k_{23} x_2^{\alpha} x_3^{\beta-1} \\ \dfrac{\partial f_3}{\partial x_3}(x_1,x_2,x_3) = \beta \left(k_{13} x_1^{\alpha} + k_{23} x_2^{\alpha}\right) x_3^{\beta-1} - \alpha \left(k_{31} x_1^{\beta} + k_{32} x_2^{\beta}\right) x_3^{\alpha-1}. \end{cases}$$

The function F is differentiable in all point (x_1, x_2, x_3) such that $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 \neq 0$ for all $\alpha > 0$ and all $\beta > 0$, and the Jacobian matrix is given by:

$$\left(DF_{(x_1,x_2,x_3)}\right) = \begin{pmatrix} -g_1(x_1,x_2,x_3) - \alpha k_{1e} x_1^{\alpha-1} & g_2(x_1,x_2,x_3) & 0 \\ g_1(x_1,x_2,x_3) & -g_2(x_1,x_2,x_3) - g_3(x_1,x_2,x_3) & g_4(x_1,x_2,x_3) \\ 0 & g_3(x_1,x_2,x_3) & -g_4(x_1,x_2,x_3) \end{pmatrix}$$

with:

$$\begin{cases} g_1(x_1,x_2,x_3 = \alpha k_{12} x_1^{\alpha-1} x_2^{\beta} - \beta k_{21} x_2^{\alpha} x_1^{\beta-1} \\ g_2(x_1,x_2,x_3) = \alpha k_{21} x_2^{\alpha-1} x_1^{\beta} - \beta k_{12} x_1^{\alpha} x_2^{\beta-1} \\ g_3(x_1,x_2,x_3) = \alpha k_{23} x_2^{\alpha-1} x_3^{\beta} - \beta k_{32} x_3^{\alpha} x_2^{\beta-1} \\ g_4(x_1,x_2,x_3) = \alpha k_{32} x_3^{\alpha-1} x_2^{\beta} - \beta k_{23} x_2^{\alpha} x_3^{\beta-1} \end{cases}$$

For the linearization of the system (2) we apply the Taylor formula in the neighborhood of the initial condition $(a, 0, 0)$.

Theorem 3.1. 1. *F is not differentiable in $(a, 0, 0)$ if $\alpha < 0$ or $\beta < 0$.*

2. *If $\alpha \geq 1$ and $\beta \geq 1$ F is differentiable in $(a, 0, 0)$. The Taylor formula applied in neighborhood of $(a, 0, 0)$ leads to:*

$$F^T(x_1,x_2,x_3) = F^T(a,0,0) + (DF)_{(a,0,0)} (x_1 - a, x_2, x_3)^T + \left(D^2 F\right)_{(x_{\theta 1}, x_{\theta 2}, x_{\theta 3})} (x_1 - a, x_2, x_3)^2$$

with: $(x_{\theta 1}, x_{\theta 2}, x_{\theta 3}) = (x_1 + \theta(x_1 - a), x_2 + \theta.x_2, x_3 + \theta.x_3)$ *with* $|\theta| < 1$.
For t sufficiently small, we propose to approach the differential system (1) on $[0, t_0]$ *by the following linear differential system:*

$$X'^T(a,0,0) + (DF)_{(a,0,0)}(x_1 - a, x_2, x_3)^T \tag{3}$$

with

$$(DF)_{(a,0,0)} = \begin{pmatrix} \alpha k_{1e} a^{\alpha-1} & 0 & 0 \\ 0 & -p_{21} & 0 \\ 0 & 0 & -p_{32} \end{pmatrix}$$

3.1. *System Delay and Linearization of the System 1*

For $t > t^* > 0$ the system $\left(S_{NL}^{(P)}\right)$ is then governed by the Cauchy problem:

$$\begin{cases} x_1'(t) = \left(k_{21}x_2^\alpha(t) + k_{31}x_3^\alpha(t)\right)x_1^\beta(t) - \left(k_{12}x_2^\beta(t) + k_{13}x_3^\beta(t) + k_{1e}\right)x_1^\alpha(t) \\ x_2'(t) = k_{12}x_1^\alpha(t)x_2^\beta(t) + k_{32}x_3^\alpha(t)x_2^\beta(t) - \left(k_{21}x_1^\beta(t) + k_{23}x_3^\beta(t)\right)x_2^\alpha(t) \\ x_3'(t) = k_{23}x_2^\alpha(t)x_3^\beta(t) - k_{32}x_3^\alpha(t)x_2^\beta(t) \\ x_1(t^*) = a_* \\ x_2(t^*) = b \\ x_3(t^*) = c \end{cases} \tag{4}$$

The Taylor formula on the interval $[t^*, t_0]$ is given by:

$$X'^T(a_*, b, c) + \left(DF_{(a_*,b,c)}(x_1 - a_*, x_2 - b, x_3 - c)^T\right).$$

with

$$\left(DF_{(a_*,b,c)}\right) = \begin{pmatrix} -g_1(a_*,b,c) - \alpha k_{1e} a_*^{\alpha-1} & g_2(a_*,b,c) & 0 \\ g_1(a_*,b,c) & -g_2(a_*,b,c) - g_3(a_*,b,c) & g_4(a_*,b,c) \\ 0 & g_3(a_*,b,c) & -g_4(a_*,b,c) \end{pmatrix}$$

such that:

$$\begin{cases} g_1(a_*,b,c) = \alpha k_{12} a_*^{\alpha-1} b^\beta - \beta k_{21} b^\alpha a_*^{\beta-1} \\ g_2(a_*,b,c) = \alpha k_{21} b^{\alpha-1} a_*^\beta - \beta k_{12} a_*^\alpha b^{\beta-1} \\ g_3(a_*,b,c) = \alpha k_{23} c^\beta b^{\alpha-1} - \beta k_{32} b^{\beta-1} c^\alpha \\ g_4(a_*,b,c) = \alpha k_{32} b^\beta c^{\alpha-1} - \beta k_{23} c^{\beta-1} b^\alpha \end{cases}$$

and

$$F^T(a_*,b,c) = \begin{pmatrix} k_{21}b^\alpha a_*^\beta - k_{12}a_*^\alpha b^\beta - k_{1e}a_*^\alpha \,;\, k_{12}a_*^\alpha b^\beta + k_{32}c^\alpha b^\beta - \left(k_{21}a_*^\beta + k_{23}c^\beta\right)b^\alpha; \\ k_{23}b^\alpha c^\beta - k_{32}c^\alpha b^\beta \end{pmatrix}$$

we pose:

$$\begin{cases} g_1(a_*,b,c) = g_1^* \\ g_2(a_*,b,c) = g_2^* \\ g_3(a_*,b,c) = g_3^* \\ g_4(a_*,b,c) = g_4^* \end{cases}$$

and

$$\begin{cases} f_1^* = k_{21}b^\alpha a_*^\beta - k_{12}a_*^\alpha b^\beta - k_{1e}a_*^\alpha \\ f_2^* = k_{12}a_*^\alpha b^\beta + k_{32}c^\alpha b^\beta - \left(k_{21}a_*^\beta + k_{23}c^\beta\right)b^\alpha \\ f_3^* = k_{23}b^\alpha c^\beta - k_{32}c^\alpha b^\beta \end{cases}$$

We can prove that there exists γ, δ and ω such that:

$$(DF_{((a_*,b,c))}).\begin{pmatrix} \gamma \\ \delta \\ \omega \end{pmatrix} = F(a_*,b,c).$$

Indeed

$$\begin{pmatrix} -g_1^* - g_2^* - \alpha k_{1e} a_*^{\alpha-1} & g_3^* & g_5^* \\ g_1^* & -g_4^* - g_3^* & g_6^* \\ g_2^* & g_4^* & -g_5^* - g_6^* \end{pmatrix} . \begin{pmatrix} \gamma \\ \delta \\ \omega \end{pmatrix} = \begin{pmatrix} f_1^* \\ f_2^* \\ f_3^* \end{pmatrix}$$

We apply the crammer method: we pose

$$M = \begin{pmatrix} -g_1^* - g_2^* - \alpha k_{1e} a_*^{\alpha-1} & g_3^* & g_5^* \\ g_1^* & -g_4^* - g_3^* & g_6^* \\ g_2^* & g_4^* & -g_5^* - g_6^* \end{pmatrix},$$

$$M_1 = \begin{pmatrix} f_1^* & g_3^* & g_5^* \\ f_2^* & -g_4^* - g_3^* & g_6^* \\ f_3^* & g_4^* & -g_5^* - g_6^* \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -g_1^* - g_2^* - \alpha k_{1e} a_*^{\alpha-1} & f_1^* & g_5^* \\ g_1^* & f_2^* & g_6^* \\ g_2^* & f_3^* & -g_5^* - g_6^* \end{pmatrix},$$

$$M_3 = \begin{pmatrix} -g_1^* - g_2^* - \alpha k_{1e} a_*^{\alpha-1} & g_3^* & f_1^* \\ g_1^* & -g_4^* - g_3^* & f_2^* \\ g_2^* & g_4^* & f_3^* \end{pmatrix},$$

we have $detM \neq 0$ so:

$$\begin{cases} \gamma = \dfrac{detM_1}{detM} \\ \delta = \dfrac{detM_2}{detM} \\ \omega = \dfrac{detM_3}{detM} \end{cases}$$

so we can write the differential system (4) in the equivalent form:

$$X'(t) = (DF_{(a_*,b,c)}).\begin{pmatrix} x_1(t) - a_* + \gamma \\ x_2(t) - b + \delta \\ x_3(t) - c + \omega \end{pmatrix}$$

therefore:

$$X'(t) = (DF_{(a_*,b,c)}).\begin{pmatrix} x_1(t) - a_* + \gamma \\ x_2(t) - b + \delta \\ x_3(t) - c + \omega \end{pmatrix}$$

the change of the state function of the tricompartmental system:

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} x_1(t) - a_* + \gamma \\ x_2(t) - b + \delta \\ x_3(t) - c + \omega \end{pmatrix} \tag{5}$$

permits to reduce the system (4) to the canonical form:

$$\left\{\begin{array}{l} Y'(t) = (DF)_{(a_*,d,c)} .Y(t) \\ Y^T(t^*) = (\gamma_*, \delta, \omega) \end{array}\right. \tag{6}$$

The matrix $(DF)_{(a_*,b,c)}$ has the general form of a compartmental matrix, so to this matrix we can associate "formally" the compartmental linear system that we will note $\left(S_{lin}^{(TP)}\right)$ represented by the following figure:

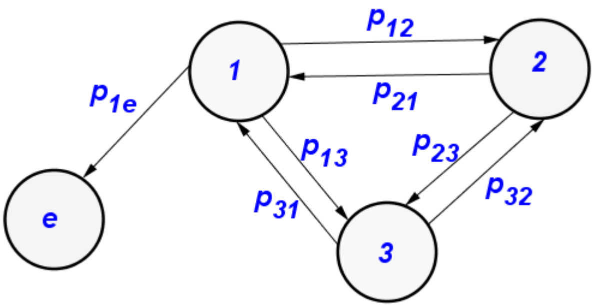

Figure 2. $\left(S_L^{(P)}\right)$:Linear Polynomial tricompartment System

with:

$$\left\{\begin{array}{l} p_{12} = \alpha k_{12} a_*^{\alpha-1} b^{\beta} - \beta k_{21} b^{\alpha} a_*^{\beta-1} \\ p_{13} = \alpha k_{13} c^{\beta} a_*^{\alpha-1} - \beta k_{31} a_*^{\beta-1} c^{\alpha} \\ p_{21} = \alpha k_{21} b^{\alpha-1} a_*^{\beta} - \beta k_{12} a_*^{\alpha} b^{\beta-1} \\ p_{23} = \alpha k_{23} c^{\beta} b^{\alpha-1} - \beta k_{32} b^{\beta-1} c^{\alpha} \\ p_{31} = \alpha k_{31} a_*^{\beta} c^{\alpha-1} - \beta k_{13} c^{\beta-1} a_*^{\alpha} \\ p_{32} = \alpha k_{32} b^{\beta} c^{\alpha-1} - \beta k_{23} c^{\beta-1} b^{\alpha} \\ p_{1e} = \alpha k_{1e} a_*^{\alpha-1} \end{array}\right.$$

Proposition 1. *The real numbers a_* , b and c such that the exchange coefficients $\{p_{ij} / \ \ i, j = 1,2,3 \ \ i \neq j\}$ is strictly than zero if and only if $\alpha > \beta > 1$.*
exist if and only if $\alpha > \beta > 1$.

Proof. Knowing that:

$$\left\{\begin{array}{l} p_{12} > 0 \\ p_{21} > 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} \alpha k_{12} a_*^{\alpha} - \beta k_{21} b^{\alpha-\beta} a_*^{\beta} > 0 \\ \alpha k_{21} b^{\alpha-\beta} a_*^{\beta} - \beta k_{12} a_*^{\alpha} > 0 \end{array}\right.$$

and

$$\left\{\begin{array}{l} p_{23} > 0 \\ p_{32} > 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} \alpha k_{23} b^{\alpha} - \beta k_{32} b^{\beta} c^{\alpha-\beta} > 0 \\ \alpha k_{32} b^{\beta} c^{\alpha-\beta} - \beta k_{23} b^{\alpha} > 0 \end{array}\right.$$

and

$$\left\{\begin{array}{l} p_{13} > 0 \\ p_{31} > 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} \alpha k_{13} a_*^{\alpha} - \beta k_{31} a_*^{\beta} c^{\alpha-\beta} > 0 \\ \alpha k_{31} c^{\alpha-\beta} a_*^{\beta} - \beta k_{13} a_*^{\alpha} > 0 \end{array}\right.$$

we pose:

$$\left\{\begin{array}{l} x = k_{12} a_*^{\alpha} \\ x' = k_{23} b^{\alpha} \\ x'' = k_{13} a_*^{\alpha} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} y = k_{21} b^{\alpha-\beta} a_*^{\beta} \\ y' = k_{32} b^{\beta} c^{\alpha-\beta} \\ y'' = k_{31} a_*^{\beta} c^{\alpha-\beta} \end{array}\right.$$

then

$$\begin{cases} p_{12} > 0 \\ p_{21} > 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha x - \beta y > 0 \\ \alpha y - \beta x > 0 \end{cases} \qquad (S)$$

$$\begin{cases} p_{23} > 0 \\ p_{32} > 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha x' - \beta y' > 0 \\ \alpha y' - \beta x' > 0 \end{cases} \qquad (S')$$

$$\begin{cases} p_{13} > 0 \\ p_{31} > 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha x'' - \beta y'' > 0 \\ \alpha y'' - \beta x'' > 0 \end{cases} \qquad (S'')$$

then
if $\alpha \leq \beta$ the set of solutions of the systems (S); (S') and (S'') is empty .
if $\alpha > \beta > 1$ the set of solutions of the systems (S); (S') and (S'') is not empty.
□

4. Calculation of the Exchange Coefficients $\{p_{ij} /\ i,j = 1,2,3\ i \neq j\}$ of the System $\left(S_{lin}^{(TP)}\right)$ and the Excretion Coefficient p_{1e}

Note the compartmental matrix of the linear model $\left(S_{lin}^{(TP)}\right)$ by:

$$A = \begin{pmatrix} -p_{12} - p_{13} - p_{1e} & p_{21} & p_{31} \\ p_{12} & -p_{21} - p_{23} & p_{32} \\ p_{13} & p_{23} & -p_{32} - p_{31} \end{pmatrix}$$

The matrix A being tridiagonal and compartmental, its eigenvalues noted $\lambda_i\ \{i \in \{1,2,3\}\}$ are real, distinct and strictly negative. The general solution of the system is written in the form

$$y_j(t) = \sum_{i=1}^{3} \beta_i^j exp\left(\lambda_i t\right) \quad \forall i \in \{1,2,3\}$$

where $\beta_i^j\ (i \in \{1,2,3\})$ is the j th column of the matrix B of the elementary masses, associated with the i compartment. The measurements made on the first and the second compartment make the minimization of the functional J introduced by Y.Cherruault [4] possible:

$$J\left(\beta_k^i, \lambda_k, 1 \leq i \leq 2, 1 \leq k \leq 3\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{2} \left(x_i(t_j) - \sum_{k=1}^{3} \beta_k^i e^{\lambda_k t_j} \right)^2 \right) \qquad (7)$$

we put:

$$\min J\left(\beta_k^i, \lambda_k, 1 \leq i \leq 2, 1 \leq k \leq 3\right) = J\left(\beta_k^{i*}, \lambda_k^*, 1 \leq i \leq 2, 1 \leq k \leq 3\right)$$

the functions $\varphi_i,\ \ i \in \{1,2,3\}$ defined by:

$$\varphi_i\left(t\right) = \exp\left(\lambda_i^* t\right) \quad , \forall t \geq t^*, \ \ \forall i \in \{1,2,3\}$$

being linearly independent we can conclude that for every integer i in $\{1,2,3\}$ we have:The matrix A being tridiagonal and compartmental, its eigenvalues noted $\lambda_i\ \{i \in \{1,2,3\}\}$ are real, distinct and strictly negative. The general solution of the system is written in the form

$$y_j(t) = \sum_{i=1}^{n} \beta_i^j exp\left(\lambda_i t\right) \quad \forall i \in \{1,2,3\}$$

where $\beta_i^j\,(i \in \{1,2,3\})$ is the j th column of the matrix B of the elementary masses, associated with the i compartment. The measurements made on the first and the second compartment make the minimization of the functional J introduced by Y.Cherruault [4] possible:

$$J\left(\beta_k^i, \lambda_k, 1 \le i \le 2, 1 \le k \le 3\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{2}\left(x_i(t_j) - \sum_{k=1}^{3}\beta_k^i e^{\lambda_k t_j}\right)^2\right) \tag{8}$$

we put:

$$\min J\left(\beta_k^i, \lambda_k, 1 \le i \le 2, 1 \le k \le 3\right) = J\left(\beta_k^{i*}, \lambda_k^*, 1 \le i \le 2, 1 \le k \le 3\right)$$

the functions $\varphi_i,\ \ i \in \{1,2,3\}$ defined by:

$$\varphi_i\left(t\right) = \exp\left(\lambda_i^* t\right) \ \ , \forall t \ge t^*,\ \ \forall i \in \{1,2,3\}$$

being linearly independent we can conclude that for every integer i in $\{1,2,3\}$ we have:

$$\lambda i^* \beta_i^{1*} = \left(-p_{12} - p_{13} - p_{1e}\right)\beta_i^{1*} + p_{21}\beta_i^{2*} + p_{31}\beta_i^3 \tag{9}$$

$$\lambda_i^* \beta_i^{2*} = p_{12}\beta_i^{1*} - \left(p_{21} - p_{23}\right)\beta_i^{2*} + p_{32}\beta_i^3 \tag{10}$$

$$\lambda_i^* \beta_i^3 = p_{13}\beta_i^{1*} + p_{23}\beta_i^{2*} - \left(p_{31} + p_{32}\right)\beta_i^3 \tag{11}$$

4.1. Calculating of Excretion Coefficient p_{1e}

We have a relationship (1) for all $t \ge 0$:

$$x_1'(t) + x_2'(t) + x_3'(t) = -k_{1e}x_1^{\alpha}(t)$$

for $t = t^*$ and $t = t_0$ such as $t* \ne t_0 \ne 0$ we have:

$$x_1'(t^*) + x_2'(t^*) + x_3'(t^*) = -k_{1e}x_1^{\alpha}(t^*) \tag{12}$$

$$x_1'(t_0) + x_2'(t_0) + x_3'(t_0) = -k_{1e}x_1^{\alpha}(t_0) \tag{13}$$

therefore:

$$\frac{x_1'(t^*) + x_2'(t^*) + x_3'(t^*)}{x_1'(t_0) + x_2'(t_0) + x_3'(t_0)} = \frac{x_1^{\alpha}(t^*)}{x_1^{\alpha}(t_0)}$$

so:

$$\left(x_1'(t^*) + x_2'(t^*)\right)x_1^{\alpha}(t_0) + x_3'(t^*)x_1^{\alpha}(t_0) = \left(x_1'(t_0) + x_2'(t_0)\right)x_1^{\alpha}(t^*) + x_3'(t_0)x_1^{\alpha}(t^*)$$

so:

$$x_3'(t^*)x_1^{\alpha}(t_0) - x_3'(t_0)x_1^{\alpha}(t^*) \ \ = \ \ \underbrace{\left(x_1'(t_0) + x_2'(t_0)\right) - \left(x_1'(t^*) + x_2'(t^*)\right)}_{\xi}$$

the relation (12) which is equivalent to:

$$\left(x_1'(t^*) + x_2'(t^*)\right)x_1^{\alpha}(t_0) + x_3'(t^*)x_1^{\alpha}(t_0) = -k_{1e}x_1^{\alpha}(t^*)x_1^{\alpha}(t_0) \tag{14}$$

and the relation (13) which is equivalent to:

$$\left(x_1'(t_0) + x_2'(t_0)\right)x_1^{\alpha}(t^*) + x_3'(t_0)x_1^{\alpha}(t^*) = -k_{1e}x_1^{\alpha}(t_0)x_1^{\alpha}(t^*) \tag{15}$$

therefore:

$$\left(x_1'(t^*)+x_2'(t^*)\right)x_1^{\alpha}(t_0)-\left(x_1'(t_0)+x_2'(t_0)\right)x_1^{\alpha}(t^*)+\xi=k_{1e}\left(x_1^{\alpha}(t_0)x_1^{\alpha}(t^*)-x_1^{\alpha}(t^*)x_1^{\alpha}(t_0)\right)$$

then

$$k_{1e}=\frac{\left(x_1'(t^*)+x_2'(t^*)\right)x_1^{\alpha}(t_0)-\left(x_1'(t_0)+x_2'(t_0)\right)x_1^{\alpha}(t^*)+\xi}{\left(x_1^{\alpha}(t_0)x_1^{\alpha}(t^*)-x_1^{\alpha}(t^*)x_1^{\alpha}(t_0)\right)} \tag{16}$$

and we have

$$p_{1e}=\alpha k_{1e}a_*^{\alpha-1} \tag{17}$$

Theorem 4.1. *adding the equations (9), (10) and (11) we get:*

$$\beta_i^3=-\beta_i^{1*}-\beta_i^{2*}-p_{1e}\frac{\beta_i^{1*}}{\lambda_i^*} \qquad \forall i\in\{1;2;3\}$$

4.2. Calculating the Exchange Coefficients $\{p_{ij}/\ \ i,j=1,2,3\ \ i\neq j\}$

The relation (9) allows to write:

$$\lambda_1^*\beta_1^{1*}=\left(-p_{12}-p_{13}-p_{1e}\right)\beta_1^{1*}+p_{21}\beta_1^{2*}+p_{31}\beta_1^{3*} \tag{18}$$

$$\lambda_2^*\beta_2^{1*}=\left(-p_{12}-p_{13}-p_{1e}\right)\beta_2^{1*}+p_{21}\beta_2^{2*}+p_{31}\beta_2^{3*} \tag{19}$$

$$\lambda_3^*\beta_3^{1*}=\left(-p_{12}-p_{13}-p_{1e}\right)\beta_3^{1*}+p_{21}\beta_3^{2*}+p_{31}\beta_3^{3*} \tag{20}$$

by multiplying both sides of the relation (18) by the number β_2^{1*} and the relation (19) by the number $-\beta_1^{1*}$ and by adding the two relations side by side, we obtain

$$\left(\lambda_1^*-\lambda_2^*\right)\beta_1^{1*}\beta_2^{1*}=p_{21}\left(\beta_1^{2*}\beta_2^{1*}-\beta_2^{2*}\beta_1^{1*}\right)+p_{31}\left(\beta_1^{3*}\beta_2^{1*}-\beta_1^{1*}\beta_2^{3*}\right) \tag{21}$$

and by multiplying both sides of the relation (19) by the number β_3^{1*} and the relation (20) by the number $-\beta_2^{1*}$ and by adding the two relations side by side, we obtain

$$\left(\lambda_2^*-\lambda_3^*\right)\beta_2^{1*}\beta_3^{1*}=p_{21}\left(\beta_2^{2*}\beta_3^{1*}-\beta_3^{2*}\beta_2^{1*}\right)+p_{31}\left(\beta_2^{3*}\beta_3^{1*}-\beta_3^{3*}\beta_2^{1*}\right) \tag{22}$$

which is equivalent to:

$$\begin{pmatrix}\left(\lambda_1^*-\lambda_2^*\right)\beta_1^{1*}\beta_2^{1*}\\ \left(\lambda_2^*-\lambda_3^*\right)\beta_2^{1*}\beta_3^{1*}\end{pmatrix}=\underbrace{\begin{pmatrix}\beta_1^{2*}\beta_2^{1*}-\beta_2^{2*}\beta_1^{1*} & \beta_1^{3*}\beta_2^{1*}-\beta_1^{1*}\beta_2^{3*}\\ \beta_2^{2*}\beta_3^{1*}-\beta_3^{2*}\beta_2^{1*} & \beta_2^{3*}\beta_3^{1*}-\beta_3^{3*}\beta_2^{1*}\end{pmatrix}}_{K}\begin{pmatrix}p_{21}\\ p_{31}\end{pmatrix} \tag{23}$$

(24)

if $detK\neq 0$ then

$$\begin{cases}p_{21}=\dfrac{detK_1}{detK}\\ p_{31}=\dfrac{detK_2}{detK}\end{cases} \tag{25}$$

such as

$$K_1=\begin{pmatrix}\left(\lambda_1^*-\lambda_2^*\right)\beta_1^{1*}\beta_2^{1*} & \beta_1^{3*}\beta_2^{1*}-\beta_1^{1*}\beta_2^{3*}\\ \left(\lambda_2^*-\lambda_3^*\right)\beta_2^{1*}\beta_3^{1*} & \beta_2^{3*}\beta_3^{1*}-\beta_3^{3*}\beta_2^{1*}\end{pmatrix} K_2=\begin{pmatrix}\beta_1^{2*}\beta_2^{1*}-\beta_2^{2*}\beta_1^{1*} & \left(\lambda_1^*-\lambda_2^*\right)\beta_1^{1*}\beta_2^{1*}\\ \beta_2^{2*}\beta_3^{1*}-\beta_3^{2*}\beta_2^{1*} & \left(\lambda_2^*-\lambda_3^*\right)\beta_2^{1*}\beta_3^{1*}\end{pmatrix}$$

and the relation (10) allows to write:

$$\lambda_1^*\beta_1^{2*} = p_{12}\beta_1^{1*} - (p_{21} - p_{23})\,\beta_1^{2*} + p_{32}\beta_1^{3*} \tag{26}$$

$$\lambda_2^*\beta_2^{2*} = p_{12}\beta_2^{1*} - (p_{21} - p_{23})\,\beta_2^{2*} + p_{32}\beta_2^{3*} \tag{27}$$

$$\lambda_3^*\beta_3^{2*} = p_{12}\beta_3^{1*} - (p_{21} - p_{23})\,\beta_3^{2*} + p_{32}\beta_3^{3*} \tag{28}$$

by multiplying both sides of the relation (26) by the number β_2^{2*} and the relation (27) by the number $-\beta_1^{2*}$ and by adding the two relations side by side, we obtain

$$(\lambda_1^* - \lambda_2^*)\,\beta_1^{2*}\beta_2^{2} = p_{12}\left(\beta_1^{*}\beta_2^{2*} - \beta_2^{1*}\beta_1^{2*}\right) + p_{32}\left(\beta_1^{3*}\beta_2^{2*} - \beta_1^{2*}\beta_2^{3*}\right) \tag{29}$$

by multiplying both sides of the relation (27) by the number β_3^{2*} and the relation (28) by the number $-\beta_2^{2*}$ and by adding the two relations side by side, we obtain

$$(\lambda_2^* - \lambda_3^*)\,\beta_2^{2*}\beta_3^{2*} = p_{12}\left(\beta_2^{1*}\beta_3^{2*} - \beta_3^{1*}\beta_2^{2*}\right) + p_{32}\left(\beta_2^{3*}\beta_3^{2*} - \beta_3^{3*}\beta_2^{2*}\right) \tag{30}$$

so

$$\begin{pmatrix} (\lambda_1^* - \lambda_2^*)\,\beta_1^{2*}\beta_2^{2} \\ (\lambda_2^* - \lambda_3^*)\,\beta_2^{2*}\beta_3^{2*} \end{pmatrix} = \underbrace{\begin{pmatrix} \beta_1^{*}\beta_2^{2*} - \beta_2^{1*}\beta_1^{2*} & \beta_1^{3*}\beta_2^{2*} - \beta_1^{2*}\beta_2^{3*} \\ \beta_2^{1*}\beta_3^{2*} - \beta_3^{1*}\beta_2^{2*} & \beta_2^{3*}\beta_3^{2*} - \beta_3^{3*}\beta_2^{2*} \end{pmatrix}}_{K'} \begin{pmatrix} p_{12} \\ p_{32} \end{pmatrix}$$

if $detK' \neq 0$ then

$$\begin{cases} p_{12} = \dfrac{detK_1'}{detK} \\ p_{32} = \dfrac{detK_2'}{detK} \end{cases}$$

such as

$$K_1' = \begin{pmatrix} (\lambda_1^* - \lambda_2^*)\,\beta_1^{2*}\beta_2^{2} & \beta_1^{3*}\beta_2^{2*} - \beta_1^{2*}\beta_2^{3*} \\ (\lambda_2^* - \lambda_3^*)\,\beta_2^{2*}\beta_3^{2*} & \beta_2^{3*}\beta_3^{2*} - \beta_3^{3*}\beta_2^{2*} \end{pmatrix}$$

and

$$K_2' = \begin{pmatrix} \beta_1^{*}\beta_2^{2*} - \beta_2^{1*}\beta_1^{2*} & (\lambda_1^* - \lambda_2^*)\,\beta_1^{2*}\beta_2^{2} \\ \beta_2^{1*}\beta_3^{2*} - \beta_3^{1*}\beta_2^{2*} & (\lambda_2^* - \lambda_3^*)\,\beta_2^{2*}\beta_3^{2*} \end{pmatrix}$$

and the relation (11) allows to write:

$$\lambda_1^*\beta_1^{3*} = p_{13}\beta_1^{1*} + p_{23}\beta_1^{2*} - (p_{31} + p_{32})\,\beta_1^{3*} \tag{31}$$

$$\lambda_2^*\beta_2^{3*} = p_{13}\beta_2^{1*} + p_{23}\beta_2^{2*} - (p_{31} + p_{32})\,\beta_2^{3*} \tag{32}$$

$$\lambda_3^*\beta_3^{3*} = p_{13}\beta_3^{1*} + p_{23}\beta_3^{2*} - (p_{31} + p_{32})\,\beta_3^{3*} \tag{33}$$

by multiplying both sides of the relation (31) by the number β_2^{3*} and the relation (32) by the number $-\beta_1^{3*}$ and by adding the two relations side by side, we obtain

$$(\lambda_1^* - \lambda_2^*)\,\beta_1^{3*}\beta_2^{3*} = p_{13}\left(\beta_1^{1*}\beta_2^{2*} - \beta_2^{1*}\beta_1^{3*}\right) + p_{23}\left(\beta_1^{2*}\beta_2^{3*} - \beta_2^{2*}\beta_2^{3*}\right) \tag{34}$$

and by multiplying both sides of the relation (32) by the number β_3^{3*} and the relation (33) by the number $-\beta_2^{3*}$ and by adding the two relations side by side, we obtain

$$\left(\lambda_2^* - \lambda_3^*\right)\beta_2^{3*}\beta_3^{3*} = p_{13}\left(\beta_2^{1*}\beta_3^{3*} - \beta_3^{1}\beta_2^{3*}\right) + p_{23}\left(\beta_2^{2*}\beta_3^{3*} - \beta_3^{2*}\beta_2^{3*}\right) \tag{35}$$

so

$$\begin{pmatrix} \left(\lambda_1^* - \lambda_2^*\right)\beta_1^{3*}\beta_2^{3*} \\ \\ \left(\lambda_2^* - \lambda_3^*\right)\beta_2^{3*}\beta_3^{3*} \end{pmatrix} = \underbrace{\begin{pmatrix} \beta_1^{1*}\beta_2^{2*} - \beta_2^{1*}\beta_1^{3*} & \beta_1^{2*}\beta_2^{3*} - \beta_2^{2*}\beta_2^{3*} \\ \\ \beta_2^{1*}\beta_3^{3*} - \beta_3^{1}\beta_2^{3*} & \beta_2^{2*}\beta_3^{3*} - \beta_3^{2*}\beta_2^{3*} \end{pmatrix}}_{K''} \begin{pmatrix} p_{13} \\ \\ p_{32} \end{pmatrix}$$

if $detK'' \neq 0$ then

$$\begin{cases} p_{13} = \dfrac{detK_1''}{detK''} \\ p_{23} = \dfrac{detK_2''}{detK''} \end{cases}$$

such as

$$K_1'' = \begin{pmatrix} \left(\lambda_1^* - \lambda_2^*\right)\beta_1^{3*}\beta_2^{3*} & \beta_1^{2*}\beta_2^{3*} - \beta_2^{2*}\beta_2^{3*} \\ \\ \left(\lambda_2^* - \lambda_3^*\right)\beta_2^{3*}\beta_3^{3*} & \beta_2^{2*}\beta_3^{3*} - \beta_3^{2*}\beta_2^{3*} \end{pmatrix}$$

and

$$K_2'' = \begin{pmatrix} \beta_1^{1*}\beta_2^{2*} - \beta_2^{1*}\beta_1^{3*} & \left(\lambda_1^* - \lambda_2^*\right)\beta_1^{3*}\beta_2^{3*} \\ \\ \beta_2^{1*}\beta_3^{3*} - \beta_3^{1}\beta_2^{3*} & \left(\lambda_2^* - \lambda_3^*\right)\beta_2^{3*}\beta_3^{3*} \end{pmatrix}$$

Let's put:

$$\begin{cases} p_{1e} = \eta_1^* \\ p_{12} = \eta_2^* \\ p_{21} = \eta_3^* \\ p_{23} = \eta_4^* \\ p_{32} = \eta_5^* \\ p_{13} = \eta_6^* \\ p_{31} = \eta_7^* \end{cases}$$

4.3. Calculating the Initial Condition c

The initial condition c is determined by:

$$c = \left(\eta_7^* + \eta_5^*\right)^{-1}\left((\alpha+\beta)\left(\lambda_1^*\beta_1^{3*}e^{\lambda_1^* t^*} + \lambda_2^*\beta_2^{3*}e^{\lambda_2^* t^*} + \lambda_3^*\beta_3^{3*}e^{\lambda_3^* t^*}\right) + a_*\eta_6^* + b\eta_4^*\right)$$

Proof. We have

$$x_3'(t^*) = k_{13}a_*^{\alpha}c^{\beta} + k_{23}b^{\alpha}c^{\beta} - \left(k_{31}c^{\alpha}b^{\beta} + k_{32}c^{\alpha}a_*^{\beta}\right)$$

and we have

$$y_3'(t^*) = \lambda_1^*\beta_1^{3*}e^{\lambda_1^*} + \lambda_2^*\beta_2^{3*}e^{\lambda_2^* t^*} + \lambda_3^*\beta_2^{3*}e^{\lambda_3^* t^*}$$

and consequently

$$k_{13}a_*^{\alpha}c^{\beta} + k_{23}b^{\alpha}c^{\beta} - \left(k_{31}c^{\alpha}b^{\beta} + k_{32}c^{\alpha}a_*^{\beta}\right) = \lambda_1^*\beta_1^{3*}e^{\lambda_1^*} + \lambda_2^*\beta_2^{3*}e^{\lambda_2^* t^*} + \lambda_3^*\beta_2^{3*}e^{\lambda_3^* t^*}$$

furthermore we have:

$$\begin{cases} \eta_4^* = \alpha k_{23} c^{\beta} b^{\alpha-1} - \beta k_{32} b^{\beta-1} c^{\alpha} \\ \eta_5^* = \alpha k_{32} b^{\beta} c^{\alpha-1} - \beta k_{23} c^{\beta-1} b^{\alpha} \\ \eta_6^* = \alpha k_{13} c^{\beta} a_*^{\alpha-1} - \beta k_{31} a_*^{\beta-1} c^{\alpha} \\ \eta_7^* = \alpha k_{31} a_*^{\beta} c^{\alpha-1} - \beta k_{13} c^{\beta-1} a_*^{\alpha} \end{cases}$$

which equivalent:

$$\begin{cases} beta_4^* = \alpha k_{23} c^{\beta} b^{\alpha} - \beta k_{32} b^{\beta} c^{\alpha} \\ c\eta_5^* = \alpha k_{32} b^{\beta} c^{\alpha} - \beta k_{23} c^{\beta} b^{\alpha} \\ a_* \eta_6^* = \alpha k_{13} c^{\beta} a_*^{\alpha} - \beta k_{31} a_*^{\beta} c^{\alpha} \\ c\eta_7^* = \alpha k_{31} a_*^{\beta} c^{\alpha} - \beta k_{13} c^{\beta} a_*^{\alpha} \end{cases}$$

and consequently:

$$\begin{cases} b\eta_4^* - c\eta_5^* = (\alpha+\beta)\left(k_{23} c^{\beta} b^{\alpha} - k_{32} c^{\alpha} b^{\beta}\right) \\ a_* \eta_6^* - c\eta_7^* = (\alpha+\beta)\left(k_{13} c^{\beta} a_*^{\alpha} - k_{31} a_*^{\beta} c^{\alpha}\right) \end{cases}$$

which implies:

$$b\eta_4^* + a_* \eta_6^* - c\left(\eta_7^* + \eta_5^*\right) = (\alpha+\beta)\left(k_{23} c^{\beta} b^{\alpha} + k_{13} c^{\beta} a_*^{\alpha} - \left(k_{31} a_*^{\beta} c^{\alpha} + k_{32} c^{\alpha} b^{\beta}\right)\right)$$

so

$$c = \left(\eta_7^* + \eta_5^*\right)^{-1}\left((\alpha+\beta)\left(\underbrace{\left(k_{31} a_*^{\beta} c^{\alpha} + k_{32} c^{\alpha} b^{\beta}\right) - \left(k_{23} c^{\beta} b^{\alpha} + k_{13} c^{\beta} a_*^{\alpha}\right)}_{y_3'(t^*)}\right) + b\eta_4^* + a_* \eta_6^*\right)$$

finally, we get:

$$c = \left(\eta_7^* + \eta_5^*\right)^{-1}\left((\alpha+\beta)\left(\lambda_1^* \beta_1^{3*} e^{\lambda_1^* t^*} + \lambda_2^* \beta_2^{3*} e^{\lambda_2^* t^*} + \lambda_3^* \beta_3^{3*} e^{\lambda_3^* t^*}\right) + a_* \eta_6^* + b\eta_4^*\right)$$

□

4.4. Calculating the Exchange Coefficients $k_{ij} \quad \{i,j = 1,2,3\}$

Proposition 2. *Let* $p^* = \begin{pmatrix} \beta_1^{1*} & \beta_1^{2*} & \beta_1^{3*} \\ \beta_2^{1*} & \beta_2^{2*} & \beta_2^{3*} \\ \beta_3^{1*} & \beta_3^{2*} & \beta_3^{3*} \end{pmatrix}$ *the associated partial matrix of measures to the system* $\left(S_{lin}^{(p)}\right)$ *identified by (6) if* $\alpha > \beta > 1$ *then the system nonlinear* $\left(S_{NL}^{(p)}\right)$ *is additionally identified*

k_{1e} *has already calculated*

and

$$\begin{cases} k_{12} = \dfrac{detQ_1}{detQ} \\ \\ k_{21} = \dfrac{detQ_2}{detQ} \end{cases} \qquad \begin{cases} k_{13} = \dfrac{detQ_1'}{detQ'} \\ \\ k_{31} = \dfrac{detQ_2'}{detQ'} \end{cases} \qquad \begin{cases} k_{23} = \dfrac{detQ_1''}{detQ''} \\ \\ k_{21} = \dfrac{detQ_2''}{detQ''} \end{cases}$$

such as

$$Q = \begin{pmatrix} \alpha a_*^{\alpha-1} b^{\beta} & -\beta b^{\alpha} a_*^{\beta-1} \\ -\beta a_*^{\alpha} b^{\beta-1} & \alpha a_*^{\beta} b_*^{\alpha-1} \end{pmatrix} Q_1 = \begin{pmatrix} \eta_2^* & -\beta b^{\alpha} a_*^{\beta-1} \\ \eta_3^* & \alpha a_*^{\beta} b_*^{\alpha-1} \end{pmatrix} Q = \begin{pmatrix} \alpha a_*^{\alpha-1} b^{\beta} & \eta_2^* \\ -\beta a_*^{\alpha} b^{\beta-1} & \eta_3^* \end{pmatrix}$$

$$Q' = \begin{pmatrix} \alpha c^{\beta} a_*^{\alpha-1} & -\beta a_*^{\beta-1} c^{\alpha} \\ -\beta c^{\beta-1} a_*^{\alpha} & \alpha a_*^{\beta} c^{\alpha-1} \end{pmatrix} Q'_1 = \begin{pmatrix} \eta_6^* & -\beta a_*^{\beta-1} c^{\alpha} \\ \eta_7^* & \alpha a_*^{\beta} c^{\alpha-1} \end{pmatrix} Q'_2 = \begin{pmatrix} \alpha c^{\beta} a_*^{\alpha-1} & \eta_6^* \\ -\beta c^{\beta-1} a_*^{\alpha} & \eta_7^* \end{pmatrix}$$
$$Q'' = \begin{pmatrix} \alpha c^{\beta} b^{\alpha-1} & -\beta b^{\beta-1} c^{\alpha} \\ -\beta c^{\beta-1} b^{\alpha} & \alpha b^{\beta} c^{\alpha-1} \end{pmatrix} Q''_1 = \begin{pmatrix} \eta_4^* & -\beta b^{\beta-1} c^{\alpha} \\ \eta_5^* & \alpha b^{\beta} c^{\alpha-1} \end{pmatrix} Q''_2 = \begin{pmatrix} \alpha c^{\beta} b^{\alpha-1} & \eta_4^* \\ -\beta c^{\beta-1} b^{\alpha} & \eta_5^* \end{pmatrix}$$

Proof. We have $\alpha > \beta$ which implies that $detQ$, $detQ'$ and $detQ''$ do not equal zero and we have

$$\begin{cases} \eta_2^* = \alpha k_{12} a_*^{\alpha-1} b^{\beta} - \beta k_{21} b^{\alpha} a_*^{\beta-1} \\ \eta_3^* = \alpha k_{21} b^{\alpha-1} a_*^{\beta} - \beta k_{12} a_*^{\alpha} b^{\beta-1} \\ \eta_4^* = \alpha k_{23} c^{\beta} b^{\alpha-1} - \beta k_{32} b^{\beta-1} c^{\alpha} \\ \eta_5^* = \alpha k_{32} b^{\beta} c^{\alpha-1} - \beta k_{23} c^{\beta-1} b^{\alpha} \\ \eta_6^* = \alpha k_{13} c^{\beta} a_*^{\alpha-1} - \beta k_{31} a_*^{\beta-1} c^{\alpha} \\ \eta_7^* = \alpha k_{31} a_*^{\beta} c^{\alpha-1} - \beta k_{13} c^{\beta-1} a_*^{\alpha} \end{cases}$$

we apply Crammer's method, we easily find the solutions □

5. Stability of the Linearization Method

For all $\{i, j = 1;2;3 \ \ with \ \ j > i\}$ we set $\overline{p}_{ij} = \vartheta_i^{j*}$ and $\overline{p}_{ji} = \vartheta_j^{i*}$ such that $\overline{p}_{ij} \ \ \{i, j = 1;2;3 \ \ i \neq j\}$ the exchange coefficients of a real linear compartmental system and note that the exchange coefficients of a real nonlinear compartmental system by $\overline{k}_{ij} \ \ \{i, j = 1;2;3\}$
And note that ε_i^j the errors made on the calculation of $p_{ij} \ \ \{i; j = 1;2;3 \ \ j > i\}$ and ε_j^i the errors made on the calculation of $p_{ji} \ \ \{i; j = 1;2;3 \ \ j > i\}$
and we set $a_* = \xi_1$, $b = \xi_2$ and $c = \xi_3$.

Proposition 3. *We can approximate the exchange coefficients of nonlinear polynomial system by:*

$$\begin{cases} \overline{k}_{ij} = \dfrac{\alpha \xi_i \vartheta_i^* + \beta \xi_j \vartheta_j^*}{(\alpha^2 - \beta^2) \xi_j^{\beta} \xi_i^{\alpha}} \\ \overline{k}_{ji} = \dfrac{\beta \xi_i \vartheta_i^* + \xi_j \alpha \vartheta_j^*}{(\alpha^2 - \beta^2) \xi_j^{\alpha} \xi_i^{\beta}} \end{cases} \tag{36}$$

which represent the respective approximations of the exchange coefficients $k_{ij} \quad \{i, j = 1, 2, 3; i \neq j\}$

$$\begin{cases} |k_{ij} - \overline{k}_{ij}| \leq \dfrac{(\alpha \xi_i + \xi_j \beta) \ max \left(\varepsilon_i^j, \varepsilon_j^i \right)}{(\alpha^2 - \beta^2) \xi_j^{\beta} \xi_i^{\alpha}} \\ |k_{ji} - \overline{k}_{ji}| \leq \dfrac{(\beta \xi_i + \xi_j \alpha) \ max \left(\varepsilon_i^j, \varepsilon_j^i \right)}{(\alpha^2 - \beta^2) \xi_j^{\alpha} \xi_i^{\beta}} \end{cases} \tag{37}$$

Proof. ϑ_i^{j*} being an approximation of p_{ij} then there is $\varepsilon_i^{j'}$ ($|\varepsilon_i^{j'}| \leq \varepsilon_i^j$) such that:

$$\vartheta_i^* + \varepsilon_i^{j'} = \alpha k_{ij} \xi_j^{\beta} \xi_i^{\alpha-1} - \beta k_{ji} \xi_j^{\alpha} \xi_i^{\beta-1} \tag{38}$$

and consequently:

$$\alpha \xi_j^{-1} \xi_i (\vartheta_i^* + \varepsilon_i^{j'}) = \alpha^2 k_{ij} \xi_j^{\beta-1} \xi_i^{\alpha} - \alpha \beta k_{ji} \xi_j^{\alpha-1} \xi_i^{\beta} \tag{39}$$

□

Proposition 4. *ϑ_j^* being an approximation of p_{ji} then there exists $\varepsilon_j^{i'}$ ($|\varepsilon_j^{i'}| \leq \varepsilon_j^{i}$) such that:*

$$\vartheta_j^* + \varepsilon_j^{i'} = \alpha k_{ji}\xi_j^{\alpha-1}\xi_i^{\beta} - \beta k_{ij}\xi_j^{\beta-1}\xi_i^{\alpha} \tag{40}$$

and consequently

$$\beta(\vartheta_j^* + \varepsilon_j^{i'}) = \alpha\beta k_{ji}\xi_j^{\alpha-1}\xi_i^{\beta} - \beta^2 k_{ij}\xi_j^{\beta-1}\xi_i^{\alpha} \tag{41}$$

by adding the two relations (39) and (41) side by side, we obtain

$$\alpha\xi_j^{-1}\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \beta(\vartheta_j^* + \varepsilon_j^{i'}) = \alpha^2 k_{ij}\xi_j^{\beta-1}\xi_i^{\alpha} - \beta^2 k_{ij}\xi_j^{\beta-1}\xi_i^{\alpha}$$

Which is equivalent to:

$$\alpha\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \beta\xi_j(\vartheta_j^* + \varepsilon_j^{i'}) = k_{ij}(\alpha^2 - \beta^2)\xi_j^{\beta}\xi_i^{\alpha}$$

so

$$k_{ij} = \frac{\alpha\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \beta\xi_j(\vartheta_j^* + \varepsilon_j^{i'})}{(\alpha^2 - \beta^2)\xi_j^{\beta}\xi_i^{\alpha}}$$

the relation (38) which is equivalent to:

$$\beta\xi_i\xi_j^{-1}(\vartheta_i^* + \varepsilon_i^{j'}) = \beta\alpha k_{ij}\xi_j^{\beta-1}\xi_i^{\alpha} - \beta^2 k_{ji}\xi_j^{\alpha-1}\xi_i^{\beta} \tag{42}$$

the relation (40) which is equivalent to:

$$\alpha(\vartheta_j^* + \varepsilon_j^{i'}) = \alpha^2 k_{ji}\xi_j^{\alpha-1}\xi_i^{\beta} - \beta\alpha k_{ij}\xi_j^{\beta-1}\xi_i^{\alpha} \tag{43}$$

and by adding the two relations (42) and (43) side by side, we obtain

$$\beta\xi_i\xi_j^{-1}(\vartheta_i^* + \varepsilon_i^{j'}) + \alpha(\vartheta_j^* + \varepsilon_j^{i'}) = \alpha^2 k_{ji}\xi_j^{\alpha-1}\xi_i^{\beta} - \beta^2 k_{ji}\xi_j^{\alpha-1}\xi_i^{\beta}$$

which is equivalent to:

$$\begin{aligned}\beta\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \xi_j\alpha(\vartheta_j^* + \varepsilon_j^{i'}) &= \alpha^2 k_{ji}\xi_j^{\alpha}\xi_i^{\beta} - \beta^2 k_{ji}\xi_j^{\alpha}\xi_i^{\beta} \\ &= k_{ji}(\alpha^2 - \beta^2)\xi_j^{\alpha}\xi_i^{\beta}\end{aligned}$$

as a result

$$k_{ji} = \frac{\beta\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \xi_j\alpha(\vartheta_j^* + \varepsilon_j^{i'})}{(\alpha^2 - \beta^2)\xi_j^{\alpha}\xi_i^{\beta}}$$

so

$$\bar{k}_{ij} = \lim_{(\varepsilon_i^{j'},\varepsilon_j^{i'})\longrightarrow(0,0)} \frac{\alpha\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \beta\xi_j(\vartheta_j^* + \varepsilon_j^{i'})}{(\alpha^2 - \beta^2)\xi_j^{\beta}\xi_i^{\alpha}} = \frac{\alpha\xi_i\vartheta_i^* + \beta\xi_j\vartheta_j^*}{(\alpha^2 - \beta^2)\xi_j^{\beta}\xi_i^{\alpha}}.$$

Proposition 5.

$$\bar{k}_{ji} = \lim_{(\varepsilon_i^{j'},\varepsilon_j^{i'})\longrightarrow(0,0)} \frac{\beta\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \xi_j\alpha(\vartheta_j^* + \varepsilon_j^{i'})}{(\alpha^2 - \beta^2)\xi_j^{\alpha}\xi_i^{\beta}} = \frac{\beta\xi_i\vartheta_i^* + \xi_j\alpha\vartheta_j^*}{(\alpha^2 - \beta^2)\xi_j^{\alpha}\xi_i^{\beta}}$$

therefore

$$|k_{ij} - \bar{k}_{ij}| = \left| \frac{\alpha\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \beta\xi_j(\vartheta_j^* + \varepsilon_j^{i'})}{(\alpha^2 - \beta^2)\xi_j^{\beta}\xi_i^{\alpha}} - \frac{\alpha\xi_i\vartheta_i^* + \xi_j\beta\vartheta_j^*}{(\alpha^2 - \beta^2)\xi_j^{\beta}\xi_i^{\alpha}} \right|$$

$$= \left| \frac{\alpha\xi_i\varepsilon_i^{j'} + \xi_j\beta\varepsilon_j^{i'}}{(\alpha^2-\beta^2)\xi_j^{\beta}\xi_i^{\alpha}} \right| \leq \left| \frac{(\alpha\xi_i + \xi_j\beta)\, max\left(\varepsilon_i^{j'}, \varepsilon_j^{i'}\right)}{(\alpha^2-\beta^2)\xi_j^{\beta}\xi_i^{\alpha}} \right|$$

$$\leq \frac{(\alpha\xi_i + \xi_j\beta)\, max\left(\varepsilon_i^{j}, \varepsilon_j^{i}\right)}{(\alpha^2-\beta^2)\xi_j^{\beta}\xi_i^{\alpha}}$$

and

$$|k_{ji} - \overline{k}_{ji}| = \left| \frac{\beta\xi_i(\vartheta_i^* + \varepsilon_i^{j'}) + \alpha\xi_j(\vartheta_j^* + \varepsilon_j^{i'})}{(\alpha^2-\beta^2)\xi_j^{\alpha}\xi_i^{\beta}} - \frac{\beta\xi_i\vartheta_i^* + \xi_j\alpha\vartheta_j^*}{(\alpha^2-\beta^2)\xi_j^{\alpha}\xi_i^{\beta}} \right|$$

$$= \left| \frac{\beta\xi_i\varepsilon_i^{j'} + \xi_j\alpha\varepsilon_j^{i'}}{(\alpha^2-\beta^2)\xi_j^{\alpha}\xi_i^{\beta}} \right| \leq \left| \frac{(\beta\xi_i + \xi_j\alpha)\, max\left(\varepsilon_i^{j'}, \varepsilon_i^{i'}\right)}{(\alpha^2-\beta^2)\xi_j^{\alpha}\xi_i^{\beta}} \right|$$

$$\leq \frac{(\beta\xi_i + \xi_j\alpha)\, max\left(\varepsilon_i^{j}, \varepsilon_j^{i}\right)}{(\alpha^2-\beta^2)\xi_j^{\alpha}\xi_i^{\beta}}$$

Theorem 5.1. *see [7] We have:*

$$p_{1e} = \overline{p}_{1e} \tag{44}$$

so

$$k_{1e} = \frac{\overline{p}_{1e}}{\alpha a_*^{\alpha-1}}.$$

Conclusion 6. *The linear model associated to the non linear polynomial tricompartmental general system of $(\alpha+\beta)$ order involves four important difficulties:*

1. *The initial condition at time $t = 0$ does not permit to give a complete information about the model $(S_{\mathrm{NL}}^{(P)})$. A temporization t^* is introduced to suppress this difficulty.*
2. *If this temporization is not modulated, the linear model is not necessarily real. We have shown that the measures done on the compartment 1 and on the compartment 2 permit to choose one measure at instant $t_{i1} = t^*$ such that we can develop a linearization method.*
3. *The nonhomogeneous condition $x_3(t^*) = c$ being unknown is identified form measures done on compartment 1 and on compartment 2.*
4. *The linearization method is stable.*

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