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Abstract: This paper presents an analysis of the number of zeros in the binary representation of natural numbers. The primary method of analysis involves the use of the concept of the fractional part of a number, which naturally emerges in the determination of binary representation. This idea is grounded in the fundamental property of the Riemann zeta function, constructed using the fractional part of a number. Understanding that the ratio between the fractional and integer parts of a number, analogous to the Riemann zeta function, reflects the profound laws of numbers becomes the key insight of this work. The findings suggest a new perspective on the interrelation between elementary properties of numbers and more complex mathematical concepts, potentially opening new directions in number theory and analysis.

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1. Introduction

We will use the following well-known fact: if, for the members of the Collatz sequence, zeros predominate in their binary representation, then these members will lead to a decrease in the subsequent members according to the Collatz rule. A striking example is when the initial number in the Collatz sequence is equal to 2^n. Let's write the solution of the equation $n = 2^x$ in the form $x = \{x\} + [x]$ and note that the smaller x, the more zeros in the corresponding binary representation for n. Developing this idea, we come to the following steps.

- Analysis of the binary representation of simple cases of natural numbers.
- Creation of a process for decomposing an arbitrary natural number into powers of two.
- Analysis of the proximity of the process to binary decomposition at the completion of decomposition at each stage.
- Calculation of the number of zeros in the binary decomposition of a natural number.
- Estimation of the Collatz sequence members depending on the number of ones in the binary decomposition.

2. Results

This document reveals a comprehensive solution to the Collatz Conjecture, as first proposed in [1]. The Collatz Conjecture, a well-known unsolved problem in mathematics, questions whether iterative application of two basic arithmetic operations can invariably convert any positive integer into 1. It deals with integer sequences generated by the following rule: if a term is even, the subsequent term is half of it; if odd, the next term is the previous term tripled plus one. The conjecture posits that all such sequences culminate in 1, regardless of the initial positive integer. Named after mathematician Lothar Collatz, who introduced the concept in 1937, this conjecture is also known as the 3n + 1 problem, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, or the Syracuse problem. The sequence is often termed the hailstone sequence due to its fluctuating nature, resembling the movement of hailstones. Paul Erdős and Jeffrey Lagarias have commented on the complexity and mathematical depth of the Collatz Conjecture, highlighting its challenging nature. Consider an operation applied to any positive integer:

- Divide it by two if it's even.
- Triple it and add one if it's odd.

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A sequence is formed by continuously applying this operation, starting with any positive integer, where each step's result becomes the next input. The Collatz Conjecture asserts that this sequence will always reach 1 Recent substantial advancements in addressing the Collatz problem have been documented in works [2]. Now let's move on to our research, which we will conduct according to the announced plan. For this, we will start with the following

Theorem 1. *Let*

$$\begin{aligned} M &\in \mathbb{N}, \\ [\alpha_j] - [\alpha_{j+1}] &= \delta_j > 0, \\ \epsilon_1 &< 0.45, \\ |F_j(x)| &< |x|, \\ \alpha_j &= [\alpha_j] + \epsilon_j, \\ \epsilon_j &< 1, \\ \sigma_j &= 1 - \epsilon_j. \end{aligned}$$

Then for $\delta_j = 1$

$$\sigma_j = 2^{-1}\sigma_{j+1}\left(1 - \frac{\sigma_{j+1}\ln 2}{2}\right) + F_j\left(\frac{\sigma_{j+1}^3}{12}\right), \tag{1}$$

and for $\delta_j > 1$

$$\sigma_j = 2^{-\delta_j}\sigma_{j+1} + 1 - \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} - 2^{-2\delta_j}\frac{\sigma_{j+1}^2 \ln 2}{4} + 2^{-2\delta_j}R_j\left(\frac{\ln^2 2\sigma_{j+1}^3}{8}\right). \tag{2}$$

Proof. Consider

$$\begin{aligned} M - M = 0 &= \sum_{i=1}^{j} 2^{[\alpha_i]} + 2^{\alpha_{j+1}} - \left[\sum_{i=1}^{j-1} 2^{[\alpha_i]} + 2^{\alpha_j}\right] \\ &= 2^{[\alpha_j]} + 2^{\alpha_{j+1}} - 2^{\alpha_j} \\ 2^{\alpha_j} &= 2^{[\alpha_j]} + 2^{\alpha_{j+1}} = 2^{[\alpha_j]} + 2^{[\alpha_{j+1}]-[\alpha_j]+[\alpha_j]+\epsilon_{j+1}}. \end{aligned}$$

Then, we proceed to functional relations between σ_j and σ_{j+1}:

$$\begin{aligned} 2^{\epsilon_j} &= 2^{-\delta_j+\epsilon_{j+1}} + 1 \\ \Rightarrow 2^{1-\sigma_j} &= 2^{-\delta_j+1-\sigma_{j+1}} + 1 \\ \Rightarrow \ln(2^{1-\sigma_j}) &= \ln 2 - \sigma_j \ln 2 = \ln(2^{-\delta_j+1-\sigma_{j+1}} + 1). \end{aligned}$$

Evaluating for $\delta_j = 1$, we get:

$$\begin{aligned} \ln(2^{-\delta_j+1-\sigma_{j+1}} + 1)\Big|_{\delta_j=1} &= \ln(2^{-\sigma_{j+1}} + 1) \\ &= \ln 2 + \ln\left(1 - \frac{\sigma_{j+1}\ln 2}{2} + \frac{\sigma_{j+1}^2 \ln^2 2}{4} + F_j\left(\frac{\sigma_{j+1}^3}{12}\right)\right). \end{aligned}$$

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Continuing the computations for $\delta_j > 1$, we obtain:

$$\ln(2^{-\delta_j+1-\sigma_{j+1}}+1) = \ln\left(1+2^{-\delta_j+1} - 2^{-\delta_j+1}\frac{\sigma_{j+1}\ln 2}{2} + 2^{-\delta_j+1}F_j\left(\sigma_{j+1}^2 + 2^{-\delta_j+1}\right)\right)$$
$$= 2^{-\delta_j} - 2^{-2\delta_j+1} - 2^{-\delta_j}\frac{\sigma_{j+1}\ln 2}{2} + 2^{-2\delta_j}F_j\left(\sigma_{j+1}^2\right).$$

Thus, we obtain the final formulas. □

Theorem 2. *Let*

$$M = 3^n = 2^{[\alpha]+\{\alpha\}} = \sum_{i=1}^{n^*} \gamma_i 2^i,$$
$$1 - \{\alpha\} > 0.55, \quad n^* = \left[n\frac{\ln(3)}{\ln(2)}\right], \tag{3}$$

then

$$\sum_{\gamma_i=0} 1 \geq \frac{n^*}{2}.$$

Proof. Let

$$3^n = 2^\alpha \Rightarrow \alpha = \frac{n}{\ln 3/\ln 2} \Rightarrow 3^n = 2^{[\alpha]+\{\alpha\}}.$$

Using Theorem 1, we construct the sequence

$$\epsilon_i,\, m_i,\, \epsilon_1 = \{\alpha\},$$
$$2^{\epsilon_1} = \sum_{k=0}^{i-1} 2^{[\alpha_k]-\alpha_1} + 2^{\alpha_i-\alpha_1}.$$

Suppose the binary decomposition process, according to formula (1), stops at the j-th step. It immediately follows that the remaining terms of the decomposition are zeros, and we immediately achieve the truth of the Theorem's statement. Therefore, we consider the case when the generation of the decomposition according to formula (1) does not stop, and j reaches n. This means that all $\sigma_j > 0, j < n$.

Let's conduct a more detailed analysis of the number of zeros and ones in our binary representation. Introduce the following notation:
l- the number of zeros in the binary representation.
m- the number of ones in the binary representation.
n- the binary decomposition bit size, then
n=l+m.

$$\delta_j = 1, \alpha_j = 0, \beta_j = \left((1 - \frac{\ln 2\sigma_{j+1}}{2})/2 + F_j\left(\frac{\sigma_{j+1}^2}{12}\right)\right)^{-1}$$

$$\delta_j > 1, \alpha_j = -2^{\delta_j}\left(1 - \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + 2^{-\delta_j}R_j\left(\frac{\ln^2 2\sigma_{j+1}^3}{8} + \frac{2^{-2\delta_j+1}}{\ln 2}\right)\right), \beta_j = 2^{\delta_j}$$

To solve the following equations

$$\sigma_{j+1} = \alpha_j + \beta_j\sigma_j$$

we introduce the notation λ_m- the number of ones after the appearance of $\alpha_m > 0$ and before the next appearance of zero in the binary decomposition and

$$\gamma_m = \prod_{k=m}^{m+\lambda_m-1} \beta_k,\; \alpha_{m+\lambda_m+1} > 0$$

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Consider the set $\lambda_1, \lambda_2, ...\lambda_n,.$ by definition $\lambda_1 \geq 1$
Define:

$$m(\lambda_*, i) = \inf_k \{k | \lambda_k > 1 + i\}$$

$$m(\lambda^*, i) = \inf_k \{k | \lambda_k > 1 + i\}$$

if the set k satisfying the condition is not empty. Let's perform a series of transformations to understand the next steps.

$$\sigma_{n+1} = \alpha_n + \beta_1\gamma_1 \frac{\alpha_1}{\beta_1\gamma_1} \prod_{k=0}^{n-2} \gamma_{n-k}\beta_{n-k} + \sum_{m=1}^{n-2} \gamma_{n-m}\beta_{n-m} \frac{\alpha_{n-m}}{\beta_{n-m}\gamma_{n-m}} \prod_{k=0}^{m-1} \beta_{n-k}\gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k}\gamma_{n-k}$$

$$\sigma_{n+1} = \alpha_n + \frac{\alpha_1}{\beta_1\gamma_1} \prod_{k=0}^{n-1} \beta_{n-k}\gamma_{n-k} + \sum_{m=1}^{n-2} \frac{\alpha_{n-m}}{\beta_{n-m}\gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k}\gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k}\gamma_{n-k} \tag{4}$$

With the consideration of the definition of $m(\lambda_*, i)$, in case of existence $\lambda_i > 1$

$$\sigma_{n+1} = \alpha_n + \frac{\alpha_1}{\beta_1\gamma_1} \prod_{k=0}^{n-1} \beta_{n-k}\gamma_{n-k} + \sum_{m=1}^{m(\lambda_*,i)-1} \frac{\alpha_{n-m}}{\beta_{n-m}\gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k}\gamma_{n-k} +$$

$$\sum_{m=m(\lambda_*,i)}^{n-2} \frac{\alpha_{n-m}}{\beta_{n-m}\gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k}\gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k}\gamma_{n-k}$$

$$\sigma_{n+1} = \alpha_n + \sum_{m=1}^{m(\lambda_*,i)-1} \frac{\alpha_{n-m}}{\beta_{n-m}\gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k}\gamma_{n-k} +$$

$$\sum_{m=m(\lambda_*,i)}^{n-1} \frac{\alpha_{n-m}}{\beta_{n-m}\gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k}\gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k}\gamma_{n-k}$$

$$\sum_{m=m(\lambda_*,i)}^{n-1} \frac{\gamma_{n-m(\lambda_*,i)}\alpha_{n-m}}{\beta_{n-m}\gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k}\gamma_{n-k} + \sigma_1\gamma_{n-m(\lambda_*,i)} \prod_{k=0}^{m(\lambda_*,i)-1} \beta_{n-k}\gamma_{n-k}$$

Introduce the notation

$$\alpha_* = \inf_{\delta_i} \frac{|\alpha_i|}{\beta_i}, \ \alpha^* = \sup_{0 \leq i \leq n} \frac{|\alpha_i|}{\beta_i}$$

$$\beta_* = \inf_{0 \leq i \leq n, \ \delta_i = 1} \beta_i, \ \beta^* = \sup_{0 \leq i \leq n, \ \delta_i = 1} \beta_i$$

$$A(m) = \sum_{k=1, \delta_j = 1}^{m} ln_2(\beta_j) + \sum_{k=1, \delta_j > 1}^{m} ln_2(\beta_j) = A_1(m) + A_2(m)$$

by definition α_i, γ_i

$$1 < \alpha_* < \alpha^* < 1.3$$

$$2 < \beta* < \beta^* < 2/(1 - \ln 2/2)$$

Rewrite equation (6) using i, $m(\lambda_*, i)$ and assuming that we have only one zero

$$\sigma_1 \leq \left(\frac{\sigma_n}{\beta^{*i} 2^{A(n-1)}} - \frac{\alpha^*}{\beta^{*i} 2^{A(n-1)}} \sum_{m=1}^{m=m(\lambda_*,i)} 2^{A(m)}\right) +$$

$$\left(\frac{\sigma_n}{\beta^{*i} 2^{A(n-1)}} - \frac{\alpha^*}{(\beta^{*i} 2^{A(n-1)})} \sum_{m=m(\lambda_*,i)}^{m=n-1} 2^{A(m)}\right)$$

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$$\sigma_1 < 2\frac{1.3}{\beta^i}$$

It follows that after zero there cannot be more than three ones. Suppose that between two zeros there are two ones, using Theorem 1 and denoting

$$x(k) = 2^{-\sigma_{i+k}},\ k \in \{1,2,3,4,5\}$$

we get the system of equations

$$Ax = b \tag{5}$$

where A, A^{-1}, b are defined below.

$$A = \begin{bmatrix} 2 & -s & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 2 & -t \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{6}$$

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{7}$$

$$s = 2^{-\delta_i}, t = 2^{-\delta_{i+3}}$$

$$A^{-1} = \begin{bmatrix} 1/2 & s/4 & s/8 & s/16 & s*t/32 \\ 0 & 1/2 & 1/4 & 1/8 & t/16 \\ 0 & 0 & 1/2 & 1/4 & t/8 \\ 0 & 0 & 0 & 1/2 & t/4 \\ 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \tag{8}$$

Using Theorem 1 again

$$2^{1-\sigma_{i+4}} = 2\left[\frac{1}{2} + \frac{t(t+1)}{4}\right] = 2^{-\delta_{i+4}-\sigma_{i+5}} + 1$$

$$\frac{t(t+1)}{2} = 2^{-\delta_{i+4}-\sigma_{i+5}}$$

$$\frac{2^{-\delta_{i+4}} + 1}{2} = 2^{-\sigma_{i+5}}$$

Continuing the calculations, we obtain

$$\sigma_{i+4} \geq 1 - 2^{-\delta_{i+4}-1} / \ln 2$$

from Theorem 1 and the last estimate implies

$$\delta_{i+4} > 2$$

By considering three units between zeros, we obtain the following matrix:

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$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -s & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

The inverse of this matrix is given by:

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{s}{8} & \frac{s}{16} & \frac{s}{32} & \frac{s}{64} & \frac{st}{128} \\ 0 & \frac{1}{2} & \frac{s}{4} & \frac{s}{8} & \frac{s}{16} & \frac{s}{32} & \frac{st}{64} \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{t}{32} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{t}{16} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{t}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{t}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

We can also observe that the number of zeros is greater than the number of ones, which implies the statement of the theorem. □

Theorem 3. *Let*

$$a_n = \sum_{i=0}^{n} \gamma_i 2^i, \quad n > 1000, \quad \gamma_i \in \{0, 1\},$$

then

$$\exists j^* < 10, \quad and \quad a_{4n-j^*} < a_n.$$

Proof. Introduce operators defined as follows:

$$Pf = \frac{f}{2}, \quad Tf = 3f + 1, \quad Zf = 3f,$$

$$T_i \in \{P, T\}, \quad R_i \in \{Z, P\}.$$

Consider all possible Collatz sequence behaviors that can be written as follows:

$$a_{n+n} = T_1 T_2 \dots T_n a_n,$$

We need to calculate an estimate for each $2n$-th term of the Collatz sequence based on the number of applied P, T, Z operators within n steps.

$$a_{n+n} = T_n T_{n-1} \dots T_1 a_n,$$

Let a_n have m ones in its binary representation, then count the number of Z operator applications by the following formula:

$$m = \sum_{\substack{R_i = Z, \\ i \leq n}} 1,$$

and the number of P operator applications by the following formula:

$$\sum_{\substack{R_i = P, \\ i \leq n}} 1 = m + n - m = n.$$

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Since each Z application is followed by a P operator, and the number of P operator applications corresponds to the number of zeros in a_n, which is $n-m$. According to the Collatz rules, after n steps we have:

$$a_{n+n} = \frac{3^m}{2^n} a_n + T_n T_{n-1} \dots T_1 1 = \frac{3^m}{2^n} a_n + B_n,$$

$$B_n \le 2^{-n+m} \sum_{j=1}^{m} \frac{3^j}{2^j} a_n < 2^{-n+m} \cdot 3^m / 2^m \cdot a_n \le 2^{-2n+1} \cdot 3^m \cdot a_n.$$

According to the last formula, we see that the growth of each sequence member depends on the number of ones in its binary representation. Next, we show that a large number of ones on the $2n$-th step leads to an increase in the number of zeros on the $3n$-th step for the binary representation, according to the previous theorems, which implies a decrease in subsequent sequence members:

$$a_{2n} = 3^m a_n \cdot 2^{-n} + B_n = 3^m + 3^m (a_n - 2^n) + B_n,$$

Repeating the reasoning of Theorem 2, consider the equation

$$2^x = a_{2n} = 3^m a_n \cdot 2^{-n} + B_n = 3^m + 3^m (a_n - 2^n) \cdot 2^{-n} + B_n,$$

$$x \ln 2 = m \ln(3) + \ln\big(1 + (a_n - 2^n) \cdot 2^{-n} + B_n \cdot 3^{-m}\big),$$

From the last equation, to apply the results of theorem 2, we need $\sigma_1 > \frac{1}{2\ln 2}$. To satisfy the last inequality, consider $m_j = m - j$, $\theta = (a_n - 2^n) \cdot 2^{-n}$,

$$\{x\} = \min_{j<10} \left\{ \frac{(m-j)\ln(3)}{\ln(2)} + \frac{\ln(1+\theta)}{\ln 2} + F_j\left(\frac{1}{2^n \ln 2}\right) \right\},$$

Consider $p = (m-j)\frac{\ln 3}{\ln 2} = (2k+l)1.5849625007\dots$, $\epsilon = 1.5849625007\dots - 1.5$, we get

$$p = (2k+l)(1.5 + \epsilon + \frac{\ln(1+\theta)}{\ln 2}) = 3k + (2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2},$$

$$\{p\} = \{1.5 \cdot l + (2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{1.5 \cdot l + \{(2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\}\},$$

Choosing l from even numbers less than 10, if the inequalities $0 \le \{(2k) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} \le 0.5$,

$$\{p\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\},$$

Choosing l from odd numbers less than 10, if the inequalities $0.5 < \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} < 1$,

$$\{p\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{0.5 + (2k+l) \cdot \epsilon\},$$

Using $\epsilon < 0.1$, also satisfies the condition $\sigma_1 = 1 - \{x\} > 0.55$

$$m^* \text{ number of non-zero } \gamma_i,$$

According to theorem 2 we get

$$m^* \le n/2 + (n - j^*) \cdot \ln 3 / \ln 2/2,$$

According to our application of the Collatz rules, we have an element a_{4n-j^*}, and the order of its binary representation is

$$n_2 = n + (n - j^*) \cdot \ln 3 / \ln 2/2,$$

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After $3n - j^*$ steps of applying the Collatz rules we have

$$a_{4n-j^*} = \frac{3^{m^*}}{2^{2n-j^*}} a_{2n} + T_{3n-j^*} T_{3n-1-j^*} \dots T_1 1 = \frac{3^{m^*}}{2^{2n}} a_{2n} + B_{3n},$$

$$a_{4n-j^*} = \frac{3^{m^*}}{2^{2n}} a_{2n} + T_{3n-j^*} T_{3n-j^*-1} \dots T_1 1 = \frac{3^{m^*}}{2^{2n}} \left(\frac{3^m}{2^{n-j^*}} a_n + B_n \right) + B_{3n-j^*},$$

$$a_{4n-j^*} = 3^{m^*+m} \cdot 2^{-3n-j^*} a_n + 3^{m^*} \cdot 2^{-2n-j^*} B_n + B_{3n-j^*},$$

$$a_{4n-j^*} \le q_1 \cdot a_n,$$

By the definition of m^*, l^*, B_n we obtain

$$q_1 < 1,$$

Using $n > 1000$, it follows that $q_1 < 1 \Rightarrow a_{4n-j^*} < a_n$. □

Theorem 4. *Let*

$$a_n = \sum_{i=0}^{n} \gamma_i 2^i, \quad n > 1000, \quad \gamma_i \in \{0, 1\},$$

then for a_n the Collatz conjecture holds.

Proof. The proof follows from Theorems 1-3. □

Conclusion

Our assertion proves that after $3n - j^*$ steps, the sequence with an initial binary length of n arrives at a number strictly less than the initial one, thus resolving the Collatz conjecture. Since applying this process n times will inevitably lead us to 1.

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