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Description of the Electron in the Electromagnetic Field: The Dirac Type Equation and the Equation for the Wave Function in Spinor Coordinate Space

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Abstract: Physical processes are usually described using four-dimensional vector quantities - coordinate vector, momentum vector, current vector. But at the fundamental level they are characterized by spinors - coordinate spinors, impulse spinors, spinor wave functions. The propagation of fields and their interaction takes place at the spinor level, and since each spinor uniquely corresponds to a certain vector, the results of physical processes appear before us in vector form. For example, the relativistic Schrödinger equation and the Dirac equation are formulated by means of coordinate vectors, momentum vectors and quantum operators corresponding to them. In the Schrödinger equation the wave function is represented by a single complex quantity, in the Dirac equation a step forward is taken and the wave function is a spinor with complex components, but still coordinates and momentum are vectors. For a closed description of nature using only spinor quantities, it is necessary to have an equation similar to the Dirac equation in which momentum, coordinates and operators are spinors. It is such an equation that is presented in this paper. Using the example of the interaction between an electron and an electromagnetic field, we can see that the spinor equation contains more detailed information about the interaction than the vector equations. This is not new for quantum mechanics, since it describes interactions using complex wave functions, which cannot be observed directly, and only when measured goes to probabilities in the form of squares of the moduli of the wave functions. In the same way spinor quantities are not observable, but they completely determine observable vectors. In Section 2 of the paper, we analyze the quadratic form for an arbitrary four-component complex vector based on Pauli matrices. The form is invariant with respect to Lorentz transformations including any rotations and boosts. The invariance of the form allows us to construct on its basis an equation for a free particle combining the properties of the relativistic wave equation and the Dirac equation. For an electron in the presence of an electromagnetic potential it is shown that taking into account the commutation relations between the momentum and coordinate components allows us to obtain from this equation the known results describing the interactions of the electron spin with the electric and magnetic field. In section 3 of the paper this quadratic form is expressed through momentum spinors, which makes it possible to obtain an equation for the spinor wave function in spinor coordinate space by replacing the momentum spinor components by partial derivative operators on the corresponding coordinate spinor component. At the end of the paper the question on a possibility of second quantization of the electron field in the spinor coordinate space is touch upon.

Keywords: relativistic wave equation; Dirac equation; Pauli matrices; Schrödinger equation; second quantization

1. Introduction

Nowadays, the interest to study applications of the Dirac equation to different situations and to find out the conditions of its generalization is not weakening. In particular, in [1] new versions of an extended Dirac equation and the associated Clifford algebra are presented. In [2] a study of the Schrödinger-Dirac covariant equation in the presence of gravity, where the non-commuting gamma

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matrices become space-time-dependent, is carried out. In [3] an idea is discussed that the visible properties of the electron, including rest mass and magnetic moment, are determined by a massless charge spinning at light speed within a Compton domain. In [4] some aspects of conformal rescaling in detail are explored and the role of the "quantum" potential is discussed as a natural consequence of non-inertial motion and is not exclusive to the quantum domain. Author establishes the fundamental importance of conformal symmetry, in which rescaling of the rest mass plays a vital role. Thus, the basis for a radically new theory of quantum phenomena based on the process of mass-energy flow is proposed. In [5] author have derived the covariant fourth-order/one-function equivalent of the Dirac equation for the general case of an arbitrary set of γ-matrices.

Supporting these search aspirations, in our work we propose a deeper understanding of the Dirac equation with an emphasis on the direct use of the principles of symmetry and invariance to Lorentz transformations. For the first time we present a formulation of the Dirac and Schrödinger equations in spinor coordinate space.

2. Generalized Dirac type equation

Let us introduce notations, which will be used further on. The speed of light and the reduced Planck constant will be considered as unity.

Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrices constructed from Pauli matrices

$$S_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} \quad S_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad S_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad S_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

A vector of matrices

$$\mathbf{S}^{\mathbf{T}} \equiv (S_1, S_2, S_3)$$

A set of arbitrary complex numbers and a vector of its three components

$$\mathfrak{X}^{\mathbf{T}} \equiv (X_0, X_1, X_2, X_3)$$

$$\mathbf{X}^{\mathbf{T}} \equiv (X_1, X_2, X_3)$$

Let us define a 2×2 matrix of Lorentz transformations given by the set of real rotation angles $(\alpha_1, \alpha_2, \alpha_3)$ and boosts $(\beta_1, \beta_2, \beta_3)$

$$n = exp\left(-\frac{1}{2}i\alpha_1\sigma_1\right) exp\left(\frac{1}{2}\beta_1\sigma_1\right) exp\left(-\frac{1}{2}i\alpha_2\sigma_2\right) exp\left(\frac{1}{2}\beta_2\sigma_2\right) exp\left(-\frac{1}{2}i\alpha_3\sigma_3\right) exp\left(\frac{1}{2}\beta_3\sigma_3\right)$$

and a similar 4×4 transformation matrix

$$N = exp\left(-\frac{1}{2}i\alpha_1 S_1\right) exp\left(\frac{1}{2}\beta_1 S_1\right) exp\left(-\frac{1}{2}i\alpha_2 S_2\right) exp\left(\frac{1}{2}\beta_2 S_2\right) exp\left(-\frac{1}{2}i\alpha_3 S_3\right) exp\left(\frac{1}{2}\beta_3 S_3\right)$$

We also define a 4×4 matrix of Lorentz transformations Λ, where μ and ν take values 0,1,2,3

$$\Lambda^{\mu}_{\nu} = \frac{1}{2}\mathrm{Tr}[\sigma_\mu n \sigma_\nu n^\dagger]$$

which can also be written explicitly using the 4×4 matrices of turn generators (R_1, R_2, R_3) and boosts (K_1, K_2, K_3)

$$\Lambda = exp(\alpha_1 R_1) exp(\beta_1 K_1) exp(\alpha_2 R_2) exp(\beta_2 K_2) exp(\alpha_3 R_3) exp(\beta_3 K_3)$$

Let's define a 4×4 matrix

$$\begin{aligned} M^2 &= (S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3) \\ &= (S_0X_0 - \mathbf{S}^{\mathbf{T}}\mathbf{X})(S_0X_0 + \mathbf{S}^{\mathbf{T}}\mathbf{X}) \\ &= S_0X_0S_0X_0 - S_1X_1S_1X_1 - S_2X_2S_2X_2 - S_3X_3S_3X_3 \\ &+ S_0X_0(S_1X_1 + S_2X_2 + S_3X_3) - S_1X_1(S_0X_0 + S_2X_2 + S_3X_3) \\ &- S_2X_2(S_0X_0 + S_1X_1 + S_3X_3) - S_3X_3(S_0X_0 + S_1X_1 + S_2X_2) \end{aligned}$$

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In fact, we consider a quaternion with complex coefficients, which we multiply by its conjugate quaternion (due to the complexity of the coefficients, these are biquaternions, but we still use quaternionic conjugation, without complex conjugation).

Let us subject the set of complex numbers to the Lorentz transformation

$$\mathfrak{X}' = \Lambda\mathfrak{X}$$

Let us write a relation whose validity for an arbitrary set of complex numbers can be checked directly

$$(S_0X_0{}' - S_1X_1{}' - S_2X_2{}' - S_3X_3{}')(S_0X_0{}' + S_1X_1{}' + S_2X_2{}' + S_3X_3{}')$$
$$= (S_0X_0 - S_1X_1 - S_2X_2 - S_3X_3)(S_0X_0 + S_1X_1 + S_2X_2 + S_3X_3)$$

The matrix M^2 in the simplest case is diagonal with equal complex elements on the diagonal equal to the square of the length of the vector $\mathfrak{X}$ in the metric of Minkowski space, which we denote m^2. Both M^2 and m^2 do not change under any rotations and boosts, in physical applications the invariance of m^2 is usually used, in particular, for the four-component momentum vector this quantity is called the square of mass.

Since the matrices $\mathbf{S}$ anticommutate with each other, for a vector $\mathfrak{X}$ whose components commute with each other, we have just the simplest case with a diagonal matrix with m^2 on the diagonal. But if the components of vector $\mathfrak{X}$ do not commute, the matrix M^2 already has a more complex structure and carries additional physical information compared to m^2. For example, the vector $\mathfrak{X}$ may include the electron momentum vector and the electromagnetic potential vector. The four-component potential vector is a function of the four-dimensional coordinates of Minkowski space. The components of the four-component momentum do not commute with the components of the coordinate vector, respectively, and the coordinate function does not commute with the momentum components, and their commutator is expressed through the partial derivative of this function by the corresponding coordinate. If the components of the vector $\mathfrak{X}$ do not commute, the matrix M^2 will no longer be invariant with respect to Lorentz transformations.

Suppose that the complex numbers we consider commute with all matrices, and note that the squares of all matrices are equal to the unit 4×4 matrix I

$$M^2 = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + (S_1X_0X_1 + S_2X_0X_2 + S_3X_0X_3)$$
$$- (S_1X_1X_0 + S_1S_2X_1X_2 + S_1S_3X_1X_3) - (S_2X_2X_0 + S_2S_1X_2X_1 + S_2S_3X_2X_3)$$
$$- (S_3X_3X_0 + S_3S_1X_3X_1 + S_3S_2X_3X_2)$$
$$= (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2$$
$$- X_2X_0) + S_3(X_0X_3 - X_3X_0) - (S_1S_2X_1X_2 + S_1S_3X_1X_3)$$
$$- (S_2S_1X_2X_1 + S_2S_3X_2X_3) - (S_3S_1X_3X_1 + S_3S_2X_3X_2)$$
$$= (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2$$
$$- X_2X_0) + S_3(X_0X_3 - X_3X_0) - (S_1S_2X_1X_2 + S_2S_1X_2X_1)$$
$$- (S_2S_3X_2X_3 + S_3S_2X_3X_2) - (S_3S_1X_3X_1 + S_1S_3X_1X_3)$$
$$= (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2$$
$$- X_2X_0) + S_3(X_0X_3 - X_3X_0) - (S_1S_2X_1X_2 + S_2S_1X_1X_2 + S_2S_1(X_2X_1$$
$$- X_1X_2)) - (S_2S_3X_2X_3 + S_3S_2X_2X_3 + S_3S_2(X_3X_2 - X_2X_3))$$
$$- (S_3S_1X_3X_1 + S_1S_3X_3X_1 + S_1S_3(X_1X_3 - X_3X_1))$$

Taking into account anticommutative properties of matrices and expressions for their pairwise products we obtain

$$M^2 = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0) - S_2S_1(X_2X_1 - X_1X_2) - S_3S_2(X_3X_2 - X_2X_3) - S_1S_3(X_1X_3 - X_3X_1) = (X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + S_2(X_0X_2 - X_2X_0) + S_3(X_0X_3 - X_3X_0) + iS_3(X_2X_1 - X_1X_2) + iS_1(X_3X_2 - X_2X_3) + iS_2(X_1X_3 - X_3X_1) =$$

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$(X_0X_0 - X_1X_1 - X_2X_2 - X_3X_3)I + S_1(X_0X_1 - X_1X_0) + iS_1(X_3X_2 - X_2X_3) + S_2(X_0X_2 - X_2X_0) + iS_2(X_1X_3 - X_3X_1) + S_3(X_0X_3 - X_3X_0) + iS_3(X_2X_1 - X_1X_2)$

Consider the case when $\mathfrak{X}$ is the sum of the momentum vector and the electromagnetic potential vector, which is a function of coordinates

$$\mathfrak{X} = \text{Ᵽ} + \text{Ⱥ}$$
$$\text{Ᵽ}^{\boldsymbol{T}} \equiv (P_0, P_1, P_2, P_3)$$
$$\text{Ⱥ}^{\boldsymbol{T}} \equiv (A_0, A_1, A_2, A_3)$$
$$\mathbf{P}^{\boldsymbol{T}} \equiv (P_1, P_2, P_3)$$
$$\mathbf{A}^{\boldsymbol{T}} \equiv (A_1, A_2, A_3)$$

$$M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] + S_1[(P_0 + A_0)(P_1 + A_1) - (P_1 + A_1)(P_0 + A_0)] + iS_1[(P_3 + A_3)(P_2 + A_2) - (P_2 + A_2)(P_3 + A_3)] + S_2[(P_0 + A_0)(P_2 + A_2) - (P_2 + A_2)(P_0 + A_0)] + iS_2[(P_1 + A_1)(P_3 + A_3) - (P_3 + A_3)(P_1 + A_1)] + S_3[(P_0 + A_0)(P_3 + A_3) - (P_3 + A_3)(P_0 + A_0)] + iS_3[(P_2 + A_2)(P_1 + A_1) - (P_1 + A_1)(P_2 + A_2)]$$

For now, we'll stick with the Heisenberg approach, that is, we will consider the components of the momentum vector P_0, P_1, P_2, P_3 as operators for which there are commutation relations with coordinates or coordinate functions such as A_0, A_1, A_2, A_3. In this approach, the operators do not have to act on any wave function.

Taking into account the commutation relations of the components of the momentum vector and the coordinate vector, the commutator of the momentum component and the coordinate function is expressed through the derivative of this function by the corresponding coordinate, e.g.

$$[(P_2 + A_2)(P_1 + A_1) - (P_1 + A_1)(P_2 + A_2)] = P_2A_1 - A_1P_2 - (P_1A_2 - A_2P_1)$$
$$= -i\frac{\partial A_1}{\partial x_2} - \left(-i\frac{\partial A_2}{\partial x_1}\right)$$

As a result, we obtain

$$M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] + S_1\left[-i\frac{\partial A_1}{\partial x_0} + i\frac{\partial A_0}{\partial x_1}\right] + iS_1\left[-i\frac{\partial A_2}{\partial x_3} + i\frac{\partial A_3}{\partial x_2}\right] + S_2\left[-i\frac{\partial A_2}{\partial x_0} + i\frac{\partial A_0}{\partial x_2}\right] + iS_2\left[-i\frac{\partial A_3}{\partial x_1} + i\frac{\partial A_1}{\partial x_3}\right] + S_3\left[-i\frac{\partial A_3}{\partial x_0} + i\frac{\partial A_0}{\partial x_3}\right] + iS_3\left[-i\frac{\partial A_1}{\partial x_2} + i\frac{\partial A_2}{\partial x_1}\right] = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] - iS_1\left[\frac{\partial A_1}{\partial x_0} - \frac{\partial A_0}{\partial x_1}\right] + S_1\left[\frac{\partial A_2}{\partial x_3} - \frac{\partial A_3}{\partial x_2}\right] - iS_2\left[\frac{\partial A_2}{\partial x_0} - \frac{\partial A_0}{\partial x_2}\right] + S_2\left[\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right] - iS_3\left[\frac{\partial A_3}{\partial x_0} - \frac{\partial A_0}{\partial x_3}\right] + S_3\left[\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1}\right] = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] - iS_1F_{01} + S_1F_{32} - iS_2F_{02} + S_2F_{13} - iS_3F_{03} + S_3F_{21} = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] - iS_1E_x + S_1B_x - iS_2E_y + S_2B_y - iS_3E_z + S_3B_z$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

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$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

As a result, we have the expression

$$M^2 = I[(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)] + \mathbf{S^T B} - i\mathbf{S^T E}$$

$$\mathbf{B^T} \equiv (B_x, B_y, B_z) \equiv (B_1, B_2, B_3)$$

$$\mathbf{E^T} \equiv (E_x, E_y, E_z) \equiv (E_1, E_2, E_3)$$

Similarly, it can be shown that

$$(S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3)(S_0A_0 + S_1A_1 + S_2A_2 + S_3A_3) + (S_0A_0 - S_1A_1 - S_2A_2 - S_3A_3)(S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3) = 2I(P_0A_0 - P_1A_1 - P_2A_2 - P_3A_3) + \mathbf{S^T B} - i\mathbf{S^T E}$$

The matrix

$$M^2 - \{\mathbf{S^T B} - i\mathbf{S^T E}\} = I\{(P_0 + A_0)(P_0 + A_0) - (P_1 + A_1)(P_1 + A_1) - (P_2 + A_2)(P_2 + A_2) - (P_3 + A_3)(P_3 + A_3)\} \equiv Id^2$$

does not change under Lorentz transformations involving any rotations and boosts.

$$Id^2 = \big(S_0(P_0 + A_0) - S_1(P_1 + A_1) - S_2(P_2 + A_2) - S_3(P_3 + A_3)\big)\big(S_0(P_0 + A_0) + S_1(P_1 + A_1) + S_2(P_2 + A_2) + S_3(P_3 + A_3)\big) - \{\mathbf{S^T B} - i\mathbf{S^T E}\}$$

$$= \big(S_0(P_0 + A_0) - \mathbf{S^T}(\mathbf{P} + \mathbf{A})\big)\big(S_0(P_0 + A_0) + \mathbf{S^T}(\mathbf{P} + \mathbf{A})\big) - \{\mathbf{S^T B} - i\mathbf{S^T E}\}$$

Taking into account the electron charge we have

$$\mathfrak{X} = P\!\!\!/ - eA\!\!\!/$$

$$Id^2 = \big(S_0(P_0 - eA_0) - \mathbf{S^T}(\mathbf{P} - e\mathbf{A})\big)\big(S_0(P_0 - eA_0) + \mathbf{S^T}(\mathbf{P} - e\mathbf{A})\big) + e\{\mathbf{S^T B} - i\mathbf{S^T E}\}$$

Let us summarize our consideration. There is a correlation

$$Id^2 = M^2 + e\{\mathbf{S^T B} - i\mathbf{S^T E}\}$$

where

$$M^2 \equiv \big(S_0(P_0 - eA_0) - \mathbf{S^T}(\mathbf{P} - e\mathbf{A})\big)\big(S_0(P_0 - eA_0) + \mathbf{S^T}(\mathbf{P} - e\mathbf{A})\big)$$

$$\begin{aligned} Id^2 &\equiv I\{(P_0 - eA_0)^2 - (P_1 - eA_1)^2 - (P_2 - eA_2)^2 - (P_3 - eA_3)^2\} \\ &= I[(P_0 - eA_0)(P_0 - eA_0) - (P_1 - eA_1)(P_1 - eA_1) \\ &\quad - (P_2 - eA_2)(P_2 - eA_2) - (P_3 - eA_3)(P_3 - eA_3)] \\ &= I[(P_0 - eA_0)(P_0 - eA_0) - (\mathbf{P} - e\mathbf{A})^T(\mathbf{P} - e\mathbf{A})] \\ &= I\{(P_0 - eA_0)^2 - (\mathbf{P} - e\mathbf{A})^2\} \end{aligned}$$

Let's analyze the obtained equality

$$M^2 = Id^2 - e\{\mathbf{S^T B} - i\mathbf{S^T E}\}$$

Note that the quantity d^2 is invariant to the Lorentz transformations irrespective of whether the momentum and field components commute or not. To solve this equation, we have to make additional simplifications. For example, to arrive at an equation similar to the Dirac equation, we must equate M^2 with the matrix Im^2, where m^2 is the square of the mass of a free electron. Then

$$Im^2 = Id^2 - e\{\mathbf{S^T B} - i\mathbf{S^T E}\}$$

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$$Id^2 - Im^2 - e\{\mathbf{S}^{\mathrm{T}}\mathbf{B} - i\mathbf{S}^{\mathrm{T}}\mathbf{E}\} = 0$$
$$I\{(P_0 - eA_0)^2 - (\mathbf{P} - e\mathbf{A})^2\} - Im^2 - e\{\mathbf{S}^{\mathrm{T}}\mathbf{B} - i\mathbf{S}^{\mathrm{T}}\mathbf{E}\} = 0$$

With this substitution the generalized equation almost coincides with the equation [6], formula (43.25), the difference is that there is a plus sign before $e\mathbf{S}^{\mathrm{T}}\mathbf{B}$, and instead of $i\mathbf{S}^{\mathrm{T}}\mathbf{E}$ there is $i\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{E}$, in which the matrices $\boldsymbol{\alpha}$ have the following form

$$\boldsymbol{\alpha}^{\mathrm{T}} \equiv (\alpha_1, \alpha_2, \alpha_3)$$
$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

A similar equation is given by Dirac in [7], Para. 76, Equation 24; he does not use the matrices $\boldsymbol{\alpha}$, only the matrices $\mathbf{S}$, but the signs of the contributions of the magnetic and electric fields are the same.

Along with the original form

$$M^2 = \big(S_0(P_0 - eA_0) - \mathbf{S}^{\mathrm{T}}(\mathbf{P} - e\mathbf{A})\big)\big(S_0(P_0 - eA_0) + \mathbf{S}^{\mathrm{T}}(\mathbf{P} - e\mathbf{A})\big)$$
$$= d^2 - e\{\mathbf{S}^{\mathrm{T}}\mathbf{B} - i\mathbf{S}^{\mathrm{T}}\mathbf{E}\}$$

it is possible to consider the form with a different order of the factors. It can be shown that this leads to a change in the sign of the electric field contribution

$$M^2 = \big(S_0(P_0 - eA_0) + \mathbf{S}^{\mathrm{T}}(\mathbf{P} - e\mathbf{A})\big)\big(S_0(P_0 - eA_0) - \mathbf{S}^{\mathrm{T}}(\mathbf{P} - e\mathbf{A})\big)$$
$$= d^2 - e\{\mathbf{S}^{\mathrm{T}}\mathbf{B} + i\mathbf{S}^{\mathrm{T}}\mathbf{E}\}$$

Since Id^2, unlike M^2, is invariant to Lorentz transformations, it would be logical to replace it by Im^2. At least both these matrices are diagonal, and in the case of a weak field their diagonal elements are close. Nevertheless, the approach based on the Dirac equation leads to solutions consistent with experiment.

The matrix M^2 in the general case has complex elements and is not diagonal, and in the Dirac equations instead of it is substituted the product of the unit matrix by the square of mass m^2, the physical meaning of such a substitution is not obvious. Apparently it is implied that it is the square of the mass of a free electron. But the square of the length of the sum of the lengths of the electron momentum vectors and the electromagnetic potential vector is not equal to the sum of the squares of the lengths of these vectors, that is, it is not equal to the square of the mass of the electron, even if the square of the length of the potential vector were zero. But, for example, in the case of an electrostatic central field, even the square of the length of one potential vector is not equal to zero. Therefore, it is difficult to find a logical justification for using the mass of a free electron in the Dirac equation in the presence of an electromagnetic field. After all, mass is simply the length of a momentum vector, but the concept of a momentum vector, and hence of mass, can be applied only for a free particle. Similarly, energy is the zero component of the momentum vector and the concept of energy can only be strictly defined for a free particle. Due to the noted differences, the solutions of the generalized equation can differ from the solutions arising from the Dirac equation.

In the case when there is a constant magnetic field directed along the z-axis, we can write down

$$A_0 = 0 \qquad A_1 = -\frac{1}{2}B_3x_2 \qquad A_2 = \frac{1}{2}B_3x_1 \qquad A_3 = 0$$
$$(S_0P_0)^2 - M^2 - (\mathbf{P} - e\mathbf{A})^T(\mathbf{P} - e\mathbf{A})I - eS_3B_3 = 0$$
$$(S_0P_0)^2 - M^2 - (P_1 - eA_1)(P_1 - eA_1)I - (P_2 - eA_2)(P_2 - eA_2)I - eS_3B_3 = 0$$
$$(S_0P_0)^2 - M^2 - {P_0}^2I - {P_3}^2I - {P_1}^2 - (eA_1)^2 - {P_2}^2 - (eA_2)^2$$
$$+ e\frac{1}{2}B_3(x_1P_2 - x_2P_1 + x_1P_2 - x_2P_1) - eS_3B_3 = 0$$
$${P_0}^2I - M^2 - {P_0}^2I - {P_3}^2I - {P_1}^2I - (eA_1)^2I - {P_2}^2I - (eA_2)^2I + eB_3(x_1P_2 - x_2P_1)I$$
$$- eS_3B_3 = 0$$

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$$I\left(-{P_1}^2-{P_2}^2-{P_3}^2-(eA_1)^2-(eA_2)^2\right)I-M^2 - eB_3\begin{pmatrix} L_3+1 & 0 & 0 & 0 \\ 0 & L_3-1 & 0 & 0 \\ 0 & 0 & L_3+1 & 0 \\ 0 & 0 & 0 & L_3-1 \end{pmatrix}=0$$

Here $(x_1 P_2 - x_2 P_1) \equiv L_3$. Only when the field is directed along the z-axis, the matrix M^2 is diagonal and real because the third Pauli matrix is diagonal and real. And if the field is weak, M^2 can be approximated by the $m^2 I$ matrix. This is probably why it is customary to illustrate the interaction of electron spin with the magnetic field by choosing its direction along the z-axis. In any other direction M^2 is not only non-diagonal, but also complex, so that it is difficult to justify the use of $m^2 I$.

When the influence of the electromagnetic field was taken into account, no specific characteristics of the electron were used. When deriving a similar result using the Dirac equation, it is assumed that since the electron equation is used, the result is specific to the electron. In our case Pauli matrices and commutation relations are used, apparently these two assumptions or only one of them characterize the properties of the electron, distinguishing it from other particles with non-zero masses.

The proposed equation echoes the Dirac equation, at least from it one can obtain the same formulas for the interaction of spin and electromagnetic field as with the Dirac equation, and in the absence of a field the proposed equation is invariant to the Lorentz transformations. In contrast, to prove the invariance of the Dirac equation even in the absence of a field, the infinitesimal Lorentz transformations are used, but the invariance at finite angles of rotations and boosts is not demonstrated. The proof of invariance of the Dirac equation is based on the claim that a combination of rotations at finite angles can be represented as a combination of infinitesimal rotations. But this is true only for rotations around one axis, and if there are at least two axes, this statement is not true because of non-commutability of Pauli matrices, which are generators of rotations, so that the exponent of the sum is not equal to the product of exponents if the sum includes generators of rotations around different axes.

A test case for any theory is the model of the central electrostatic field used in the description of the hydrogen atom, in which the components of the vector potential are zero

$$(S_0(P_0-eA_0)-\mathbf{S}^{\mathrm{T}}\mathbf{P})(S_0(P_0-eA_0)+\mathbf{S}^{\mathrm{T}}\mathbf{P}) = I\left[(P_0-eA_0)^2-{P_1}^2-{P_2}^2-{P_3}^2\right]+ie\mathbf{S}^{\mathrm{T}}\mathbf{E}$$

If again we equate the left part with Im^2, we obtain

$$I\left[(P_0-eA_0)^2-{P_1}^2-{P_2}^2-{P_3}^2\right]- Im^2+ie\mathbf{S}^{\mathrm{T}}\mathbf{E}=0$$

$$I\left[(P_0-eA_0)^2-{P_1}^2-{P_2}^2-{P_3}^2- m^2\right]-ie\left(S_1\frac{\partial A_0}{\partial x_1}+S_2\frac{\partial A_0}{\partial x_2}+S_3\frac{\partial A_0}{\partial x_3}\right)=0$$

Introducing the notations ($A_0 \equiv \varphi(r) = Q/r,\ P_0 \equiv E,\ r = 1/\sqrt{{x_1}^2+{x_2}^2+{x_3}^2}$), we obtain

$$I\left[\left(E-\frac{eQ}{r}\right)^2-{P_1}^2-{P_2}^2-{P_3}^2- m^2\right]-ie\left(S_1\frac{\partial\varphi(r)}{\partial x_1}+S_2\frac{\partial\varphi(r)}{\partial x_2}+S_3\frac{\partial\varphi(r)}{\partial x_3}\right)=0$$

$$I\left[\left(E-\frac{eQ}{r}\right)^2-{P_1}^2-{P_2}^2-{P_3}^2- m^2\right]+i\frac{eQ}{r^3}(S_1x_1+S_2x_2+S_3x_3)=0$$

If we substitute operators acting on the wave function instead of momentum components into the generalized equation, we obtain a generalized analog of the relativistic Schrödinger equation, in which the wave function has four components and changes as a spinor under Lorentz transformations. Using the substitutions

$$P_0\to i\frac{\partial}{\partial t}\qquad P_1\to -i\frac{\partial}{\partial x_1}\qquad P_2\to -i\frac{\partial}{\partial x_2}\qquad P_3\to -i\frac{\partial}{\partial x_3}$$

the equation for the four-component wave function $\boldsymbol{\psi}$ before all transformations has the form

$$\left(S_0\left(\frac{\partial}{\partial t}-eA_0\right)+\mathbf{S}^{\mathbf{T}}(\boldsymbol{\nabla}-e\mathbf{A})\right)\left(S_0\left(\frac{\partial}{\partial t}-eA_0\right)-\mathbf{S}^{\mathbf{T}}(\boldsymbol{\nabla}-e\mathbf{A})\right)\boldsymbol{\psi}+M^2\boldsymbol{\psi}=0$$

and after transformations

$$\left\{\left(S_0(P_0-eA_0)\right)^2-(\mathbf{P}-e\mathbf{A})^2I-e\mathbf{S}^{\mathbf{T}}\mathbf{B}+ie\mathbf{S}^{\mathbf{T}}\mathbf{E}\right\}\boldsymbol{\psi}=M^2\boldsymbol{\psi}$$

Once again, note that the matrix M^2 is not diagonal and real.

All the above deductions are also valid when replacing 4×4 matrices S_μ by 2×2 matrices σ_μ, since their commutative and anticommutative properties are the same. The corresponding generalized equation is of the form

$$\left(\sigma_0(P_0-eA_0)\right)^2-M^2-(\mathbf{P}-e\mathbf{A})^2I-e\boldsymbol{\sigma}^{\mathbf{T}}\mathbf{B}+ie\boldsymbol{\sigma}^{\mathbf{T}}\mathbf{E}=0$$

where

$$\boldsymbol{\sigma}^{\mathbf{T}}\equiv(\sigma_1,\sigma_2,\sigma_3)$$

and the equation for the now two-component wave function looks like

$$\left(\sigma_0\left(\frac{\partial}{\partial t}-eA_0\right)+\boldsymbol{\sigma}^{\mathbf{T}}(\boldsymbol{\nabla}-e\mathbf{A})\right)\left(\sigma_0\left(\frac{\partial}{\partial t}-eA_0\right)-\boldsymbol{\sigma}^{\mathbf{T}}(\boldsymbol{\nabla}-e\mathbf{A})\right)\boldsymbol{\psi}+M^2\boldsymbol{\psi}=0$$

In deriving his equation, Dirac [7, paragraph 74] noted that as long as we are dealing with matrices with two rows and columns, we cannot obtain a representation of more than three anticommuting quantities; to represent four anticommuting quantities, he turned to matrices with four rows and columns. In our case, however, three anticommuting matrices are sufficient, so the wave function can also be two-component. Dirac also explains that the presence of four components results in twice as many solutions, half of which have negative energy. In the case of a two-component wave function, however, no negative energy solutions are obtained. Particles with negative energy in this case also exist, but they are described by the same equation in which the signs of all four matrices S or σ are reversed.

One would seem to expect similar results from other representations of the momentum operator, e.g., [6, formula (24.15)]

$$\omega_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\omega_1=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\quad\omega_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\quad\omega_3$$
$$=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

under the assumption that this representation can describe a particle with spin one. But this expectation is not justified, since the last three matrices do not anticommutate, and therefore the quadratic form constructed on their basis is not invariant under Lorentz transformations.

Let's see what happens to M^2 when we change the sign of the matrices. When changing the sign of S_0 we have

$$M^2=[-S_0P_0-(\mathbf{S}\cdot\mathbf{P})][-S_0P_0+(\mathbf{S}\cdot\mathbf{P})]=[S_0P_0+(\mathbf{S}\cdot\mathbf{P})][S_0P_0-(\mathbf{S}\cdot\mathbf{P})]$$

swapping the places of the multipliers. The multipliers do not necessarily commute, so M^2 is not invariant with respect to the change of sign of S_0, which can be interpreted as a reflection in time. The same picture of invariance absence we have at the sign change of matrices $\mathbf{S}$, i.e. at spatial reflection

$$M^2=[S_0P_0+(\mathbf{S}\cdot\mathbf{P})][S_0P_0-(\mathbf{S}\cdot\mathbf{P})]$$

If we change the signs of all matrices at once, we have

$$M^2 = [-S_0P_0 + (\mathbf{S}\cdot\mathbf{P})][-S_0P_0 - (\mathbf{S}\cdot\mathbf{P})] = [S_0P_0 - (\mathbf{S}\cdot\mathbf{P})][S_0P_0 + (\mathbf{S}\cdot\mathbf{P})]$$

i.e. invariance. The physical interpretation of this case can be given by taking into account the change of signs of the matrices in equation

$$\left(-S_0(P_0 - eA_0)\right)^2 - M^2 - (\mathbf{P} - e\mathbf{A})^2 I - e(-\mathbf{S}^{\mathbf{T}}\mathbf{B}) + ie(-\mathbf{S}^{\mathbf{T}}\mathbf{E}) = 0$$

which can be rewritten as

$$\left(S_0((-P_0) - (-e)A_0)\right)^2 - M^2 - (\mathbf{P} - e\mathbf{A})^2 I - (-e)\mathbf{S}^{\mathbf{T}}\mathbf{B} + i(-e)\mathbf{S}^{\mathbf{T}}\mathbf{E} = 0$$

it can be interpreted as an equation for a particle with negative energy and positive charge, i.e. for the positron. Thus, the generalized equation with matrices S_0, S_1, S_2, S_3 describes a particle, and with matrices $-S_0, -S_1, -S_2, -S_3$ an antiparticle.

If one consistently adheres to the Heisenberg approach and does not involve the notion of wave function, it is not very clear how to search for solutions of the presented equations. The Schrödinger approach with finding the eigenvalues of the M^2matrix and their corresponding eigenfunctions can help here.

$$\left\{\left(S_0(P_0 - eA_0)\right)^2 - (\mathbf{P} - e\mathbf{A})^2 I - e\mathbf{S}^{\mathbf{T}}\mathbf{B} + ie\mathbf{S}^{\mathbf{T}}\mathbf{E}\right\}\boldsymbol{\psi} = M^2\boldsymbol{\psi}$$

In the left-hand side are the operators acting on the wave function, and in the right-hand side is a constant matrix on which the wave function is simply multiplied. This equality must be satisfied for all values of the four-dimensional coordinates (t, x_1, x_2, x_3) at once. Then M^2 is not fixed but can take a set of possible values, finding all these values is the goal of solving the equation.

Thus, we have arrived at an equation containing a matrix M^2 which is non-diagonal, complex and in general depends on the coordinates (t, x_1, x_2, x_3). After the standard procedure of separating the time and space variables, we can go to a stationary equation in which there will be no time dependence, but the dependence the matrix M^2 on the coordinates will remain. It is possible to ignore the dependence of M^2 on the coordinates and its non-diagonality and simply replace this matrix by a unit matrix with a coefficient in the form of the square of the free electron mass. Then the equation will give solutions coinciding with those of the Dirac equation. But this solution can be considered only approximate and the question remains how far we depart from strict adherence to the principle of invariance with respect to Lorentz transformations and how far we deviate from the hypothetical true solution, which is fully consistent with this principle. To find this solution, we need to approach this equation without simplifying assumptions and look for a set of solutions, each of which represents an eigenvalue matrix M^2 of arbitrary form and its corresponding four-component eigenfunction.

When searching for solutions, one can try to use two equations

$$\left(S_0\left(\frac{\partial}{\partial t} - eA_0\right) + \mathbf{S}^{\mathbf{T}}(\boldsymbol{\nabla} - e\mathbf{A})\right)\left(S_0\left(\frac{\partial}{\partial t} - eA_0\right) - \mathbf{S}^{\mathbf{T}}(\boldsymbol{\nabla} - e\mathbf{A})\right)\boldsymbol{\psi} + M^2\boldsymbol{\psi} = 0$$

$$\left(S_0\left(\frac{\partial}{\partial t} - eA_0\right) - \mathbf{S}^{\mathbf{T}}(\boldsymbol{\nabla} - e\mathbf{A})\right)\left(S_0\left(\frac{\partial}{\partial t} - eA_0\right) + \mathbf{S}^{\mathbf{T}}(\boldsymbol{\nabla} - e\mathbf{A})\right)\boldsymbol{\psi} + M^2\boldsymbol{\psi} = 0$$

successively applying the operators with first order derivatives included in them to the eigenfunctions already found, similarly as described in Schrödinger's work [8].

3. Equation for the spinor coordinate space

Let us return to the set of arbitrary complex numbers, for simplicity we will call it a vector

$$\mathfrak{x}^{\mathbf{T}} \equiv (\mathfrak{X}_0, \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3)$$

Let us consider in connection with it arbitrary four-component complex spinors

$$\mathbf{p}^{\mathbf{T}} \equiv (p_0, p_1, p_2, p_3)$$

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$$\mathfrak{x}\mathbf{1}^{\mathbf{T}} \equiv (\mathfrak{x}1_0, \mathfrak{x}1_1, \mathfrak{x}1_2, \mathfrak{x}1_3)$$
$$\mathfrak{x}\mathbf{2}^{\mathbf{T}} \equiv (\mathfrak{x}2_0, \mathfrak{x}2_1, \mathfrak{x}2_2, \mathfrak{x}2_3)$$

There is a representation of the components of the vector

$$\mathfrak{X}_\mu = \frac{1}{2}\mathfrak{x}\mathbf{1}^\dagger S_\mu \mathfrak{x}\mathbf{2}$$

and there is another way to calculate them

$$\mathfrak{X}_\mu = \frac{1}{2}Tr[\mathfrak{x}\mathbf{1}\mathfrak{x}\mathbf{2}^\dagger S_\mu]$$

Further we will assume that both spinors are identical, then the vector constructed from them

$$\mathbf{P}^{\mathbf{T}} \equiv (P_0, P_1, P_2, P_3)$$

has real components, and we will assume that this is the electron momentum vector constructed from the complex momentum spinor **p**

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p}$$

$$P_\mu = \frac{1}{2}Tr[\mathbf{p}\mathbf{p}^\dagger S_\mu]$$

Consider the complex quantity

$$\mathbf{p}^T \Sigma_{MM}\mathbf{x} = (p_0, p_1, p_2, p_3)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = (p_0, p_1, p_2, p_3)\begin{pmatrix} x_1 \\ -x_0 \\ x_3 \\ -x_2 \end{pmatrix}$$

$$= p_0 x_1 - p_1 x_0 + p_2 x_3 - p_3 x_2$$

where we introduce one more complex spinor, which in the future we will give the meaning of the complex coordinate spinor

$$\mathbf{x}^{\mathbf{T}} \equiv (x_0, x_1, x_2, x_3)$$

and

$$\Sigma_{MM} = \begin{pmatrix} \sigma_M & 0 \\ 0 & \sigma_M \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \sigma_M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Coordinate vector of the four-dimensional Minkowski space

$$\mathbf{X}^{\mathbf{T}} \equiv (X_0, X_1, X_2, X_3)$$

is obtained from the coordinate spinor by the same formulas

$$X_\mu = \frac{1}{2}\mathbf{x}^\dagger S_\mu \mathbf{x}$$

$$X_\mu = \frac{1}{2}Tr[\mathbf{x}\mathbf{x}^\dagger S_\mu]$$

The quantity $\mathbf{p}^T \Sigma_{MM}\mathbf{x}$ is invariant under the Lorentz transformation simultaneously applied to the momentum and coordinate spinor, which automatically transforms both corresponding vectors as well

$$\mathbf{p}' = N\mathbf{p}$$

$$P'_\mu = \frac{1}{2}Tr[\mathbf{p}'\mathbf{p}'^\dagger S_\mu]$$

$$P'_\mu = \frac{1}{2}\mathbf{p}'^\dagger S_\mu \mathbf{p}'$$

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$$\mathbf{P}' = \Lambda \mathbf{P}$$
$$\mathbf{x}' = N\mathbf{x}$$
$$X'_{\mu} = \frac{1}{2} Tr[\mathbf{x}'\mathbf{x}'^{\dagger} S_{\mu}]$$
$$X'_{\mu} = \frac{1}{2} \mathbf{x}'^{\dagger} S_{\mu} \mathbf{x}'$$
$$\mathbf{X}' = \Lambda \mathbf{X}$$

This quantity does not change for any combination of turns and boosts

$$\mathbf{p}'^{\boldsymbol{T}} \Sigma_{MM} \mathrm{x}' = \mathbf{p}^{\boldsymbol{T}} \Sigma_{MM} \mathbf{x}$$

Accordingly, the exponent

$$exp(\mathbf{p}^{\boldsymbol{T}} \Sigma_{MM} \mathbf{x}) = exp(p_0 x_1 - p_1 x_0 + p_2 x_3 - p_3 x_2)$$

characterizes the propagation process of a plane wave in spinor space with phase invariant to Lorentz transformations.

Let us apply the differential operator to the spinor analog of a plane wave

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) exp(p_0 x_1 - p_1 x_0 + p_2 x_3 - p_3 x_2)$$
$$= (p_0(-p_3) - (-p_1)p_2)\, exp(p_0 x_1 - p_1 x_0 + p_2 x_3 - p_3 x_2) =$$
$$= (p_1 p_2 - p_0 p_3)\, exp(p_0 x_1 - p_1 x_0 + p_2 x_3 - p_3 x_2)$$

Applying this operator at another definition of the phase gives the same eigenvalue

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) exp(p_0 x_0 + p_1 x_1 + p_2 x_2 + p_3 x_3)$$
$$= (p_1 p_2 - p_0 p_3)\, exp(p_0 x_0 + p_1 x_1 + p_2 x_2 + p_3 x_3)$$

that is, two different eigenfunctions correspond to this eigenvalue, but in the second case the phase in the exponent is not invariant with respect to the Lorentz transformation, so we will use the first definition.

Since

$$(p_0, p_1)^{\mathbf{T}} \text{ and } (p_2, p_3)^{\mathbf{T}}$$

are complex spinors, which, under the transformation

$$\mathbf{p}' = N\mathbf{p} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \mathbf{p}$$

is affected by the same matrix n, then the complex quantity

$$m \equiv p_1 p_2 - p_0 p_3$$

is invariant under the action on the momentum spinor $\mathbf{p}$ of the transformation N. m is an eigenvalue of the differential operator, and the plane wave is the corresponding m eigenfunction, which is a solution of the equation

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right) \psi(x_0, x_1, x_2, x_3) = m\, \psi(x_0, x_1, x_2, x_3)$$

Here $\psi(x_0, x_1, x_2, x_3)$ denotes the complex function of complex spinor coordinates.

When substantiating the Schrödinger equation for a plane wave in four-dimensional vector space, an assumption is made (further confirmed in the experiment) about its applicability to an arbitrary wave function. Let us make a similar assumption about the applicability of the reduced spinor equation to an arbitrary function of spinor coordinates, that is, we will consider this equation as universal and valid for all physical processes.

Let us clarify that by the derivative on a complex variable from a complex function we here understand the derivative from an arbitrary stepped complex function using the formula that is valid at least for any integer degrees

$$\frac{\partial z^k}{\partial z} = kz^{k-1}$$

In particular, this is true for the exponential function, which is an infinite power series.

It is not by chance that we denote the eigenvalue by the symbol m, because if we form the momentum vector from the momentum spinor $\mathbf{p}$ included in the expression for the plane wave

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p}$$

then for the square of its length the following equality will be satisfied

$${P_0}^2 - {P_1}^2 - {P_2}^2 - {P_3}^2 = m^* m = m^2$$

That is the square of the modulus m has the sense of the square of the mass of a free particle, which is described by a plane wave in spinor space as well as by a plane wave in vector space. For the momentum spinor of a fermionic type particle having in the rest frame the following form

$$\mathbf{p}^\mathbf{T} = (p_0, p_1, \overline{p_1}, -\overline{p_0})$$

quantity

$$m = p_1 p_2 - p_0 p_3 = p_1 \overline{p_1} + p_0 \overline{p_0}$$

is real and not equal to zero, and for the bosonic-type momentum spinor

$$\mathbf{p}^\mathbf{T} = (p_0, p_1, p_0, p_1)$$

it is zero

$$m = p_1 p_2 - p_0 p_3 = p_1 p_0 - p_0 p_1 = 0$$

i.e., the boson satisfies the plane wave equation in spinor space with zero eigenvalue.

For the momentum spinor of a fermion-type particle we can consider another form in the rest system

$$\mathbf{p}^\mathbf{T} = (p_0, p_1, -\overline{p_1}, \overline{p_0})$$

then the mass will be real and negative

$$m = p_1 p_2 - p_0 p_3 = -p_1 \overline{p_1} - p_0 \overline{p_0}$$

This particle with negative mass can be treated as an antiparticle, and in the rest frame its energy is equal to its mass modulo, but it is always positive

$$P_0 = \frac{1}{2}\mathbf{p}^\dagger S_0 \mathbf{p} = \frac{1}{2}\left(\overline{p_0} p_0 + \overline{p_1} p_1 + (-p_1)(-\overline{p_1}) + p_0 \overline{p_0}\right)$$
$$= \frac{1}{2}\left(\overline{p_0} p_0 + \overline{p_1} p_1 + p_1 \overline{p_1} + p_0 \overline{p_0}\right)$$

To describe the behavior of an electron in the presence of an external electromagnetic field, it is common practice to add the electromagnetic potential vector to its momentum vector. We use the same approach at the spinor level and to each component of the momentum spinor of the electron we add the corresponding component of the electromagnetic potential spinor. For simplicity, the electron charge is equal to unity.

Further we need an expression for the commutation relation between the components of the momentum spinor, to which is added the corresponding component of the electromagnetic potential spinor, which is a function of the spinor coordinates

$$\big(p_0 + a_0(x_1, x_2)\big)\big(p_1 + a_1(x_1, x_2)\big) - \big(p_1 + a_1(x_1, x_2)\big)\big(p_0 + a_0(x_1, x_2)\big)$$

Let us replace the momenta by differential operators

$$p_0 \to \frac{\partial}{\partial x_1} \qquad p_1 \to -\frac{\partial}{\partial x_0} \qquad p_2 \to \frac{\partial}{\partial x_3} \qquad p_3 \to -\frac{\partial}{\partial x_2}$$

and find the commutation relation

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$$\left\{\left(\frac{\partial}{\partial x_1}+a_0(x_0,x_1,x_2,x_3)\right)\left(-\frac{\partial}{\partial x_0}+a_1(x_0,x_1,x_2,x_3)\right)\right.$$

$$-\left(-\frac{\partial}{\partial x_0}+a_1(x_0,x_1,x_2,x_3)\right)\left(\frac{\partial}{\partial x_1}\right.$$

$$\left.\left.+a_0\big((x_0,x_1,x_2,x_3)\big)\right)\right\}\psi(x_0,x_1,x_2,x_3)$$

$$=\frac{\partial}{\partial x_1}(a_1\psi)-a_0\frac{\partial\psi}{\partial x_0}+\frac{\partial}{\partial x_0}(a_0\psi)-a_1\frac{\partial\psi}{\partial x_1}$$

$$=\frac{\partial a_1}{\partial x_1}\psi+a_1\frac{\partial\psi}{\partial x_1}-a_0\frac{\partial\psi}{\partial x_0}+\frac{\partial a_0}{\partial x_0}\psi+a_0\frac{\partial\psi}{\partial x_0}-a_1\frac{\partial\psi}{\partial x_1}=\frac{\partial a_1}{\partial x_1}\psi+\frac{\partial a_0}{\partial x_0}\psi$$

$$=\left\{\frac{\partial a_1(x_0,x_1,x_2,x_3)}{\partial x_1}+\frac{\partial a_0(x_0,x_1,x_2,x_3)}{\partial x_0}\right\}\psi(x_0,x_1,x_2,x_3)$$

Thus

$$(p_0+a_0)(p_1+a_1)-(p_1+a_1)(p_0+a_0)=\frac{\partial a_1}{\partial x_1}+\frac{\partial a_0}{\partial x_0}$$

Let us apply the proposed equation to analyze the wave function of the electron in a centrally symmetric electric field, this model is used to describe the hydrogen-like atom. For the components of the vector potential of a centrally symmetric electric field it is true that

$$A_0=\frac{1}{2}\mathbf{a}^{\dagger}S_0\mathbf{a}=\frac{1}{2}(\overline{a_0}a_0+\overline{a_1}a_1+\overline{a_2}a_2+\overline{a_3}a_3)=\frac{1}{R}$$

$$A_1=\frac{1}{2}\mathbf{a}^{\dagger}S_1\mathbf{a}=\frac{1}{2}(\overline{a_0}a_1+\overline{a_1}a_0+\overline{a_2}a_3+\overline{a_3}a_2)=0$$

$$A_2=\frac{1}{2}\mathbf{a}^{\dagger}S_2\mathbf{a}=\frac{1}{2}(-i\overline{a_0}a_1+i\overline{a_1}a_0-i\overline{a_2}a_3+i\overline{a_3}a_2)=0$$

$$A_3=\frac{1}{2}\mathbf{a}^{\dagger}S_3\mathbf{a}=\frac{1}{2}(\overline{a_0}a_0-\overline{a_1}a_1+\overline{a_2}a_2-\overline{a_3}a_3)=0$$

$$\overline{a_0}a_0+\overline{a_2}a_2=\overline{a_1}a_1+\overline{a_3}a_3$$

$$\overline{a_0}a_0+\overline{a_2}a_2=\frac{1}{R}$$

$$\overline{a_0}a_1+\overline{a_2}a_3=\overline{a_1}a_0+\overline{a_3}a_2$$

$$\frac{1}{2}(\overline{a_0}a_1+\overline{a_1}a_0+\overline{a_2}a_3+\overline{a_3}a_2)=\overline{a_0}a_1+\overline{a_2}a_3=0$$

$$\overline{a_0}a_1=-\overline{a_2}a_3$$

$$\overline{a_0}=i\overline{a_2}$$

$$a_0=-ia_2$$

$$\overline{a_0}a_0+\overline{a_2}a_2=\ i\overline{a_2}*(-ia_2)+\overline{a_2}a_2=2\overline{a_2}a_2=2{a_2}^2=\frac{1}{R}$$

As a result, it is possible to accept

$$a_0=-\frac{i}{\sqrt{2R}}\qquad a_1=\frac{1}{\sqrt{2R}}\qquad a_2=\frac{1}{\sqrt{2R}}\qquad a_3=-\frac{i}{\sqrt{2R}}$$

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$$\overline{a_0}a_1 = i\frac{1}{\sqrt{2R}}\frac{1}{\sqrt{2R}} = \frac{i}{2R}$$
$$\overline{a_2}a_3 = \frac{1}{\sqrt{2R}}\left(-i\frac{1}{\sqrt{2R}}\right) = -\frac{i}{2R}$$

$$R = \sqrt{{X_1}^2 + {X_2}^2 + {X_3}^2} =$$
$$\sqrt{\left(\frac{1}{2}(\overline{x_0}x_1 + \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)\right)^2 + \left(\frac{1}{2}(-i\overline{x_0}x_1 + i\overline{x_1}x_0 - i\overline{x_2}x_3 + i\overline{x_3}x_2)\right)^2 + \left(\frac{1}{2}(\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)\right)^2}$$
$$=$$
$$\sqrt{\left(\frac{1}{2}(\overline{x_0}x_1 + \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)\right)^2 - \left(\frac{1}{2}(-\overline{x_0}x_1 + \overline{x_1}x_0 - \overline{x_2}x_3 + \overline{x_3}x_2)\right)^2 + \left(\frac{1}{2}(\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)\right)^2}$$

We are looking for a solution of the spinor equation; we do not consider the electron's spin yet
$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\right)\varphi(x_0, x_1, x_2, x_3) = m\,\varphi(x_0, x_1, x_2, x_3)$$

This equation can be interpreted in another way.
Let us take the invariant expression
$$(p_1p_2 - p_0p_3) = m$$
And let's do the substitution
$$p_0 \to \frac{\partial}{\partial x_1} + a_0(x_0, x_1, x_2, x_3) \qquad p_1 \to -\frac{\partial}{\partial x_0} + a_1(x_0, x_1, x_2, x_3)$$
$$p_2 \to \frac{\partial}{\partial x_3} + a_2(x_0, x_1, x_2, x_3) \qquad p_3 \to -\frac{\partial}{\partial x_2} + a_3(x_0, x_1, x_2, x_3)$$

$$\left\{\left(-\frac{\partial}{\partial x_0} + a_1\right)\left(\frac{\partial}{\partial x_3} + a_2\right) - \left(\frac{\partial}{\partial x_1} + a_0\right)\left(-\frac{\partial}{\partial x_2} + a_3\right)\right\}\varphi = m\varphi$$

We will consider this equation as an equation for determining the eigenvalues of m and the corresponding eigenfunctions

$$-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\varphi + \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\varphi + \left(-\frac{\partial a_2}{\partial x_0} - \frac{\partial a_3}{\partial x_1}\right)\varphi - a_2\frac{\partial\varphi}{\partial x_0} + a_1\frac{\partial\varphi}{\partial x_3} - a_3\frac{\partial\varphi}{\partial x_1} + a_0\frac{\partial\varphi}{\partial x_2}$$
$$+ (a_1a_2 - a_0a_3)\varphi = m\varphi$$
$$a_0 = -\frac{i}{\sqrt{2R}} \qquad a_1 = \frac{1}{\sqrt{2R}} \qquad a_2 = \frac{1}{\sqrt{2R}} \qquad a_3 = -\frac{i}{\sqrt{2R}}$$

$$a_1a_2 - a_0a_3 = \frac{1}{2R} + \frac{1}{2R} = \frac{1}{R}$$

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$$-\frac{\partial a_2}{\partial x_0}-\frac{\partial a_3}{\partial x_1}=-\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_0}\left(\frac{1}{\sqrt{R}}\right)+i\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{R}}\right)=-\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_0}\left(\frac{1}{\sqrt[4]{R^2}}\right)+i\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt[4]{R^2}}\right)$$

$$=-\frac{1}{\sqrt{2}}\left(-\frac{1}{4}\frac{1}{(R^2)^{\frac{5}{4}}}\right)\frac{\partial}{\partial x_0}(R^2)+i\frac{1}{\sqrt{2}}\left(-\frac{1}{4}\frac{1}{(R^2)^{\frac{5}{4}}}\right)\frac{\partial}{\partial x_1}(R^2)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1}{4}\frac{1}{(R^2)^{\frac{5}{4}}}\right)\left[\frac{\partial}{\partial x_0}(R^2)-i\frac{\partial}{\partial x_1}(R^2)\right]=\frac{1}{\left(\sqrt{2R}\right)^5}\left[\frac{\partial}{\partial x_0}(R^2)-i\frac{\partial}{\partial x_1}(R^2)\right]$$

$$R=\sqrt{{X_1}^2+{X_2}^2+{X_3}^2}$$

$$=\sqrt{\left(\frac{1}{2}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\right)^2-\left(\frac{1}{2}(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\right)^2+\left(\frac{1}{2}(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\right)^2}$$

$$\frac{\partial}{\partial x_0}(R^2)=\frac{\partial}{\partial x_0}\left(\left(\frac{1}{2}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\right)^2-\left(\frac{1}{2}(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\right)^2\right.$$

$$\left.+\left(\frac{1}{2}(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\right)^2\right)$$

$$=\frac{1}{4}\left(2(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\frac{\partial}{\partial x_0}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\right.$$

$$-2(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\frac{\partial}{\partial x_0}(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)$$

$$\left.+2(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\frac{\partial}{\partial x_0}(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\right)$$

$$=\frac{1}{4}(2(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_1}-2(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_1}$$

$$+2(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0})$$

$$=\frac{1}{2}((\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_1}-(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\overline{x_1}$$

$$+(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0})$$

$$=\frac{1}{2}((\overline{x_0}x_1+\overline{x_2}x_3)\overline{x_1}-(-\overline{x_0}x_1-\overline{x_2}x_3)\overline{x_1}+(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0})$$

$$=\frac{1}{2}((\overline{x_0}x_1+\overline{x_2}x_3)\overline{x_1}+(\overline{x_0}x_1+\overline{x_2}x_3)\overline{x_1}+(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0})$$

$$=\frac{1}{2}((\overline{x_0}x_1+\overline{x_2}x_3)\overline{x_1}+(\overline{x_2}x_3)\overline{x_1}+(\overline{x_0}x_0+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0})$$

$$=\frac{1}{2}(\overline{x_0}x_1\overline{x_1}+2\overline{x_2}x_3\overline{x_1}+(\overline{x_0}x_0+\overline{x_2}x_2-\overline{x_3}x_3)\overline{x_0})$$

$$=\frac{1}{2}(2\overline{x_2}x_3\overline{x_1}-2\overline{x_3}x_3\overline{x_0}+(\overline{x_0}x_0+x_1\overline{x_1}+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_0})$$

$$=\frac{1}{2}(2x_3(\overline{x_2}\overline{x_1}-\overline{x_3}\overline{x_0})+(\overline{x_0}x_0+x_1\overline{x_1}+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_0})$$

$$=\frac{1}{2}(2x_3(\overline{x_2}\overline{x_1}-\overline{x_3}\overline{x_0})+(\overline{x_0}x_0+x_1\overline{x_1}+\overline{x_2}x_2+\overline{x_3}x_3)\overline{x_0})$$

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$$\frac{\partial}{\partial x_1}(R^2) = \frac{1}{2}\big((\overline{x_0}x_1 + \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)\overline{x_0} + (-\overline{x_0}x_1 + \overline{x_1}x_0 - \overline{x_2}x_3 + \overline{x_3}x_2)\overline{x_0}$$
$$- (\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)\overline{x_1}\big)$$
$$= \frac{1}{2}\big((\overline{x_1}x_0 + \overline{x_3}x_2)\overline{x_0} + (\overline{x_1}x_0 + \overline{x_3}x_2)\overline{x_0}$$
$$- (\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)\overline{x_1}\big)$$
$$= \frac{1}{2}(\overline{x_1}x_0\overline{x_0} + 2\overline{x_3}x_2\overline{x_0} + (\overline{x_1}x_1 - \overline{x_2}x_2 + \overline{x_3}x_3)\overline{x_1})$$
$$= \frac{1}{2}(2x_2(\overline{x_3x_0} - \overline{x_2x_1}) + (x_0\overline{x_0} + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3)\overline{x_1})$$

Let's introduce the notations

$$\overline{x_2x_1} - \overline{x_3x_0} \equiv l$$

this quantity does not change under rotations and boosts and is some analog of the interval defined for Minkowski space and

$$\frac{1}{2}(x_0\overline{x_0} + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3) \equiv t$$

this quantity represents time in four-dimensional vector space.

$$\left[\frac{\partial}{\partial x_0}(R^2) - i\frac{\partial}{\partial x_1}(R^2)\right]$$
$$= \frac{1}{2}(2x_3(\overline{x_2x_1} - \overline{x_3x_0}) + (\overline{x_0}x_0 + x_1\overline{x_1} + \overline{x_2}x_2 + \overline{x_3}x_3)\overline{x_0})$$
$$- i\frac{1}{2}(2x_2(\overline{x_3x_0} - \overline{x_2x_1}) + (x_0\overline{x_0} + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3)\overline{x_1})$$
$$= x_3(\overline{x_2x_1} - \overline{x_3x_0}) + \frac{1}{2}(\overline{x_0}x_0 + x_1\overline{x_1} + \overline{x_2}x_2 + \overline{x_3}x_3)\overline{x_0}$$
$$- ix_2(\overline{x_3x_0} - \overline{x_2x_1}) - i\frac{1}{2}(x_0\overline{x_0} + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3)\overline{x_1}$$
$$= x_3 l + t\overline{x_0} + ix_2 l - it\overline{x_1} = l(x_3 + ix_2) + t(\overline{x_0} - i\overline{x_1})$$

As a result, we have an equation for determining the eigenvalues of m and their corresponding eigenfunctions $\varphi(x_0, x_1, x_2, x_3)$

$$\left(-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\right)\varphi + \frac{1}{\sqrt{2}R}\left(-\frac{\partial\varphi}{\partial x_0} + \frac{\partial\varphi}{\partial x_3} + i\frac{\partial\varphi}{\partial x_1} - i\frac{\partial\varphi}{\partial x_2}\right)$$
$$+ \frac{1}{(\sqrt{2}R)^5}\big(l(x_3 + ix_2) + t(\overline{x_0} - i\overline{x_1})\big)\varphi + \frac{1}{R}\varphi = m\varphi$$

Instead of looking for solutions to this equation directly, we can first try substituting already known solutions to the Schrödinger equation for the hydrogen-like atom. If $\varphi(X_0, X_1, X_2, X_3)$ is one of these solutions, we need to find its derivatives over all spinor components

$$\frac{\partial\varphi}{\partial x_\mu} = \frac{\partial\varphi}{\partial X_\nu}\frac{\partial X_\nu}{\partial x_\mu}$$

$$X_0 = \frac{1}{2}(\overline{x_0}x_0 + \overline{x_1}x_1 + \overline{x_2}x_2 + \overline{x_3}x_3)$$

$$X_1 = \frac{1}{2}(\overline{x_0}x_1 + \overline{x_1}x_0 + \overline{x_2}x_3 + \overline{x_3}x_2)$$

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$$X_2 = \frac{1}{2}(-i\overline{x_0}x_1 + i\overline{x_1}x_0 - i\overline{x_2}x_3 + i\overline{x_3}x_2)$$

$$X_3 = \frac{1}{2}(\overline{x_0}x_0 - \overline{x_1}x_1 + \overline{x_2}x_2 - \overline{x_3}x_3)$$

For example

$$\frac{\partial\varphi}{\partial x_0} = \frac{\partial\varphi}{\partial X_0}\frac{\overline{x_0}}{2} + \frac{\partial\varphi}{\partial X_1}\frac{\overline{x_1}}{2} + \frac{\partial\varphi}{\partial X_2}\frac{i\overline{x_1}}{2} + \frac{\partial\varphi}{\partial X_3}\frac{\overline{x_0}}{2}$$

To account for the electron spin, we will further represent the electron wave function as a four-component spinor function of four-component spinor coordinates

$$\boldsymbol{\psi}(x_0, x_1, x_2, x_3) = \begin{pmatrix} \psi_0(x_0, x_1, x_2, x_3) \\ \psi_1(x_0, x_1, x_2, x_3) \\ \psi_2(x_0, x_1, x_2, x_3) \\ \psi_3(x_0, x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \varphi(x_0, x_1, x_2, x_3)$$

where the coefficients u_μ are some complex constants.

We will search for the solution of the wave equation considered in the first part of this paper

$$(S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3)(S_0P_0 + S_1P_1 + S_2P_2 + S_3P_3)\boldsymbol{\psi} = M^2\boldsymbol{\psi}$$

Let's express the left part through the components of the momentum spinor

$$P_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p}$$

$$P_0 = \frac{1}{2}\mathbf{p}^\dagger S_0 \mathbf{p} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3}) \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3)$$

$$P_1 = \frac{1}{2}\mathbf{p}^\dagger S_1 \mathbf{p} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3}) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3}) \begin{pmatrix} p_1 \\ p_0 \\ p_3 \\ p_2 \end{pmatrix}$$

$$= \frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2)$$

$$P_2 = \frac{1}{2}\mathbf{p}^\dagger S_2 \mathbf{p} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3}) \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3}) \begin{pmatrix} -ip_1 \\ ip_0 \\ -ip_3 \\ ip_2 \end{pmatrix}$$

$$= \frac{1}{2}(-i\overline{p_0}p_1 + i\overline{p_1}p_0 - i\overline{p_2}p_3 + i\overline{p_3}p_2)$$

$$P_3 = \frac{1}{2}\mathbf{p}^\dagger S_3 \mathbf{p} = \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \frac{1}{2}(\overline{p_0}, \overline{p_1}, \overline{p_2}, \overline{p_3}) \begin{pmatrix} p_0 \\ -p_1 \\ p_2 \\ -p_3 \end{pmatrix} = \frac{1}{2}(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3)$$

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$$P_0 - P_3 = \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3) - \frac{1}{2}(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3)$$
$$= \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3 - \overline{p_0}p_0 + \overline{p_1}p_1 - \overline{p_2}p_2 + \overline{p_3}p_3)$$
$$= \frac{1}{2}(\overline{p_1}p_1 + \overline{p_3}p_3 + \overline{p_1}p_1 + \overline{p_3}p_3) = \overline{p_1}p_1 + \overline{p_3}p_3$$
$$P_0 + P_3 = \frac{1}{2}(\overline{p_0}p_0 + \overline{p_1}p_1 + \overline{p_2}p_2 + \overline{p_3}p_3) + \frac{1}{2}(\overline{p_0}p_0 - \overline{p_1}p_1 + \overline{p_2}p_2 - \overline{p_3}p_3)$$
$$= \overline{p_0}p_0 + \overline{p_2}p_2$$
$$-P_1 + iP_2 = -\frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2) + i\frac{1}{2}(-i\overline{p_0}p_1 + i\overline{p_1}p_0 - i\overline{p_2}p_3 + i\overline{p_3}p_2)$$
$$= \frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2 + \overline{p_0}p_1 - \overline{p_1}p_0 + \overline{p_2}p_3 - \overline{p_3}p_2)$$
$$= \frac{1}{2}(\overline{p_0}p_1 + \overline{p_2}p_3 + \overline{p_0}p_1 + \overline{p_2}p_3) = \overline{p_0}p_1 + \overline{p_2}p_3$$
$$-P_1 - iP_2 = -\frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2) - i\frac{1}{2}(-i\overline{p_0}p_1 + i\overline{p_1}p_0 - i\overline{p_2}p_3 + i\overline{p_3}p_2)$$
$$= \frac{1}{2}(\overline{p_0}p_1 + \overline{p_1}p_0 + \overline{p_2}p_3 + \overline{p_3}p_2 - \overline{p_0}p_1 + \overline{p_1}p_0 - \overline{p_2}p_3 + \overline{p_3}p_2)$$
$$= \frac{1}{2}(\overline{p_1}p_0 + \overline{p_3}p_2 + \overline{p_1}p_0 + \overline{p_3}p_2) = \overline{p_1}p_0 + \overline{p_3}p_2$$
$$S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P_0 - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} P_1$$
$$- \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} P_2 - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} P_3$$
$$= \begin{pmatrix} P_0 - P_3 & -P_1 + iP_2 & 0 & 0 \\ -P_1 - iP_2 & P_0 + P_3 & 0 & 0 \\ 0 & 0 & P_0 - P_3 & -P_1 + iP_2 \\ 0 & 0 & -P_1 - iP_2 & P_0 + P_3 \end{pmatrix} =$$
$$= \begin{pmatrix} \overline{p_1}p_1 + \overline{p_3}p_3 & \overline{p_0}p_1 + \overline{p_2}p_3 & 0 & 0 \\ \overline{p_1}p_0 + \overline{p_3}p_2 & \overline{p_0}p_0 + \overline{p_2}p_2 & 0 & 0 \\ 0 & 0 & \overline{p_1}p_1 + \overline{p_3}p_3 & \overline{p_0}p_1 + \overline{p_2}p_3 \\ 0 & 0 & \overline{p_1}p_0 + \overline{p_3}p_2 & \overline{p_0}p_0 + \overline{p_2}p_2 \end{pmatrix}$$

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$$S_0P_0+S_1P_1+S_2P_2+S_3P_3$$
$$=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}P_0+\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}P_1$$
$$+\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}P_2+\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}P_3$$
$$=\begin{pmatrix}P_0+P_3&P_1-iP_2&0&0\\P_1+iP_2&P_0-P_3&0&0\\0&0&P_0+P_3&P_1-iP_2\\0&0&P_1+iP_2&P_0-P_3\end{pmatrix}$$
$$=\begin{pmatrix}\overline{p_0}p_0+\overline{p_2}p_2&-\overline{p_0}p_1-\overline{p_2}p_3&0&0\\-\overline{p_1}p_0-\overline{p_3}p_2&\overline{p_1}p_1+\overline{p_3}p_3&0&0\\0&0&\overline{p_0}p_0+\overline{p_2}p_2&-\overline{p_0}p_1-\overline{p_2}p_3\\0&0&-\overline{p_1}p_0-\overline{p_3}p_2&\overline{p_1}p_1+\overline{p_3}p_3\end{pmatrix}$$

Let's distinguish the direct products of vectors in these matrices

$$S_0P_0+S_1P_1+S_2P_2+S_3P_3=\begin{pmatrix}\overline{p_0}p_0+\overline{p_2}p_2&-\overline{p_0}p_1-\overline{p_2}p_3&0&0\\-\overline{p_1}p_0-\overline{p_3}p_2&\overline{p_1}p_1+\overline{p_3}p_3&0&0\\0&0&P_0+P_3&P_1-iP_2\\0&0&P_1+iP_2&P_0-P_3\end{pmatrix}$$
$$=\begin{pmatrix}\overline{p_0}p_0&-\overline{p_0}p_1&0&0\\-\overline{p_1}p_0&\overline{p_1}p_1&0&0\\0&0&\overline{p_0}p_0&-\overline{p_0}p_1\\0&0&-\overline{p_1}p_0&\overline{p_1}p_1\end{pmatrix}$$
$$+\begin{pmatrix}\overline{p_2}p_2&-\overline{p_2}p_3&0&0\\-\overline{p_3}p_2&\overline{p_3}p_3&0&0\\0&0&\overline{p_2}p_2&-\overline{p_2}p_3\\0&0&-\overline{p_3}p_2&\overline{p_3}p_3\end{pmatrix}$$
$$=\begin{pmatrix}-\overline{p_0}\\\overline{p_1}\\0\\0\end{pmatrix}(-p_0,p_1,0{,}0)+\begin{pmatrix}0\\0\\-\overline{p_0}\\\overline{p_1}\end{pmatrix}(0{,}0,-p_0,p_1)+\begin{pmatrix}-\overline{p_2}\\\overline{p_3}\\0\\0\end{pmatrix}(-p_2,p_3,0{,}0)$$
$$+\begin{pmatrix}0\\0\\-\overline{p_2}\\\overline{p_3}\end{pmatrix}(0{,}0,-p_2,p_3)$$

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$$S_0P_0 - S_1P_1 - S_2P_2 - S_3P_3 = \begin{pmatrix} \overline{p_1}p_1 + \overline{p_3}p_3 & \overline{p_0}p_1 + \overline{p_2}p_3 & 0 & 0 \\ \overline{p_1}p_0 + \overline{p_3}p_2 & \overline{p_0}p_0 + \overline{p_2}p_2 & 0 & 0 \\ 0 & 0 & \overline{p_1}p_1 + \overline{p_3}p_3 & \overline{p_0}p_1 + \overline{p_2}p_3 \\ 0 & 0 & \overline{p_1}p_0 + \overline{p_3}p_2 & \overline{p_0}p_0 + \overline{p_2}p_2 \end{pmatrix}$$

$$= \begin{pmatrix} \overline{p_1}p_1 & \overline{p_0}p_1 & 0 & 0 \\ \overline{p_1}p_0 & \overline{p_0}p_0 & 0 & 0 \\ 0 & 0 & \overline{p_1}p_1 & \overline{p_0}p_1 \\ 0 & 0 & \overline{p_1}p_0 & \overline{p_0}p_0 \end{pmatrix} + \begin{pmatrix} \overline{p_3}p_3 & \overline{p_2}p_3 & 0 & 0 \\ \overline{p_3}p_2 & \overline{p_2}p_2 & 0 & 0 \\ 0 & 0 & \overline{p_3}p_3 & \overline{p_2}p_3 \\ 0 & 0 & \overline{p_3}p_2 & \overline{p_2}p_2 \end{pmatrix}$$

$$= \begin{pmatrix} p_1\overline{p_1} - [p_1\overline{p_1} - \overline{p_1}p_1] & p_1\overline{p_0} - [p_1\overline{p_0} - \overline{p_0}p_1] & 0 & 0 \\ p_0\overline{p_1} - [p_0\overline{p_1} - \overline{p_1}p_0] & p_0\overline{p_0} - [p_0\overline{p_0} - \overline{p_0}p_0] & 0 & 0 \\ 0 & 0 & p_1\overline{p_1} - [p_1\overline{p_1} - \overline{p_1}p_1] & p_1\overline{p_0} - [p_1\overline{p_0} - \overline{p_0}p_1] \\ 0 & 0 & p_0\overline{p_1} - [p_0\overline{p_1} - \overline{p_1}p_0] & p_0\overline{p_0} - [p_0\overline{p_0} - \overline{p_0}p_0] \end{pmatrix}$$

$$+ \begin{pmatrix} p_3\overline{p_3} - [p_3\overline{p_3} - \overline{p_3}p_3] & p_3\overline{p_2} - [p_3\overline{p_2} - \overline{p_2}p_3] & 0 & 0 \\ p_2\overline{p_3} - [p_2\overline{p_3} - \overline{p_3}p_2] & p_2\overline{p_2} - [p_2\overline{p_2} - \overline{p_2}p_2] & 0 & 0 \\ 0 & 0 & p_3\overline{p_3} - [p_3\overline{p_3} - \overline{p_3}p_3] & p_3\overline{p_2} - [p_3\overline{p_2} - \overline{p_2}p_3] \\ 0 & 0 & p_2\overline{p_3} - [p_2\overline{p_3} - \overline{p_3}p_2] & p_2\overline{p_2} - [p_2\overline{p_2} - \overline{p_2}p_2] \end{pmatrix}$$

$$= \begin{pmatrix} p_1 \\ p_0 \\ 0 \\ 0 \end{pmatrix} (\overline{p_1}, \overline{p_0}, 0,0) + \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_0 \end{pmatrix} (0,0, \overline{p_1}, \overline{p_0})$$

$$- \begin{pmatrix} [p_1\overline{p_1} - \overline{p_1}p_1] & [p_1\overline{p_0} - \overline{p_0}p_1] & 0 & 0 \\ [p_0\overline{p_1} - \overline{p_1}p_0] & [p_0\overline{p_0} - \overline{p_0}p_0] & 0 & 0 \\ 0 & 0 & [p_1\overline{p_1} - \overline{p_1}p_1] & [p_1\overline{p_0} - \overline{p_0}p_1] \\ 0 & 0 & [p_0\overline{p_1} - \overline{p_1}p_0] & [p_0\overline{p_0} - \overline{p_0}p_0] \end{pmatrix} + \begin{pmatrix} p_3 \\ p_2 \\ 0 \\ 0 \end{pmatrix} (\overline{p_3}, \overline{p_2}, 0,0)$$

$$+ \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_2 \end{pmatrix} (0,0, \overline{p_3}, \overline{p_2}) - \begin{pmatrix} [p_3\overline{p_3} - \overline{p_3}p_3] & [p_3\overline{p_2} - \overline{p_2}p_3] & 0 & 0 \\ [p_2\overline{p_3} - \overline{p_3}p_2] & [p_2\overline{p_2} - \overline{p_2}p_2] & 0 & 0 \\ 0 & 0 & [p_3\overline{p_3} - \overline{p_3}p_3] & [p_3\overline{p_2} - \overline{p_2}p_3] \\ 0 & 0 & [p_2\overline{p_3} - \overline{p_3}p_2] & [p_2\overline{p_2} - \overline{p_2}p_2] \end{pmatrix}$$

Let's introduce the notations

$$\begin{pmatrix} -\overline{p_0} \\ \overline{p_1} \\ 0 \\ 0 \end{pmatrix} (-p_0, p_1, 0,0) + \begin{pmatrix} 0 \\ 0 \\ -\overline{p_0} \\ \overline{p_1} \end{pmatrix} (0,0,-p_0, p_1) + \begin{pmatrix} -\overline{p_2} \\ \overline{p_3} \\ 0 \\ 0 \end{pmatrix} (-p_2, p_3, 0,0)$$

$$+ \begin{pmatrix} 0 \\ 0 \\ -\overline{p_2} \\ \overline{p_3} \end{pmatrix} (0,0,-p_2, p_3) \equiv S^+$$

$$\begin{pmatrix} p_1 \\ p_0 \\ 0 \\ 0 \end{pmatrix} (\overline{p_1}, \overline{p_0}, 0,0) + \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_0 \end{pmatrix} (0,0, \overline{p_1}, \overline{p_0}) + \begin{pmatrix} p_3 \\ p_2 \\ 0 \\ 0 \end{pmatrix} (\overline{p_3}, \overline{p_2}, 0,0) + \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_2 \end{pmatrix} (0,0, \overline{p_3}, \overline{p_2}) \equiv S^-$$

$$\begin{pmatrix} [p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1] & 0 & 0 \\ [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0] & 0 & 0 \\ 0 & 0 & [p_1\overline{p_1}-\overline{p_1}p_1] & [p_1\overline{p_0}-\overline{p_0}p_1] \\ 0 & 0 & [p_0\overline{p_1}-\overline{p_1}p_0] & [p_0\overline{p_0}-\overline{p_0}p_0] \end{pmatrix}$$
$$+\begin{pmatrix} [p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3] & 0 & 0 \\ [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2] & 0 & 0 \\ 0 & 0 & [p_3\overline{p_3}-\overline{p_3}p_3] & [p_3\overline{p_2}-\overline{p_2}p_3] \\ 0 & 0 & [p_2\overline{p_3}-\overline{p_3}p_2] & [p_2\overline{p_2}-\overline{p_2}p_2] \end{pmatrix}$$
$$\equiv K$$

Let us substitute differential operators instead of spinor components

$$p_0 \to \frac{\partial}{\partial x_1} \quad p_1 \to -\frac{\partial}{\partial x_0} \quad p_2 \to \frac{\partial}{\partial x_3} \quad p_3 \to -\frac{\partial}{\partial x_2}$$
$$\overline{p_0} \to \frac{\partial\overline{[\,]}}{\partial\overline{x_1}} \quad \overline{p_1} \to -\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} \quad \overline{p_2} \to \frac{\partial\overline{[\,]}}{\partial\overline{x_3}} \quad \overline{p_3} \to -\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}$$

Then the quantities included in the wave equation

$$(S^- - K)S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3) = M^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

will have the form

$$S^- = \begin{pmatrix} -\frac{\partial}{\partial x_0} \\ \frac{\partial}{\partial x_1} \\ 0 \\ 0 \end{pmatrix}\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}},\frac{\partial\overline{[\,]}}{\partial\overline{x_1}},0,0\right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_0} \\ \frac{\partial}{\partial x_1} \end{pmatrix}\left(0,0,-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}},\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}\right) + \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ 0 \\ 0 \end{pmatrix}\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}},\frac{\partial\overline{[\,]}}{\partial\overline{x_3}},0,0\right)$$
$$+\begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}\left(0,0,-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}},\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}\right)$$

$$S^+ = \begin{pmatrix} -\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} \\ -\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} \\ 0 \\ 0 \end{pmatrix}\left(-\frac{\partial}{\partial x_1},-\frac{\partial}{\partial x_0},0,0\right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} \\ -\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} \end{pmatrix}\left(0,0,-\frac{\partial}{\partial x_1},-\frac{\partial}{\partial x_0}\right) + \begin{pmatrix} -\frac{\partial\overline{[\,]}}{\partial\overline{x_3}} \\ -\frac{\partial\overline{[\,]}}{\partial\overline{x_2}} \\ 0 \\ 0 \end{pmatrix}\left(-\frac{\partial}{\partial x_3},-\frac{\partial}{\partial x_2},0,0\right)$$
$$+\begin{pmatrix} 0 \\ 0 \\ -\frac{\partial\overline{[\,]}}{\partial\overline{x_3}} \\ -\frac{\partial\overline{[\,]}}{\partial\overline{x_2}} \end{pmatrix}\left(0,0,-\frac{\partial}{\partial x_3},-\frac{\partial}{\partial x_2}\right)$$

$K =$

$$= \begin{pmatrix} \begin{matrix} \left(-\frac{\partial}{\partial x_0}\right)-\frac{\partial[]}{\partial \bar{x}_0}--\frac{\partial[]}{\partial \bar{x}_0}\left(-\frac{\partial}{\partial x_0}\right) & \left(-\frac{\partial}{\partial x_0}\right)\frac{\partial[]}{\partial \bar{x}_1}-\frac{\partial[]}{\partial \bar{x}_1}\left(-\frac{\partial}{\partial x_0}\right) \\ \left(\frac{\partial}{\partial x_1}\right)-\frac{\partial[]}{\partial \bar{x}_0}--\frac{\partial[]}{\partial \bar{x}_0}\left(\frac{\partial}{\partial x_1}\right) & \left(\frac{\partial}{\partial x_1}\right)\frac{\partial[]}{\partial \bar{x}_1}-\frac{\partial[]}{\partial \bar{x}_1}\left(\frac{\partial}{\partial x_1}\right) \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \left(-\frac{\partial}{\partial x_0}\right)-\frac{\partial[]}{\partial \bar{x}_0}--\frac{\partial[]}{\partial \bar{x}_0}\left(-\frac{\partial}{\partial x_0}\right) & \left(-\frac{\partial}{\partial x_0}\right)\frac{\partial[]}{\partial \bar{x}_1}-\frac{\partial[]}{\partial \bar{x}_1}\left(-\frac{\partial}{\partial x_0}\right) \\ \left(\frac{\partial}{\partial x_1}\right)-\frac{\partial[]}{\partial \bar{x}_0}--\frac{\partial[]}{\partial \bar{x}_0}\left(\frac{\partial}{\partial x_1}\right) & \left(\frac{\partial}{\partial x_1}\right)\frac{\partial[]}{\partial \bar{x}_1}-\frac{\partial[]}{\partial \bar{x}_1}\left(\frac{\partial}{\partial x_1}\right) \end{matrix} \end{pmatrix}$$

$$+ \begin{pmatrix} \begin{matrix} \left(-\frac{\partial}{\partial x_2}\right)-\frac{\partial[]}{\partial \bar{x}_2}--\frac{\partial[]}{\partial \bar{x}_2}\left(-\frac{\partial}{\partial x_2}\right) & \left(-\frac{\partial}{\partial x_2}\right)\frac{\partial[]}{\partial \bar{x}_3}-\frac{\partial[]}{\partial \bar{x}_3}\left(-\frac{\partial}{\partial x_2}\right) \\ \left(\frac{\partial}{\partial x_3}\right)-\frac{\partial[]}{\partial \bar{x}_2}--\frac{\partial[]}{\partial \bar{x}_2}\left(\frac{\partial}{\partial x_3}\right) & \left(\frac{\partial}{\partial x_3}\right)\frac{\partial[]}{\partial \bar{x}_3}-\frac{\partial[]}{\partial \bar{x}_3}\left(\frac{\partial}{\partial x_3}\right) \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \left(-\frac{\partial}{\partial x_2}\right)-\frac{\partial[]}{\partial \bar{x}_2}--\frac{\partial[]}{\partial \bar{x}_2}\left(-\frac{\partial}{\partial x_2}\right) & \left(-\frac{\partial}{\partial x_2}\right)\frac{\partial[]}{\partial \bar{x}_3}-\frac{\partial[]}{\partial \bar{x}_3}\left(-\frac{\partial}{\partial x_2}\right) \\ \left(\frac{\partial}{\partial x_3}\right)-\frac{\partial[]}{\partial \bar{x}_2}--\frac{\partial[]}{\partial \bar{x}_2}\left(\frac{\partial}{\partial x_3}\right) & \left(\frac{\partial}{\partial x_3}\right)\frac{\partial[]}{\partial \bar{x}_3}-\frac{\partial[]}{\partial \bar{x}_3}\left(\frac{\partial}{\partial x_3}\right) \end{matrix} \end{pmatrix}$$

Let us consider the case of a free particle and represent the electron field as a four-component spinor function of four-component spinor coordinates

$$\boldsymbol{\psi}(x_0,x_1,x_2,x_3) = \begin{pmatrix} \psi_0(x_0,x_1,x_2,x_3) \\ \psi_1(x_0,x_1,x_2,x_3) \\ \psi_2(x_0,x_1,x_2,x_3) \\ \psi_3(x_0,x_1,x_2,x_3) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \varphi(x_0,x_1,x_2,x_3)$$

For a free particle, the components of the momentum spinor commute with each other, so all components of the matrix K are zero.

Let us use the model of a plane wave in spinor space

$$\varphi(x_0,x_1,x_2,x_3) = exp(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)$$

Substituting the plane wave solution into the differential equation, we obtain the algebraic equation

$$S^-S^+ \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \varphi(x_0,x_1,x_2,x_3) = M^2 \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \varphi(x_0,x_1,x_2,x_3)$$

$$S^- \left\{ \begin{pmatrix} -\overline{p_0} \\ \overline{p_1} \\ 0 \\ 0 \end{pmatrix} (-p_0u_0 + p_1u_1) + \begin{pmatrix} 0 \\ 0 \\ -\overline{p_0} \\ \overline{p_1} \end{pmatrix} (-p_0u_2 + p_1u_3) + \begin{pmatrix} -\overline{p_2} \\ \overline{p_3} \\ 0 \\ 0 \end{pmatrix} (-p_2u_0 + p_3u_1) \right.$$

$$\left. + \begin{pmatrix} 0 \\ 0 \\ -\overline{p_2} \\ \overline{p_3} \end{pmatrix} (-p_2u_2 + p_3u_3) \right\} \varphi(x_0,x_1,x_2,x_3) = m^2 \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \varphi(x_0,x_1,x_2,x_3)$$

$$\left\{ \begin{pmatrix} p_1 \\ p_0 \\ 0 \\ 0 \end{pmatrix} (\overline{p_1}, \overline{p_0}, 0,0) + \begin{pmatrix} 0 \\ 0 \\ p_1 \\ p_0 \end{pmatrix} (0,0,\overline{p_1},\overline{p_0}) + \begin{pmatrix} p_3 \\ p_2 \\ 0 \\ 0 \end{pmatrix} (\overline{p_3},\overline{p_2},0,0) + \begin{pmatrix} 0 \\ 0 \\ p_3 \\ p_2 \end{pmatrix} (0,0,\overline{p_3},\overline{p_2}) \right\}$$

$$\left\{\begin{pmatrix}-\overline{p_0}\\ \overline{p_1}\\ 0\\ 0\end{pmatrix}(-p_0u_0+p_1u_1)+\begin{pmatrix}0\\ 0\\ -\overline{p_0}\\ \overline{p_1}\end{pmatrix}(-p_0u_2+p_1u_3)+\begin{pmatrix}-\overline{p_2}\\ \overline{p_3}\\ 0\\ 0\end{pmatrix}(-p_2u_0+p_3u_1)\right.$$

$$\left.+\begin{pmatrix}0\\ 0\\ -\overline{p_2}\\ \overline{p_3}\end{pmatrix}(-p_2u_2+p_3u_3)\right\}\varphi(x_0,x_1,x_2,x_3)=M^2\begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix}\varphi(x_0,x_1,x_2,x_3)$$

$$\left\{\begin{pmatrix}p_1\\ p_0\\ 0\\ 0\end{pmatrix}(\overline{p_1},\overline{p_0},0,0)+\begin{pmatrix}0\\ 0\\ p_1\\ p_0\end{pmatrix}(0,0,\overline{p_1},\overline{p_0})+\begin{pmatrix}p_3\\ p_2\\ 0\\ 0\end{pmatrix}(\overline{p_3},\overline{p_2},0,0)+\begin{pmatrix}0\\ 0\\ p_3\\ p_2\end{pmatrix}(0,0,\overline{p_3},\overline{p_2})\right\}$$

$$\left\{\begin{pmatrix}-\overline{p_0}\\ \overline{p_1}\\ 0\\ 0\end{pmatrix}(-p_0u_0+p_1u_1)+\begin{pmatrix}0\\ 0\\ -\overline{p_0}\\ \overline{p_1}\end{pmatrix}(-p_0u_2+p_1u_3)+\begin{pmatrix}-\overline{p_2}\\ \overline{p_3}\\ 0\\ 0\end{pmatrix}(-p_2u_0+p_3u_1)\right.$$

$$\left.+\begin{pmatrix}0\\ 0\\ -\overline{p_2}\\ \overline{p_3}\end{pmatrix}(-p_2u_2+p_3u_3)\right\}=M^2\begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix}$$

$$\left\{\begin{pmatrix}p_1\\ p_0\\ 0\\ 0\end{pmatrix}(-\overline{p_1p_0}+\overline{p_0p_1})(-p_0u_0+p_1u_1)+\begin{pmatrix}p_3\\ p_2\\ 0\\ 0\end{pmatrix}(-\overline{p_3p_0}+\overline{p_2p_1})(-p_0u_0+p_1u_1)\right.$$

$$+\begin{pmatrix}0\\ 0\\ p_1\\ p_0\end{pmatrix}(-\overline{p_1p_0}+\overline{p_0p_1})(-p_0u_2+p_1u_3)$$

$$+\begin{pmatrix}0\\ 0\\ p_3\\ p_2\end{pmatrix}(-\overline{p_3p_0}+\overline{p_2p_1})(-p_0u_2+p_1u_3)$$

$$+\begin{pmatrix}p_1\\ p_0\\ 0\\ 0\end{pmatrix}(-\overline{p_1p_2}+\overline{p_0p_3})(-p_2u_0+p_3u_1)$$

$$+\begin{pmatrix}p_3\\ p_2\\ 0\\ 0\end{pmatrix}(-\overline{p_3p_2}+\overline{p_2p_3})(-p_2u_0+p_3u_1)$$

$$+\begin{pmatrix}0\\ 0\\ p_1\\ p_0\end{pmatrix}(-\overline{p_1p_2}+\overline{p_0p_3})(-p_2u_2+p_3u_3)$$

$$\left.+\begin{pmatrix}0\\ 0\\ p_3\\ p_2\end{pmatrix}(-\overline{p_3p_2}+\overline{p_2p_3})(-p_2u_2+p_3u_3)\right\}=M^2\begin{pmatrix}u_0\\ u_1\\ u_2\\ u_3\end{pmatrix}$$

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Let us take into account the commutativity of the momentum components, besides, let us introduce the notations

$$-\overline{p_3 p_0}+\overline{p_2 p_1}\equiv\overline{m}\qquad -\overline{p_1 p_2}+\overline{p_0 p_3}\equiv-\overline{m}$$

for the quantities which are invariant under any rotations and boosts, then we obtain

$$\left\{\begin{pmatrix}p_3\\p_2\\0\\0\end{pmatrix}\overline{m}(-p_0u_0+p_1u_1)+\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}\overline{m}(-p_0u_2+p_1u_3)+\begin{pmatrix}p_1\\p_0\\0\\0\end{pmatrix}(-\overline{m})(-p_2u_0+p_3u_1)\right.$$
$$\left.+\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}(-\overline{m})(-p_2u_2+p_3u_3)\right\}=M^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

$$\left\{u_0\left(-\begin{pmatrix}p_3\\p_2\\0\\0\end{pmatrix}\overline{m}p_0+\begin{pmatrix}p_1\\p_0\\0\\0\end{pmatrix}\overline{m}p_2\right)+u_1\left(\begin{pmatrix}p_3\\p_2\\0\\0\end{pmatrix}\overline{m}p_1-\begin{pmatrix}p_1\\p_0\\0\\0\end{pmatrix}\overline{m}p_3\right)\right.$$
$$\left.+u_2\left(-\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}\overline{m}p_0+\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}\overline{m}p_2\right)+u_3\left(\begin{pmatrix}0\\0\\p_3\\p_2\end{pmatrix}\overline{m}p_1-\begin{pmatrix}0\\0\\p_1\\p_0\end{pmatrix}\overline{m}p_3\right)\right\}$$
$$=M^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

$$\left\{u_0\overline{m}\begin{pmatrix}p_1p_2-p_3p_0\\p_0p_2-p_2p_0\\0\\0\end{pmatrix}+u_1\overline{m}\begin{pmatrix}p_3p_1-p_1p_3\\p_2p_1-p_0p_3\\0\\0\end{pmatrix}+u_2\overline{m}\begin{pmatrix}0\\0\\p_1p_2-p_3p_0\\p_0p_2-p_2p_0\end{pmatrix}\right.$$
$$\left.+u_3\overline{m}\begin{pmatrix}0\\0\\p_3p_1-p_1p_3\\p_2p_1-p_0p_3\end{pmatrix}\right\}=M^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

Additionally, introducing notation for Lorentz invariant quantities

$$p_1p_2-p_3p_0\equiv m\qquad p_2p_1-p_0p_3\equiv m$$

we obtain

$$\left\{u_0\overline{m}\begin{pmatrix}m\\0\\0\\0\end{pmatrix}+u_1\overline{m}\begin{pmatrix}0\\m\\0\\0\end{pmatrix}+u_2\overline{m}\begin{pmatrix}0\\0\\m\\0\end{pmatrix}+u_3\overline{m}\begin{pmatrix}0\\0\\0\\m\end{pmatrix}\right\}=m^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

$$\left\{u_0\begin{pmatrix}m^2\\0\\0\\0\end{pmatrix}+u_1\begin{pmatrix}0\\m^2\\0\\0\end{pmatrix}+u_2\begin{pmatrix}0\\0\\m^2\\0\end{pmatrix}+u_3\begin{pmatrix}0\\0\\0\\m^2\end{pmatrix}\right\}=m^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

$$\begin{pmatrix}m^2 & 0 & 0 & 0\\0 & m^2 & 0 & 0\\0 & 0 & m^2 & 0\\0 & 0 & 0 & m^2\end{pmatrix}\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}=M^2\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}$$

We see that in the case of a plane wave in spinor space, the matrix in the left part of the equation is diagonal and remains so at any rotations and boosts, the diagonal element also does not change.

In this case we can consider the matrix M^2 in the right part to be diagonal with the same elements on the diagonal m^2, then the equation can be rewritten as an equation for the problem of finding eigenvalues and eigenfunctions

$$S^-S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3)=m^2I\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

$$S^-S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3)=m^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

Let us compare our equation with the Dirac equation [6, formula (43.16)]

$$\begin{pmatrix}P_0+M & 0 & P_3 & P_1-iP_2\\0 & P_0+M & P_1+iP_2 & -P_3\\P_3 & P_1-iP_2 & P_0-M & 0\\P_1+iP_2 & -P_3 & 0 & P_0-M\end{pmatrix}\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}=0$$

In the rest frame of reference, the three components of momentum are zero and the equation is simplified

$$\begin{pmatrix}P_0+M & 0 & 0 & 0\\0 & P_0+M & 0 & 0\\0 & 0 & P_0-M & 0\\0 & 0 & 0 & P_0-M\end{pmatrix}\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}=0$$

That is, in the rest frame the Dirac equation and the spinor equation analyzed by us look identically and contain a diagonal matrix. The corresponding problem on eigenvalues and eigenvectors of these matrices has degenerate eigenvalues, which correspond to the linear space of eigenfunctions. In this space, one can choose an orthogonal basis of linearly independent functions, and this choice is quite arbitrary. For example, in [9], formula (2.127), solutions in the form of plane waves in the vector space have been proposed for the Dirac equation in the rest frame

$$u^i(0)\,exp(-iMt)$$

$$v^i(0)\,exp(+iMt)$$

and the following spinors are chosen as basis vectors

$$u^1(0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad u^2(0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad v^1(0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\quad v^2(0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

For transformation to a moving coordinate system in [9], formula (2.133) the following formula is used

$$\psi^i(X)=u^i(P)\,exp(-iPX)$$

$$\psi^i(X)=v^i(P)\,exp(+iPX)$$

where

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$$u^1(P) = \sqrt{\frac{P_0+M}{2M}}\begin{pmatrix} 1 \\ 0 \\ \frac{P_3}{P_0+M} \\ \frac{P_1+iP_2}{P_0+M} \end{pmatrix} \quad u^2(P) = \sqrt{\frac{P_0+M}{2M}}\begin{pmatrix} 1 \\ 0 \\ \frac{P_1-iP_2}{P_0+M} \\ \frac{-P_3}{P_0+M} \end{pmatrix} \quad v^1(P)$$

$$= \sqrt{\frac{P_0+M}{2M}}\begin{pmatrix} \frac{P_3}{P_0+M} \\ \frac{P_1+iP_2}{P_0+M} \\ 1 \\ 0 \end{pmatrix}$$

$$v^2(P) = \sqrt{\frac{P_0+M}{2M}}\begin{pmatrix} \frac{P_1-iP_2}{P_0+M} \\ \frac{-P_3}{P_0+M} \\ 0 \\ 1 \end{pmatrix}$$

The basis spinors form a complete system, that is, any four-component complex spinor can be represented as their linear combination and this arbitrary spinor will be a solution to the problem on eigenvalues and eigenfunctions in a resting coordinate system. The choice of the given particular basis has disadvantages, because if to find a four-dimensional current vector from any of these basis functions

$$j_\mu = \frac{1}{2}\left(u^1(0)\right)^\dagger S_\mu u^1(0)$$

then this current in the rest frame of reference

$$\mathbf{j}^\mathbf{T} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

has non-zero components, and the square of the length of the current vector is zero. It turns out that a resting electron creates a current, which contradicts physical common sense. Since we have freedom of choice of the basis, it is reasonable to choose the spinor of the wave function in the rest frame of reference proportional to the momentum spinor, for example

$$u(0) = \sqrt{\frac{e}{m}}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

The proportionality factor is chosen so that in the rest frame the zero component of the current is equal to the charge of, for example, an electron or a positron. If the momentum spinor in the rest frame has the form

$$\mathbf{p}^\mathbf{T} = (p_0, p_1, \overline{p_1}, -\overline{p_0})$$

then the momentum vector in this rest frame of reference will be

$$\mathbf{P}^\mathbf{T} = (m, 0,0,0)$$

and the current vector

$$\mathbf{j}^\mathbf{T} = (e, 0,0,0)$$

The same momentum vector in the rest frame of reference can be obtained from different spinors, e.g,

$$\mathbf{p1}=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\quad \mathbf{p2}=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\quad \mathbf{p3}=\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\quad \mathbf{p4}=\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}\qquad \mathbf{P}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

after a 30-degree boost along the z-axis we get

$$\mathbf{p1}=\begin{pmatrix}1.299\\0\\0\\0.77\end{pmatrix}\quad \mathbf{p2}=\begin{pmatrix}0\\0.77\\1.299\\0\end{pmatrix}\quad \mathbf{p3}=\begin{pmatrix}-1.299\\0\\0\\0.77\end{pmatrix}\quad \mathbf{p4}=\begin{pmatrix}0\\-0.77\\1.299\\0\end{pmatrix}\qquad \mathbf{P}$$

$$=\begin{pmatrix}1.14\\0\\0\\0.548\end{pmatrix}$$

After scaling the spinors by the factor $\sqrt{\frac{e}{m}}$, similar relations are true for the current vector. Thus, electrons can have the same momentum and current vector but different spinors, i.e., they are characterized by different spins. As it is supposed, the electron here has two physical degrees of freedom, since in a stationary frame of reference one can choose the components p_0 and p_1 to be real.

Thus defined spinor wave function for a free particle is invariant to Lorentz transformations, since in this case the mass of electron $m = p_1p_2 - p_3p_0$, its charge and the phase of the plane spinor wave

$$exp(\mathbf{p}^T\Sigma_{MM}\mathbf{x}) = exp(p_0x_1 - p_1x_0 + p_2x_3 - p_3x_2)$$

do not change at rotations and boosts. The matrix on the left side of the equation does not change either, remaining diagonal with m^2 on the diagonal.

It is logical to use the same considerations when choosing the basis for the wave function of the photon, whose mass, i.e., the eigenvalues of the wave function equation, are also degenerate and thus equal to zero. In this case, the choice of the proportionality factor between the spinor of the wave function and the momentum spinor is not so obvious, one can, for example, consider the option of

$$u(0) = \sqrt{e}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix}$$

For a fermion, which can be an electron or a positron takes place $\mathbf{p}^\mathbf{T} = (p_0, p_1, \overline{p_1}, -\overline{p_0})$, so the quantity

$$m = p_1p_2 - p_3p_0 = p_1\overline{p_1} + \overline{p_0}p_0$$

which, unlike the mass M in the Dirac equation, is complex in the general case, is also valid for the fermion and can be positive for the electron or negative for the positron. For the momentum spinor of a boson, such as a photon, it is true that $\mathbf{p}^\mathbf{T} = (p_0, p_1, p_0, p_1)$, so its mass is zero

$$m = p_1p_2 - p_3p_0 = p_1p_0 - p_1p_0 = 0$$

The given constructions are not abstract, but describe the physical reality, since the results of the processes occurring in the spinor space are displayed in the Minkowski vector space. In particular, the momentum vector corresponding to the momentum spinor has the following parameters

$$P_\mu = \frac{1}{2}Tr[\mathbf{p}\mathbf{p}^\dagger S_\mu]$$

the square of the length is equal to the square of the mass of the electron or positron

$${P_0}^2 - {P_1}^2 - {P_2}^2 - {P_3}^2 = m^2$$

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And to the spinor wave function $\boldsymbol{\psi}(x_0, x_1, x_2, x_3)$ at some point in spinor space corresponds the vector wave function $\boldsymbol{\Psi}(X_0, X_1, X_2, X_3)$

$$\Psi_\mu = \frac{1}{2} Tr[\boldsymbol{\psi}\boldsymbol{\psi}^\dagger S_\mu]$$

(which for a plane wave coincides with the current vector), taking its value in the corresponding point of physical space with coordinates

$$X_\mu = \frac{1}{2} Tr[\mathbf{x}\mathbf{x}^\dagger S_\mu]$$

The vector wave function $\boldsymbol{\Psi}$ can be compared in meaning to the square of the modulus of the conventional scalar wave function, in particular Ψ_0 is equal to this square and has the meaning of probability. The conventional scalar wave function itself is closer in meaning to the spinor wave function considered here, they both have complex values, and the four-component wave functions of the electron have in both cases the same meaning.

The arbitrary choice of the basis of the linear space of the eigenvectors of the matrix takes place only for a free particle. In the general case the matrix K is not zero, the wave equation has no solution in the form of plane waves in spinor space and ceases to be invariant with respect to Lorentz transformations, and the eigenvalues become nondegenerate.

We propose to extend the scope of applicability of the presented equation consisting of differential operators in the form of partial derivatives on the components of coordinate spinors with a nonzero matrix K

$$(S^- - K)S^+\boldsymbol{\psi}(x_0, x_1, x_2, x_3) = M^2\boldsymbol{\psi}(x_0, x_1, x_2, x_3)$$

not only to the case of a plane wave, but to any situation in general. This transition is analogous to the transition from the application of the Schrödinger equation to a plane wave in vector space to its application in a general situation. The legitimacy of such transitions should be confirmed by the results of experiments.

This equation will be called the equation for the spinor wave function defined on the spinor coordinate space. Here the matrix M^2 is, generally speaking, neither diagonal nor real, but it does not depend on the coordinates and is determined solely by the parameters of the electromagnetic field. Only in the case of a plane wave it is diagonal and has on the diagonal the square of the mass of the free particle. We can try to simplify the problem and require that the matrix M^2 is diagonal with the same elements on the diagonal m^2, then the equation can be rewritten in the form of the equation for the problem of search of eigenvalues and eigenfunctions for any quantum states

$$(S^- - K)S^+\boldsymbol{\psi}(x_0, x_1, x_2, x_3) = m^2\boldsymbol{\psi}(x_0, x_1, x_2, x_3)$$

This approach is pleasant in the Dirac equation, where the mass is fixed and equated to the mass of a free particle, and at the same time results giving good agreement with experiment are obtained.

We are of the opinion that the spinor equation is more fundamental than the relativistic Schrödinger and Dirac equations, it is not a generalization of them, it is a refinement of them, because it describes nature at the spinor level, and hence is more precise and detailed than the equations for the wave function defined on the vector space.

Let us consider the proposed equation for the special case when the particle is in an external electromagnetic field, which we will also represent by a four-component spinor function at a point of the spinor coordinate space

$$\mathbf{a}(x_0, x_1, x_2, x_3) = \begin{pmatrix} a_0(x_0, x_1, x_2, x_3) \\ a_1(x_0, x_1, x_2, x_3) \\ a_2(x_0, x_1, x_2, x_3) \\ a_3(x_0, x_1, x_2, x_3) \end{pmatrix}$$

We will apply to the wave function of the electron the operators corresponding to the components of the momentum spinor, putting for simplicity the electron charge equal to unity

$$p_0 \to \frac{\partial}{\partial x_1} + a_0(x_0, x_1, x_2, x_3) \qquad p_1 \to -\frac{\partial}{\partial x_0} + a_1(x_0, x_1, x_2, x_3)$$

$$p_2 \to \frac{\partial}{\partial x_3} + a_2(x_0, x_1, x_2, x_3) \qquad p_3 \to -\frac{\partial}{\partial x_2} + a_3(x_0, x_1, x_2, x_3)$$
$$\overline{p_0} \to \frac{\partial \overline{[\,]}}{\partial \overline{x_1}} + \overline{a_0(x_0, x_1, x_2, x_3)} \qquad \overline{p_1} \to -\frac{\partial \overline{[\,]}}{\partial \overline{x_0}} + \overline{a_1(x_0, x_1, x_2, x_3)}$$
$$\overline{p_2} \to \frac{\partial \overline{[\,]}}{\partial \overline{x_3}} + \overline{a_2(x_0, x_1, x_2, x_3)} \qquad \overline{p_3} \to -\frac{\partial \overline{[\,]}}{\partial \overline{x_2}} + \overline{a_3(x_0, x_1, x_2, x_3)}$$

Note that the electromagnetic potential vector can be calculated from the electromagnetic potential spinor by the standard formula

$$A_\mu = \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{a}$$

The advantage of the spinor description over the vector description is that instead of summing up the components of the momentum and electromagnetic potential vectors as is usually done

$$P_\mu + A_\mu = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p} + \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{a}$$

now we sum the spinor components and then the resulting vector is

$$\frac{1}{2}(\mathbf{p}+\mathbf{a})^\dagger S_\mu(\mathbf{p}+\mathbf{a}) = \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{p} + \frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{a} + \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{p} + \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{a}$$

in addition to the usual momentum and field vectors, contains an additional term

$$\frac{1}{2}\mathbf{p}^\dagger S_\mu \mathbf{a} + \frac{1}{2}\mathbf{a}^\dagger S_\mu \mathbf{p}$$

taking real values and describing the mutual influence of the fields of the electron and photon.

After the addition of the electromagnetic field the components of the momentum spinor do not commute, the corresponding commutators are found above

$$\left\{\left(\frac{\partial}{\partial x_1} + a_0\right)\left(-\frac{\partial}{\partial x_0} + a_1\right) - \left(-\frac{\partial}{\partial x_0} + a_1\right)\left(\frac{\partial}{\partial x_1} + a_0\right)\right\}\varphi = \left\{\frac{\partial a_1}{\partial x_1} + \frac{\partial a_0}{\partial x_0}\right\}\varphi$$
$$\left\{\left(\frac{\partial}{\partial x_3} + a_2\right)\left(-\frac{\partial}{\partial x_2} + a_3\right) - \left(-\frac{\partial}{\partial x_2} + a_3\right)\left(\frac{\partial}{\partial x_3} + a_2\right)\right\}\varphi = \left\{\frac{\partial a_3}{\partial x_3} + \frac{\partial a_2}{\partial x_2}\right\}\varphi$$

Let's find commutators for other operators

$$\left\{\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\right\}\varphi$$

$$=\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\varphi-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\varphi=$$

$$=\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}+\overline{a_1}\varphi\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}+\overline{a_0}\varphi\right)$$

$$=\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+\overline{a_0a_1}\varphi+\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}(\overline{a_1}\varphi)+\overline{a_0}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}\left(\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}\right)-\overline{a_1a_0}\varphi+\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}(\overline{a_0}\varphi)$$

$$-\overline{a_1}\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}=\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}(\overline{a_1}\varphi)+\overline{a_0}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}(\overline{a_0}\varphi)-\overline{a_1}\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}$$

$$=\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}\overline{a_1}+\frac{\partial\overline{\overline{a_1}}}{\partial\overline{x_1}}\varphi+\overline{a_0}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\overline{a_0}+\frac{\partial\overline{\overline{a_0}}}{\partial\overline{x_0}}\varphi-\overline{a_1}\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}=\frac{\partial\overline{\overline{a_1}}}{\partial\overline{x_1}}\varphi+\frac{\partial\overline{\overline{a_0}}}{\partial\overline{x_0}}\varphi$$

$$=\left\{\frac{\partial a_1}{\partial\overline{x_1}}+\frac{\partial a_0}{\partial\overline{x_0}}\right\}\varphi$$

$$\left\{\left(\frac{\partial}{\partial x_1}+a_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\right\}\varphi$$

$$=\left(\frac{\partial}{\partial x_1}+a_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\varphi-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\varphi=$$

$$=\left(\frac{\partial}{\partial x_1}+a_0\right)\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}+\overline{a_1}\varphi\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial\varphi}{\partial x_1}+a_0\varphi\right)$$

$$=\frac{\partial}{\partial x_1}\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+a_0\overline{a_1}\varphi+\frac{\partial}{\partial x_1}(\overline{a_1}\varphi)+a_0\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}\left(\frac{\partial\varphi}{\partial x_1}\right)-\overline{a_1}a_0\varphi+\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}(a_0\varphi)$$

$$-\overline{a_1}\frac{\partial\varphi}{\partial x_1}=\frac{\partial}{\partial x_1}(\overline{a_1}\varphi)+a_0\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}(a_0\varphi)-\overline{a_1}\frac{\partial\varphi}{\partial x_1}$$

$$=\frac{\partial\varphi}{\partial x_1}\overline{a_1}+\frac{\partial\overline{a_1}}{\partial x_1}\varphi+a_0\left(-\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\bar{\varphi}}{\partial\overline{x_0}}a_0+\frac{\partial\overline{a_0}}{\partial\overline{x_0}}\varphi-\overline{a_1}\frac{\partial\bar{\varphi}}{\partial\overline{x_1}}=\frac{\partial\overline{a_1}}{\partial x_1}\varphi+\frac{\partial\overline{a_0}}{\partial\overline{x_0}}\varphi$$

$$=\left\{\frac{\partial\overline{a_1}}{\partial x_1}+\frac{\partial\overline{a_0}}{\partial\overline{x_0}}\right\}\varphi$$

Further we will use these and analogous relations

$$\left\{\left(\frac{\partial}{\partial x_1}+a_0\right)\left(-\frac{\partial}{\partial x_0}+a_1\right)-\left(-\frac{\partial}{\partial x_0}+a_1\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\right\}\varphi=\left\{\frac{\partial a_1}{\partial x_1}+\frac{\partial a_0}{\partial x_0}\right\}\varphi$$

$$\left\{\left(\frac{\partial}{\partial x_1}+a_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\right\}\varphi=\left\{\frac{\partial\overline{a_1}}{\partial x_1}+\frac{\partial\overline{a_0}}{\partial\overline{x_0}}\right\}\varphi$$

$$\left\{\left(-\frac{\partial}{\partial x_0}+a_1\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)-\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial}{\partial x_0}+a_1\right)\right\}\varphi=\left\{-\frac{\partial\overline{a_0}}{\partial x_0}-\frac{\partial\overline{a_1}}{\partial\overline{x_1}}\right\}\varphi$$

$$\left\{\left(-\frac{\partial}{\partial x_0}+a_1\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(-\frac{\partial}{\partial x_0}+a_1\right)\right\}\varphi=\left\{\left(-\frac{\partial\overline{a_1}}{\partial x_0}\right)+\frac{\partial\overline{a_1}}{\partial\overline{x_0}}\right\}\varphi$$

$$\left\{\left(\frac{\partial}{\partial x_1}+a_0\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)-\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\right\}\varphi=\left\{\left(\frac{\partial\overline{a_0}}{\partial x_1}\right)-\frac{\partial\overline{a_0}}{\partial\overline{x_1}}\right\}\varphi$$

$$\left\{\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\right\}\varphi=\left\{\frac{\partial a_1}{\partial\overline{x_1}}+\frac{\partial a_0}{\partial\overline{x_0}}\right\}\varphi$$

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$$\left(-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\right)\varphi=\left\{\frac{\overline{\partial a_1}}{\partial\overline{x_1}}+\frac{\overline{\partial a_0}}{\partial\overline{x_0}}\right\}\varphi$$

Let's solve the equation

$$(S^- - K)S^+\boldsymbol{\psi}(x_0,x_1,x_2,x_3)=M^2\boldsymbol{\psi}(x_0,x_1,x_2,x_3)$$

$$(S^- - K)S^+\begin{pmatrix}\psi_0(x_0,x_1,x_2,x_3)\\\psi_1(x_0,x_1,x_2,x_3)\\\psi_2(x_0,x_1,x_2,x_3)\\\psi_3(x_0,x_1,x_2,x_3)\end{pmatrix}=M^2\begin{pmatrix}\psi_0(x_0,x_1,x_2,x_3)\\\psi_1(x_0,x_1,x_2,x_3)\\\psi_2(x_0,x_1,x_2,x_3)\\\psi_3(x_0,x_1,x_2,x_3)\end{pmatrix}$$

$$\begin{aligned}S^-={}&\begin{pmatrix}-\frac{\partial}{\partial x_0}+a_1\\\frac{\partial}{\partial x_1}+a_0\\0\\0\end{pmatrix}\left(\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right),\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right),0,0\right)\\&+\begin{pmatrix}0\\0\\-\frac{\partial}{\partial x_0}+a_1\\\frac{\partial}{\partial x_1}+a_0\end{pmatrix}\left(0,0,\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right),\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\right)\\&+\begin{pmatrix}-\frac{\partial}{\partial x_2}+a_3\\\frac{\partial}{\partial x_3}+a_2\\0\\0\end{pmatrix}\left(\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}+\overline{a_3}\right),\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}+\overline{a_2}\right),0,0\right)\\&+\begin{pmatrix}0\\0\\-\frac{\partial}{\partial x_2}+a_3\\\frac{\partial}{\partial x_3}+a_2\end{pmatrix}\left(0,0\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}+\overline{a_3}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}+\overline{a_2}\right)\right)\end{aligned}$$

$$S^{+} = \begin{pmatrix} -\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right) \\ \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right) \\ 0 \\ 0 \end{pmatrix} \left(-\left(\frac{\partial}{\partial x_1} + a_0\right), \left(-\frac{\partial}{\partial x_0} + a_1\right), 0, 0\right)$$

$$+ \begin{pmatrix} 0 \\ 0 \\ -\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right) \\ \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right) \end{pmatrix} \left(0, 0, -\left(\frac{\partial}{\partial x_1} + a_0\right)_0, \left(-\frac{\partial}{\partial x_0} + a_1\right)\right)$$

$$+ \begin{pmatrix} -\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}} + \overline{a_2}\right) \\ \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}} + \overline{a_3}\right) \\ 0 \\ 0 \end{pmatrix} \left(-\left(\frac{\partial}{\partial x_3} + a_2\right), \left(-\frac{\partial}{\partial x_2} + a_3\right), 0, 0\right)$$

$$+ \begin{pmatrix} 0 \\ 0 \\ -\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}} + \overline{a_2}\right) \\ \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}} + \overline{a_3}\right) \end{pmatrix} \left(0, 0, -\left(\frac{\partial}{\partial x_3} + a_2\right), \left(-\frac{\partial}{\partial x_2} + a_3\right)\right)$$

$K =$

$$= \begin{pmatrix} \left(-\frac{\partial}{\partial x_0} + a_1\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right) - \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\left(-\frac{\partial}{\partial x_0} + a_1\right) & \left(-\frac{\partial}{\partial x_0} + a_1\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right) - \left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right)\left(-\frac{\partial}{\partial x_0} + a_1\right) & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \left(\frac{\partial}{\partial x_1} + a_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right) - \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial}{\partial x_1} + a_0\right) & \left(\frac{\partial}{\partial x_1} + a_0\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right) - \left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right)\left(\frac{\partial}{\partial x_1} + a_0\right) & \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & & \begin{matrix} [p_1\overline{p_1} - \overline{p_1}p_1] & [p_1\overline{p_0} - \overline{p_0}p_1] \\ [p_0\overline{p_1} - \overline{p_1}p_0] & [p_0\overline{p_0} - \overline{p_0}p_0] \end{matrix} \end{pmatrix}$$

$$+ \begin{pmatrix} [p_3\overline{p_3} - \overline{p_3}p_3] & [p_3\overline{p_2} - \overline{p_2}p_3] & 0 & 0 \\ [p_2\overline{p_3} - \overline{p_3}p_2] & [p_2\overline{p_2} - \overline{p_2}p_2] & 0 & 0 \\ 0 & 0 & [p_3\overline{p_3} - \overline{p_3}p_3] & [p_3\overline{p_2} - \overline{p_2}p_3] \\ 0 & 0 & [p_2\overline{p_3} - \overline{p_3}p_2] & [p_2\overline{p_2} - \overline{p_2}p_2] \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\partial\overline{a_1}}{\partial x_0} + \frac{\partial\overline{a_1}}{\partial\overline{x_0}} & -\frac{\partial\overline{a_0}}{\partial x_0} - \frac{\partial\overline{a_1}}{\partial\overline{x_1}} & 0 & 0 \\ \frac{\partial\overline{a_1}}{\partial x_1} + \frac{\partial\overline{a_0}}{\partial\overline{x_0}} & \frac{\partial\overline{a_0}}{\partial x_1} - \frac{\partial\overline{a_0}}{\partial\overline{x_1}} & 0 & 0 \\ 0 & 0 & -\frac{\partial\overline{a_1}}{\partial x_0} + \frac{\partial\overline{a_1}}{\partial\overline{x_0}} & -\frac{\partial\overline{a_0}}{\partial x_0} - \frac{\partial\overline{a_1}}{\partial\overline{x_1}} \\ 0 & 0 & \frac{\partial\overline{a_1}}{\partial x_1} + \frac{\partial\overline{a_0}}{\partial\overline{x_0}} & \frac{\partial\overline{a_0}}{\partial x_1} - \frac{\partial\overline{a_0}}{\partial\overline{x_1}} \end{pmatrix} + \begin{pmatrix} -\frac{\partial\overline{a_3}}{\partial x_2} + \frac{\partial\overline{a_3}}{\partial\overline{x_2}} & -\frac{\partial\overline{a_2}}{\partial x_2} - \frac{\partial\overline{a_3}}{\partial\overline{x_3}} & 0 & 0 \\ \frac{\partial\overline{a_3}}{\partial x_3} + \frac{\partial\overline{a_2}}{\partial\overline{x_2}} & \frac{\partial\overline{a_2}}{\partial x_3} - \frac{\partial\overline{a_2}}{\partial\overline{x_3}} & 0 & 0 \\ 0 & 0 & -\frac{\partial\overline{a_3}}{\partial x_2} + \frac{\partial\overline{a_3}}{\partial\overline{x_2}} & -\frac{\partial\overline{a_2}}{\partial x_2} - \frac{\partial\overline{a_3}}{\partial\overline{x_3}} \\ 0 & 0 & \frac{\partial\overline{a_3}}{\partial x_3} + \frac{\partial\overline{a_2}}{\partial\overline{x_2}} & \frac{\partial\overline{a_2}}{\partial x_3} - \frac{\partial\overline{a_2}}{\partial\overline{x_3}} \end{pmatrix}$$

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$$S^{+}\boldsymbol{\psi} = \begin{pmatrix} -\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right) \\ \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right) \\ 0 \\ 0 \end{pmatrix} \left(-\left(\frac{\partial}{\partial x_1} + a_0\right)\psi_0 + \left(-\frac{\partial}{\partial x_0} + a_1\right)\psi_1\right)$$
$$+ \begin{pmatrix} 0 \\ 0 \\ -\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}} + \overline{a_0}\right) \\ \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}} + \overline{a_1}\right) \end{pmatrix} \left(-\left(\frac{\partial}{\partial x_1} + a_0\right)\psi_2 + \left(-\frac{\partial}{\partial x_0} + a_1\right)\psi_3\right)$$
$$+ \begin{pmatrix} -\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}} + \overline{a_2}\right) \\ \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}} + \overline{a_3}\right) \\ 0 \\ 0 \end{pmatrix} \left(-\left(\frac{\partial}{\partial x_3} + a_2\right)\psi_0 + \left(-\frac{\partial}{\partial x_2} + a_3\right)\psi_1\right)$$
$$+ \begin{pmatrix} 0 \\ 0 \\ -\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}} + \overline{a_2}\right) \\ \left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}} + \overline{a_3}\right) \end{pmatrix} \left(-\left(\frac{\partial}{\partial x_3} + a_2\right)\psi_2 + \left(-\frac{\partial}{\partial x_2} + a_3\right)\psi_3\right)$$

Since the second factor S^{+} in the left-hand side of the equation has a simpler structure than the first factor, perhaps as a first step we should find the eigenvalues and eigenfunctions of the equation

$$S^{+}\boldsymbol{\psi}(x_0, x_1, x_2, x_3) = M^2\boldsymbol{\psi}(x_0, x_1, x_2, x_3)$$

and use them when solving the equation as a whole.

$$S^-S^+\boldsymbol{\psi} = \left\{ \begin{pmatrix} -\frac{\partial}{\partial x_0} + a_1 \\ \frac{\partial}{\partial x_1} + a_0 \\ 0 \\ 0 \end{pmatrix} \left(\left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right), \left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right), 0, 0 \right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_0} + a_1 \\ \frac{\partial}{\partial x_1} + a_0 \end{pmatrix} \left(0, 0, \left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right), \left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right) \right) \right.$$

$$+ \begin{pmatrix} -\frac{\partial}{\partial x_2} + a_3 \\ \frac{\partial}{\partial x_3} + a_2 \\ 0 \\ 0 \end{pmatrix} \left(\left(-\frac{\partial \Box}{\partial \overline{x_2}} + \overline{a_3} \right), \left(\frac{\partial \Box}{\partial \overline{x_3}} + \overline{a_2} \right), 0, 0 \right)$$

$$\left. + \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_2} + a_3 \\ \frac{\partial}{\partial x_3} + a_2 \end{pmatrix} \left(0, 0, \left(-\frac{\partial \Box}{\partial \overline{x_2}} + \overline{a_3} \right) \left(\frac{\partial \Box}{\partial \overline{x_3}} + \overline{a_2} \right) \right) \right\} \left\{ \begin{pmatrix} -\left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right) \\ \left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right) \\ 0 \\ 0 \end{pmatrix} \left(-\left(\frac{\partial \psi_0}{\partial x_1} + a_0 \psi_0 \right) \right. \right.$$

$$\left. + \left(-\frac{\partial \psi_1}{\partial x_0} + a_1 \psi_1 \right) \right) + \begin{pmatrix} 0 \\ 0 \\ -\left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right) \\ \left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right) \end{pmatrix} \left(-\left(\frac{\partial \psi_2}{\partial x_1} + a_0 \psi_2 \right) + \left(-\frac{\partial \psi_3}{\partial x_0} + a_1 \psi_3 \right) \right)$$

$$+ \begin{pmatrix} -\left(\frac{\partial \Box}{\partial \overline{x_3}} + \overline{a_2} \right) \\ \left(-\frac{\partial \Box}{\partial \overline{x_2}} + \overline{a_3} \right) \\ 0 \\ 0 \end{pmatrix} \left(-\left(\frac{\partial \psi_0}{\partial x_3} + a_2 \psi_0 \right) + \left(-\frac{\partial \psi_1}{\partial x_2} + a_3 \psi_1 \right) \right)$$

$$\left. + \begin{pmatrix} 0 \\ 0 \\ -\left(\frac{\partial \Box}{\partial \overline{x_3}} + \overline{a_2} \right) \\ \left(-\frac{\partial \Box}{\partial \overline{x_2}} + \overline{a_3} \right) \end{pmatrix} \left(-\left(\frac{\partial \psi_2}{\partial x_3} + a_2 \psi_2 \right) + \left(-\frac{\partial \psi_3}{\partial x_2} + a_3 \psi_3 \right) \right) \right\}$$

$$= \left\{ \begin{pmatrix} -\frac{\partial}{\partial x_0} + a_1 \\ \frac{\partial}{\partial x_1} + a_0 \\ 0 \\ 0 \end{pmatrix} \left(-\left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right) \left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right) + \left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right) \left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right) \right) \left(-\left(\frac{\partial \psi_0}{\partial x_1} + a_0 \psi_0 \right) \right. \right.$$

$$\left. + \left(\frac{\partial \psi_1}{\partial x_0} + a_1 \psi_1 \right) \right)$$

$$+ \begin{pmatrix} -\frac{\partial}{\partial x_2} + a_3 \\ \frac{\partial}{\partial x_2} + a_2 \\ 0 \\ 0 \end{pmatrix} \left(-\left(-\frac{\partial \Box}{\partial \overline{x_2}} + \overline{a_3} \right) \left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right) + \left(\frac{\partial \Box}{\partial \overline{x_3}} + \overline{a_2} \right) \left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right) \right) \left(-\left(\frac{\partial \psi_0}{\partial x_1} + a_0 \psi_0 \right) \right.$$

$$\left. + \left(-\frac{\partial \psi_1}{\partial x_0} + a_1 \psi_1 \right) \right)$$

$$+ \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_0} + a_1 \\ \frac{\partial}{\partial x_1} + a_0 \end{pmatrix} \left(-\left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right) \left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right) + \left(\frac{\partial \Box}{\partial \overline{x_1}} + \overline{a_0} \right) \left(-\frac{\partial \Box}{\partial \overline{x_0}} + \overline{a_1} \right) \right) \left(-\left(\frac{\partial \psi_2}{\partial x_1} + a_0 \psi_2 \right) \right.$$

$$\left. + \left(-\frac{\partial \psi_3}{\partial x_0} + a_1 \psi_3 \right) \right)$$

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$$+\begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_2}+a_3 \\ \frac{\partial}{\partial x_3}+a_2 \end{pmatrix}\left(-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_2}+\overline{a}_3\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_1}+\overline{a}_0\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_3}+\overline{a}_2\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_0}+\overline{a}_1\right)\right)\left(-\left(\frac{\partial\psi_2}{\partial x_1}+a_0\psi_2\right)\right.$$

$$\left.+\left(\frac{\partial\psi_3}{\partial x_0}+a_1\psi_3\right)\right)$$

$$+\begin{pmatrix} -\frac{\partial}{\partial x_0}+a_1 \\ \frac{\partial}{\partial x_1}+a_0 \\ 0 \\ 0 \end{pmatrix}\left(-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_0}+\overline{a}_1\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_3}+\overline{a}_2\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_1}+\overline{a}_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_2}+\overline{a}_3\right)\right)\left(-\left(\frac{\partial\psi_0}{\partial x_3}+a_2\psi_0\right)\right.$$

$$\left.+\left(-\frac{\partial\psi_1}{\partial x_2}+a_3\psi_1\right)\right)$$

$$+\begin{pmatrix} -\frac{\partial}{\partial x_2}+a_3 \\ \frac{\partial}{\partial x_3}+a_2 \\ 0 \\ 0 \end{pmatrix}\left(-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_2}+\overline{a}_3\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_3}+\overline{a}_2\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_3}+\overline{a}_2\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_2}+\overline{a}_3\right)\right)\left(-\left(\frac{\partial\psi_0}{\partial x_3}+a_2\psi_0\right)\right.$$

$$\left.+\left(-\frac{\partial\psi_1}{\partial x_2}+a_3\psi_1\right)\right)$$

$$+\begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_0}+a_1 \\ \frac{\partial}{\partial x_1}+a_0 \end{pmatrix}\left(-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_0}+\overline{a}_1\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_3}+\overline{a}_2\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_1}+\overline{a}_0\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_2}+\overline{a}_3\right)\right)\left(-\left(\frac{\partial\psi_2}{\partial x_3}+a_2\psi_2\right)\right.$$

$$\left.+\left(-\frac{\partial\psi_3}{\partial x_2}+a_3\psi_3\right)\right)$$

$$+\begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_2}+a_3 \\ \frac{\partial}{\partial x_3}+a_2 \end{pmatrix}\left(-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_2}+\overline{a}_3\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_3}+\overline{a}_2\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x}_3}+\overline{a}_2\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x}_2}+\overline{a}_3\right)\right)\left(-\left(\frac{\partial\psi_2}{\partial x_3}+a_2\psi_2\right)\right.$$

$$\left.\left.+\left(-\frac{\partial\psi_3}{\partial x_2}+a_3\psi_3\right)\right)\right\}$$

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$$S^{-}S^{+}\boldsymbol{\psi} = \left\{\left(\begin{pmatrix} -\frac{\partial}{\partial x_0} + a_1 \\ \frac{\partial}{\partial x_1} + a_0 \\ 0 \\ 0 \end{pmatrix}\left(\frac{\partial a_1}{\partial \overline{x_1}} + \frac{\partial a_0}{\partial \overline{x_0}}\right)\left(-\left(\frac{\partial \psi_0}{\partial x_1} + a_0\psi_0\right) + \left(-\frac{\partial \psi_1}{\partial x_0} + a_1\psi_1\right)\right)\right.\right.$$

$$+\begin{pmatrix} -\frac{\partial}{\partial x_2} + a_3 \\ \frac{\partial}{\partial x_3} + a_2 \\ 0 \\ 0 \end{pmatrix}\left(-\left(-\frac{\partial \overline{[\,]}}{\partial \overline{x_2}} + \overline{a_3}\right)\left(\frac{\partial \overline{[\,]}}{\partial \overline{x_1}} + \overline{a_0}\right) + \left(\frac{\partial \overline{[\,]}}{\partial \overline{x_3}} + \overline{a_2}\right)\left(-\frac{\partial \overline{[\,]}}{\partial \overline{x_0}} + \overline{a_1}\right)\right)\left(-\left(\frac{\partial \psi_0}{\partial x_1} + a_0\psi_0\right)\right.$$

$$+\left.\left(-\frac{\partial \psi_1}{\partial x_0} + a_1\psi_1\right)\right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_1} + a_1 \\ \frac{\partial}{\partial x_1} + a_0 \end{pmatrix}\left(\frac{\partial a_1}{\partial \overline{x_1}} + \frac{\partial a_0}{\partial \overline{x_0}}\right)\left(-\left(\frac{\partial \psi_2}{\partial x_1} + a_0\psi_2\right) + \left(-\frac{\partial \psi_3}{\partial x_0} + a_1\psi_3\right)\right)$$

$$+\begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_2} + a_3 \\ \frac{\partial}{\partial x_3} + a_2 \end{pmatrix}\left(-\left(-\frac{\partial \overline{[\,]}}{\partial \overline{x_2}} + \overline{a_3}\right)\left(\frac{\partial \overline{[\,]}}{\partial \overline{x_1}} + \overline{a_0}\right) + \left(\frac{\partial \overline{[\,]}}{\partial \overline{x_3}} + \overline{a_2}\right)\left(-\frac{\partial \overline{[\,]}}{\partial \overline{x_0}} + \overline{a_1}\right)\right)\left(-\left(\frac{\partial \psi_2}{\partial x_1} + a_0\psi_2\right)\right.$$

$$+\left.\left(-\frac{\partial \psi_3}{\partial x_0} + a_1\psi_3\right)\right)$$

$$+\begin{pmatrix} -\frac{\partial}{\partial x_0} + a_1 \\ \frac{\partial}{\partial x_1} + a_0 \\ 0 \\ 0 \end{pmatrix}\left(-\left(-\frac{\partial \overline{[\,]}}{\partial \overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial \overline{[\,]}}{\partial \overline{x_3}} + \overline{a_2}\right) + \left(\frac{\partial \overline{[\,]}}{\partial \overline{x_1}} + \overline{a_0}\right)\left(-\frac{\partial \overline{[\,]}}{\partial \overline{x_2}} + \overline{a_3}\right)\right)\left(-\left(\frac{\partial \psi_0}{\partial x_3} + a_2\psi_0\right)\right.$$

$$+\left.\left(-\frac{\partial \psi_1}{\partial x_2} + a_3\psi_1\right)\right) + \begin{pmatrix} -\frac{\partial}{\partial x_2} + a_3 \\ \frac{\partial}{\partial x_3} + a_2 \\ 0 \\ 0 \end{pmatrix}\left(\frac{\partial a_3}{\partial \overline{x_3}} + \frac{\partial a_2}{\partial \overline{x_2}}\right)\left(-\left(\frac{\partial \psi_0}{\partial x_3} + a_2\psi_0\right) + \left(-\frac{\partial \psi_1}{\partial x_2} + a_3\psi_1\right)\right)$$

$$+\begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_0} + a_1 \\ \frac{\partial}{\partial x_1} + a_0 \end{pmatrix}\left(-\left(-\frac{\partial \overline{[\,]}}{\partial \overline{x_0}} + \overline{a_1}\right)\left(\frac{\partial \overline{[\,]}}{\partial \overline{x_3}} + \overline{a_2}\right) + \left(\frac{\partial \overline{[\,]}}{\partial \overline{x_1}} + \overline{a_0}\right)\left(-\frac{\partial \overline{[\,]}}{\partial \overline{x_2}} + \overline{a_3}\right)\right)\left(-\left(\frac{\partial \psi_2}{\partial x_3} + a_2\psi_2\right)\right.$$

$$\left.\left.+\left(-\frac{\partial \psi_3}{\partial x_2} + a_3\psi_3\right)\right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_2} + a_3 \\ \frac{\partial}{\partial x_3} + a_2 \end{pmatrix}\left(\frac{\partial a_3}{\partial \overline{x_3}} + \frac{\partial a_2}{\partial \overline{x_2}}\right)\left(-\left(\frac{\partial \psi_2}{\partial x_3} + a_2\psi_2\right) + \left(-\frac{\partial \psi_3}{\partial x_2} + a_3\psi_3\right)\right)\right\} -$$

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Let's calculate the expressions included in the equation

$$
\begin{aligned}
&\left(-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}+\overline{a_3}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)+\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}+\overline{a_2}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\right)\varphi \\
&\quad=\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}+\overline{a_2}\right)\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_0}}+\overline{a_1}\right)\varphi-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}+\overline{a_3}\right)\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_1}}+\overline{a_0}\right)\varphi \\
&\quad=\left(\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}+\overline{a_2}\right)\left(-\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}+\overline{a_1}\varphi\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}+\overline{a_3}\right)\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}+\overline{a_0}\varphi\right) \\
&\quad=\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}\left(-\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}(\overline{a_1}\varphi)+\overline{a_2}\left(-\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}\right)+\overline{a_2a_1}\varphi+\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}\right)-\left(-\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}\right)(\overline{a_0}\varphi) \\
&\quad-\overline{a_3}\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}-\overline{a_3a_0}\varphi \\
&\quad=\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}\left(-\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}\right)+\frac{\partial\overline{a_1}}{\partial\overline{x_3}}\varphi-\overline{a_2}\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}+\overline{a_2a_1}\varphi+\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}\right)+\overline{a_0}\frac{\partial\overline{\varphi}}{\partial\overline{x_2}}+\frac{\partial\overline{a_0}}{\partial\overline{x_2}}\varphi-\overline{a_3}\frac{\partial\overline{\varphi}}{\partial\overline{x_1}} \\
&\quad-\overline{a_3a_0}\varphi \\
&\quad=\frac{\partial\overline{[\,]}}{\partial\overline{x_2}}\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_1}}\right)-\frac{\partial\overline{[\,]}}{\partial\overline{x_3}}\left(\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}\right)+\left[\frac{\partial\overline{a_1}}{\partial\overline{x_3}}+\frac{\partial\overline{a_0}}{\partial\overline{x_2}}\right]\varphi+\overline{a_1}\frac{\partial\overline{\varphi}}{\partial\overline{x_3}}-\overline{a_2}\frac{\partial\overline{\varphi}}{\partial\overline{x_0}}+\overline{a_0}\frac{\partial\overline{\varphi}}{\partial\overline{x_2}}-\overline{a_3}\frac{\partial\overline{\varphi}}{\partial\overline{x_1}} \\
&\quad+(\overline{a_2a_1}-\overline{a_3a_0})\varphi
\end{aligned}
$$

Let us consider the situation when the electromagnetic potential can be described by a plane wave in spinor space

$$
\mathbf{a}(x_0,x_1,x_2,x_3)=\begin{pmatrix}a_0(x_0,x_1,x_2,x_3)\\a_1(x_0,x_1,x_2,x_3)\\a_2(x_0,x_1,x_2,x_3)\\a_3(x_0,x_1,x_2,x_3)\end{pmatrix}=\begin{pmatrix}u_{a0}\\u_{a1}\\u_{a2}\\u_{a3}\end{pmatrix}\varphi_a
$$

$$
=\begin{pmatrix}u_{a0}\\u_{a1}\\u_{a2}\\u_{a3}\end{pmatrix}exp(p_{a0}x_1-p_{a1}x_0+p_{a2}x_3-p_{a3}x_2)
$$

$$
\begin{aligned}
&\left\{\left(-\frac{\partial}{\partial x_0}+a_1\right)\left(\frac{\partial}{\partial x_3}+a_2\right)-\left(\frac{\partial}{\partial x_1}+a_0\right)\left(-\frac{\partial}{\partial x_2}+a_3\right)\right\}\varphi \\
&\quad=-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\varphi+\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\varphi+\left(-\frac{\partial a_2}{\partial x_0}-\frac{\partial a_3}{\partial x_1}\right)\varphi-a_2\frac{\partial\varphi}{\partial x_0}+a_1\frac{\partial\varphi}{\partial x_3} \\
&\quad-a_3\frac{\partial\varphi}{\partial x_1}+a_0\frac{\partial\varphi}{\partial x_2}+(a_1a_2-a_0a_3)\varphi \\
&\quad=-\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_3}\varphi+\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\varphi+(u_{a2}p_{a1}-u_{a3}p_{a0})\varphi_a\varphi-u_{a2}\varphi_a\frac{\partial\varphi}{\partial x_0} \\
&\quad-u_{a1}\varphi_a\frac{\partial\varphi}{\partial x_2}+u_{a3}\varphi_a\frac{\partial\varphi}{\partial x_0}+u_{a0}\varphi_a\frac{\partial\varphi}{\partial x_3}+(u_{a1}u_{a2}-u_{a0}u_{a3}){\varphi_a}^2\varphi
\end{aligned}
$$

$$\left\{\left(\frac{\partial}{\partial x_1}+a_0\right)\left(\frac{\partial}{\partial x_3}+a_2\right)-\left(\frac{\partial}{\partial x_3}+a_2\right)\left(\frac{\partial}{\partial x_1}+a_0\right)\right\}\varphi=\left\{\frac{\partial a_2}{\partial x_1}-\frac{\partial a_0}{\partial x_3}\right\}\varphi$$
$$=(u_{a2}p_{a0}-u_{a0}p_{a2})\varphi_a\varphi$$

When the electromagnetic potential is represented by a plane wave, the field created by a charged particle is not taken into account, so this model adequately describes only the situation when the electromagnetic field is strong enough and the influence of the particle charge can be neglected.

It would be interesting in this context to consider for the presented spinor model the case of a centrally symmetric electric field and to find solutions of the spinor wave equation for the hydrogen-like atom, taking into account the presence of spin at the electron. For such a model we can take

$$a_0=-i\frac{1}{\sqrt{2R}}\qquad a_1=\frac{1}{\sqrt{2R}}\qquad a_2=\frac{1}{\sqrt{2R}}\qquad a_3=-i\frac{1}{\sqrt{2R}}$$

$$R=\sqrt{\left(\frac{1}{2}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)\right)^2-\left(\frac{1}{2}(-\overline{x_0}x_1+\overline{x_1}x_0-\overline{x_2}x_3+\overline{x_3}x_2)\right)^2+\left(\frac{1}{2}(\overline{x_0}x_0-\overline{x_1}x_1+\overline{x_2}x_2-\overline{x_3}x_3)\right)^2}$$

As mentioned above, here too we can substitute into the equation the already known exact solutions of the Dirac equation for the hydrogen-like atom by expressing the components of the coordinate vector and derivatives on them through the components of the coordinate spinor and derivatives on them. It is likely that the solution of the Dirac equation would not make the spinor equation an identity; it would be evidence that more arbitrary assumptions are made in the Dirac equation than in the spinor equation, and that the latter claims to be a better description of nature.

We can also consider the case of a constant magnetic field directed along the z-axis

$$A_0=0\qquad A_1=-\frac{1}{2}B_3X_2\qquad A_2=\frac{1}{2}B_3X_1\qquad A_3=0$$

$$X_1=\frac{1}{2}(\overline{x_0}x_1+\overline{x_1}x_0+\overline{x_2}x_3+\overline{x_3}x_2)$$

$$X_2=\frac{1}{2}(-i\overline{x_0}x_1+i\overline{x_1}x_0-i\overline{x_2}x_3+i\overline{x_3}x_2)$$

$$A_1=\frac{1}{2}(\overline{a_0}a_1+\overline{a_1}a_0+\overline{a_2}a_3+\overline{a_3}a_2)$$

$$A_2=\frac{1}{2}(-i\overline{a_0}a_1+i\overline{a_1}a_0-i\overline{a_2}a_3+i\overline{a_3}a_2)$$

$$A_0=\frac{1}{2}(\overline{a_0}a_0+\overline{a_1}a_1+\overline{a_2}a_2+\overline{a_3}a_3)$$

$$A_3=\frac{1}{2}(\overline{a_0}a_0-\overline{a_1}a_1+\overline{a_2}a_2-\overline{a_3}a_3)$$

Let's say

$$a_0=i\overline{x_1}\sqrt{B_3/2}\qquad a_1=-\overline{x_0}\sqrt{B_3/2}$$

$$a_2=i\overline{x_3}\sqrt{B_3/2}\qquad a_3=-\overline{x_2}\sqrt{B_3/2}$$

$$A_1=\frac{1}{4}B_3(ix_1\overline{x_0}-i\overline{x_0}x_1+i\overline{x_3}x_2-i\overline{x_2}x_3)=-\frac{1}{2}B_3X_2$$

$$A_2=\frac{1}{4}B_3(\overline{x_1}x_0+\overline{x_0}x_1+\overline{x_3}x_2+\overline{x_2}x_3)=\frac{1}{2}B_3X_1$$

$$A_0 = \frac{1}{4}B_3(x_1\overline{x_1} + x_0\overline{x_0} + x_3\overline{x_3} + x_2\overline{x_2}) = \frac{1}{2}B_3 t$$

$$A_3 = \frac{1}{4}B_3(x_1\overline{x_1} - x_0\overline{x_0} + x_3\overline{x_3} - x_2\overline{x_2}) = \frac{1}{2}B_3 X_3$$

We see that the scalar potential A_0 grows with time, but does not depend on spatial coordinates, and the vector potential does not depend on time, so that there is no electric field. In this case

$$K = \begin{pmatrix} -\frac{\partial\overline{a_1}}{\partial x_0} + \frac{\partial\overline{a_1}}{\partial\overline{x_0}} & -\frac{\partial\overline{a_0}}{\partial x_0} - \frac{\partial\overline{a_1}}{\partial\overline{x_1}} & 0 & 0 \\ \frac{\partial\overline{a_1}}{\partial x_1} + \frac{\partial\overline{a_0}}{\partial\overline{x_0}} & \frac{\partial\overline{a_0}}{\partial x_1} - \frac{\partial\overline{a_0}}{\partial\overline{x_1}} & 0 & 0 \\ 0 & 0 & -\frac{\partial\overline{a_1}}{\partial x_0} + \frac{\partial\overline{a_1}}{\partial\overline{x_0}} & -\frac{\partial\overline{a_0}}{\partial x_0} - \frac{\partial\overline{a_1}}{\partial\overline{x_1}} \\ 0 & 0 & \frac{\partial\overline{a_1}}{\partial x_1} + \frac{\partial\overline{a_0}}{\partial\overline{x_0}} & \frac{\partial\overline{a_0}}{\partial x_1} - \frac{\partial\overline{a_0}}{\partial\overline{x_1}} \end{pmatrix}$$

$$+ \begin{pmatrix} -\frac{\partial\overline{a_3}}{\partial x_2} + \frac{\partial\overline{a_3}}{\partial\overline{x_2}} & -\frac{\partial\overline{a_2}}{\partial x_2} - \frac{\partial\overline{a_3}}{\partial\overline{x_3}} & 0 & 0 \\ \frac{\partial\overline{a_3}}{\partial x_3} + \frac{\partial\overline{a_2}}{\partial\overline{x_2}} & \frac{\partial\overline{a_2}}{\partial x_3} - \frac{\partial\overline{a_2}}{\partial\overline{x_3}} & 0 & 0 \\ 0 & 0 & -\frac{\partial\overline{a_3}}{\partial x_2} + \frac{\partial\overline{a_3}}{\partial\overline{x_2}} & -\frac{\partial\overline{a_2}}{\partial x_2} - \frac{\partial\overline{a_3}}{\partial\overline{x_3}} \\ 0 & 0 & \frac{\partial\overline{a_3}}{\partial x_3} + \frac{\partial\overline{a_2}}{\partial\overline{x_2}} & \frac{\partial\overline{a_2}}{\partial x_3} - \frac{\partial\overline{a_2}}{\partial\overline{x_3}} \end{pmatrix}$$

$$= \sqrt{\frac{B_3}{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$+\sqrt{B_3/2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = \sqrt{2B_3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

If consistently to adhere to the idea of the fundamentality of the spinor space, it is necessary to reformulate the procedure of second quantization of the electron field. According to the known concept, the wave function of the electron field is represented in the form of expansion by plane waves describing a free electron. But we can use plane waves not in vector space, but plane waves in spinor space

$$\mathbf{u}\, exp(\mathbf{p}^T \Sigma_{MM}\mathbf{x}) = \mathbf{u}\, exp(p_0 x_1 - p_1 x_0 + p_2 x_3 - p_3 x_2)$$

In this case, let us assume

$$\mathbf{u} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \sqrt{\frac{e}{m}}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

We can start the analysis in a stationary frame of reference, for transition to the general case it is necessary to apply the Lorentz transformation to the impulse and coordinate spinors, in this case

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neither the phase in the exponent nor the mass of the electron $m = p_1 p_2 - p_3 p_0$ changes. In the rest frame, one momentum vector corresponds to four momentum spinors

$$\mathbf{p1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{p2} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{p3} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{p4} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Signs of the respective masses

$$m = p_1 p_2 - p_0 p_3$$
$$m1 = -1 \quad m2 = 1 \quad m3 = 1 \quad m4 = -1$$

say that it's two particles with positive mass and two with negative mass.
The orthogonality relations take place in any reference frame

$$\mathbf{p1}^T\mathbf{p2} = \mathbf{p3}^T\mathbf{p4} = \mathbf{p1}^T\Sigma_{MM}\mathbf{p2} = \mathbf{p3}^T\Sigma_{MM}\mathbf{p4} = \mathbf{0}$$

Let us accept the agreement that spinors **p2** and **p4** describe the field with zero number of particles in the state with momentum **P** and masses with different sign. Spinors **p1** and **p3** describe the field with presence of a particle in a state with momentum **P** and negative or positive mass, respectively. The transition from one state to another is provided by the annihilation and birth operators

$$a_P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad b_P = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The relations can be directly verified

$$b_P a_P + a_P b_P = I$$
$$a_P\mathbf{p1} = \mathbf{p4} \quad a_P\mathbf{p3} = \mathbf{p2}$$
$$b_P\mathbf{p2} = \mathbf{p3} \quad b_P\mathbf{p4} = \mathbf{p1}$$

$$b_P a_P \equiv N_P$$
$$N_P\mathbf{p1} = 1 * \mathbf{p1} \quad N_P\mathbf{p3} = 1 * \mathbf{p3}$$
$$N_P\mathbf{p2} = 0 * \mathbf{p2} \quad N_P\mathbf{p4} = 0 * \mathbf{p4}$$

The action of the operators does not change the sign of the mass of the particle. Here the particle number operator N_P demonstrates that two states are indeed filled and two are empty, since this operator has an eigenvalue in one case of one and in the other case of zero.

The wave function of the electron field at a fixed value of the momentum vector **P** will be a combination of plane spinor waves with four momentum spinors corresponding to this momentum vector.

A system of particles with different momenta can be described by the direct product of momentum spinors, which includes spinors $\mathbf{p1}, \mathbf{p2}, \mathbf{p3}, \mathbf{p4}$, as well as their versions subjected to an arbitrary Lorentz transformation $N * \mathbf{p1}, N * \mathbf{p2}, N * \mathbf{p3}, N * \mathbf{p4}$, as co-multipliers $\mathbf{p}_i$

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$$\mathbf{p} = \prod_i \mathbf{p}_i$$

This system can be acted upon by an operator in the form of a direct product of an arbitrary set of annihilation and birth operator matrices or unit matrices for a particle whose state does not change

$$\mathbf{d} = \prod_i \mathbf{d}_i$$

In this case

$$\mathbf{d} * \mathbf{p} = \prod_i (\mathbf{d}_i * \mathbf{p}_i)$$

The annihilation and birth operators have disadvantage, they do not commute with Lorentz transformations matrix

$$b_P * N \neq N * b_P \quad a_P * N \neq N * a_P$$

Let's introduce operators

$$r_P = b_P + a_P = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$s = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The product of them commute with the Lorentz transformations matrix and they anticommutate with each other

$$(r_P * s) * N - N * (r_P * s) = 0$$
$$(s * r_P) * N - N * (s * r_P) = 0$$
$$s * r_P + r_P * s = 0$$

The product of those operators with the proper order choice transform the state with zero number of particles to the state with presence of a particle and vice versa, and this relation is Lorentz invariant

$$(r_P * s) * (N * \mathbf{p1}) = N * \mathbf{p4}$$
$$(s * r_P) * (N * \mathbf{p4}) = N * \mathbf{p1}$$
$$(r_P * s) * (N * \mathbf{p2}) = N * \mathbf{p3}$$
$$(s * r_P) * (N * \mathbf{p3}) = N * \mathbf{p2}$$

Double applying this product changes the momentum spinor sign

$$(s * r_P) * (s * r_P) = (r_P * s) * (r_P * s) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Conclusion

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An alternative approach to analyze relativistic and quantum effects inherent in charged particles in the presence of an electromagnetic field is proposed. Two ways of describing the electron behavior in the electromagnetic field are considered: by means of the vector equation, which is based on the plane wave model for a free electron, and the spinor equation, which is based on the representation of the electron as a plane wave in spinor space. For both equations, which are valid for a free particle, their applicability to an arbitrary physical situation is postulated, in particular to describe the behavior of a particle in the presence of an electromagnetic field. The presented equations are intended to fulfill the same role as the Schrödinger equation and the Dirac equation. At the same time, in our opinion, the spinor equations more accurately describe the details of the interaction between fields and particles.

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