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Collatz Conjecture

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Abstract: This paper presents an analysis of the number of zeros in the binary representation of natural numbers. The primary method of analysis involves the use of the concept of the fractional part of a number, which naturally emerges in the determination of binary representation. This idea is grounded in the fundamental property of the Riemann zeta function, constructed using the fractional part of a number. Understanding that the ratio between the fractional and integer parts of a number, analogous to the Riemann zeta function, reflects the profound laws of numbers becomes the key insight of this work. The findings suggest a new perspective on the interrelation between elementary properties of numbers and more complex mathematical concepts, potentially opening new directions in number theory and analysis.

Keywords: binary representation; Collatz conjecture

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1. Introduction

We will use the following well-known fact: if, for the members of the Collatz sequence, zeros predominate in their binary representation, then these members will lead to a decrease in the subsequent members according to the Collatz rule. A striking example is when the initial number in the Collatz sequence is equal to 2^n. Let's write the solution of the equation $n = 2^x$ in the form $x = \{x\} + [x]$ and note that the smaller x, the more zeros in the corresponding binary representation for n. Developing this idea, we come to the following steps.

- Analysis of the binary representation of simple cases of natural numbers.
- Creation of a process for decomposing an arbitrary natural number into powers of two.
- Analysis of the proximity of the process to binary decomposition at the completion of decomposition at each stage.
- Calculation of the number of zeros in the binary decomposition of a natural number.
- Estimation of the Collatz sequence members depending on the number of ones in the binary decomposition.

2. Results

This document reveals a comprehensive solution to the Collatz Conjecture, as first proposed in [1]. The Collatz Conjecture, a well-known unsolved problem in mathematics, questions whether iterative application of two basic arithmetic operations can invariably convert any positive integer into 1. It deals with integer sequences generated by the following rule: if a term is even, the subsequent term is half of it; if odd, the next term is the previous term tripled plus one. The conjecture posits that all such sequences culminate in 1, regardless of the initial positive integer. Named after mathematician Lothar Collatz, who introduced the concept in 1937, this conjecture is also known as the 3n + 1 problem, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, or the Syracuse problem. The sequence is often termed the hailstone sequence due to its fluctuating nature, resembling the movement of hailstones. Paul Erdős and Jeffrey Lagarias have commented on the complexity and mathematical depth of the Collatz Conjecture, highlighting its challenging nature. Consider an operation applied to any positive integer:

- Divide it by two if it's even.

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- Triple it and add one if it's odd.

This operation is mathematically defined as:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \mod 2, \\ 3n+1, & \text{if } n \equiv 1 \mod 2. \end{cases}$$

A sequence is formed by continuously applying this operation, starting with any positive integer, where each step's result becomes the next input. The Collatz Conjecture asserts that this sequence will always reach 1 Recent substantial advancements in addressing the Collatz problem have been documented in works [2]. Now let's move on to our research, which we will conduct according to the announced plan. For this, we will start with the following

Theorem 1. *Let*

$$M = 3^n = 2^{[\alpha]+\{\alpha\}} = \sum_{i=1}^{n^*} \gamma_i 2^i,$$
$$1 - \{\alpha\} > 0.5, \quad n^* = \left[n\frac{\ln(3)}{\ln(2)} \right], \tag{1}$$

then

$$\sum_{\gamma_i=0} 1 \geq \frac{n^*}{2}.$$

Proof. Let

$$3^n = 2^\alpha \Rightarrow \alpha = n\frac{\ln(3)}{\ln(2)} \Rightarrow 3^n = 2^{[\alpha]+\{\alpha\}}.$$

Using Theorem 1, we create the sequence

$$\epsilon_i, m_i, \epsilon_1 = \{\alpha\},$$

$$2^{\epsilon_1} = \sum_{k=0}^{i-1} 2^{[\alpha_k]-\alpha_1} + 2^{\alpha_i-\alpha_1}.$$

Assuming that the process of binary decomposition according to formula (1) stops at the j-th step, it follows that the remaining terms of the decomposition are zeros, and we immediately reach the truth of the theorem statement. Therefore, we will consider the case when the generation of the decomposition according to formula (1) does not stop, and j reaches n. This means that all $\sigma_j > 0, j < n$ A more detailed analysis of the number of zeros and ones in our binary representation is conducted. The following notations are introduced:
l- the number of zeros in the binary representation.
m- the number of ones in the binary representation.
n- the order of binary decomposition, then
n=l+m.

$$\delta_j = 1, \alpha_j = 0, \beta_j = \left((1 - \frac{\ln 2\sigma_{j+1}}{2})/2 + F_j \left(\frac{\sigma_{j+1}^2}{12} \right)) \right)^{-1}$$

$$\delta_j > 1, \alpha_j = -2^{\delta_j} \left(1 - \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + 2^{-\delta_j} R_j \left(\frac{\ln^2 2\sigma_{j+1}^3}{8} + \frac{2^{-2\delta_j+1}}{\ln 2} \right) \right), \beta_j = 2^{\delta_j}$$

To solve the following equations

$$\sigma_{j+1} = \alpha_j + \beta_j \sigma_j$$

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we introduce the notations λ_k- the number of ones after the appearance of $\alpha_k > 0$ and before the next appearance of a zero in the binary decomposition and

$$\gamma_k = \prod_{m=k+1}^{k+1+\lambda_k} \beta_m, \ \alpha_{k+1} > 0$$

Consider the set

$$\lambda_1, \lambda_2, ...\lambda_n, .$$

by definition $\lambda_1 \geq 1$
Define:

$$m(\lambda_*, i) = \inf_k \{k | \lambda_{k+l} \geq 1 + i.l \geq 0\}$$

if the set of k satisfying the condition is not empty. A series of transformations are conducted for understanding the next steps.

$$\sigma_{j+1} = \alpha_j + \beta_j \gamma_j \sigma_{j-k}$$

continuing the transformations we get

$$\sigma_{n+1} = \alpha_n + \beta_1 \gamma_1 \frac{\alpha_1}{\beta_1 \gamma_1} \prod_{k=0}^{n-2} \gamma_{n-k} \beta_{n-k} + \sum_{m=1}^{n-2} \gamma_{n-m} \beta_{n-m} \frac{\alpha_{n-m}}{\beta_{n-m} \gamma_{n-m}} \prod_{k=0}^{m-1} \beta_{n-k} \gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k}$$

$$\sigma_{n+1} = \alpha_n + \frac{\alpha_1}{\beta_1 \gamma_1} \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k} + \sum_{m=1}^{n-2} \frac{\alpha_{n-m}}{\beta_{n-m} \gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k} \gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k} \tag{2}$$

Considering the definition of $m(\lambda_*, i)$, in the case of $\lambda_i > 1$

$$\sigma_{n+1} = \alpha_n + \frac{\alpha_1}{\beta_1 \gamma_1} \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k} + \sum_{m=1}^{m(\lambda_*, i)-1} \frac{\alpha_{n-m}}{\beta_{n-m} \gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k} \gamma_{n-k} +$$

$$\sum_{m=m(\lambda_*, i)}^{n-2} \frac{\alpha_{n-m}}{\beta_{n-m} \gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k} \gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k}$$

Introduce the notations

$$\alpha_* = \inf_{0 \leq i \leq n} \frac{|\alpha_i|}{\beta_i}, \ \alpha^* = \sup_{0 \leq i \leq n} \frac{|\alpha_i|}{\beta_i}$$

$$\beta_* = \inf_{0 \leq i \leq n, \ \delta_i = 1} \beta_i, \ \ \beta^* = \sup_{0 \leq i \leq n, \ \delta_i = 1} \beta_i$$

$$A(m) = \sum_{k=1, \delta_j = 1}^{m} ln_2(\beta_j) + \sum_{k=1, \delta_j > 1}^{m} ln_2(\beta_j) = A_1(m) + A_2(m)$$

Note that δ_k, σ_k appear at the points with coordinates $x(\delta_k), x(\sigma_k)$, $x(\delta_k) = x(\sigma_k)$ and by definition α_i, γ_i

$$1 < \alpha_* < \alpha^* < 1.3$$

$$2 < \beta_* < \beta^* < 2/(1 - \ln 2/2)$$

Thus all possible variants with L-zeros will be defined by all possible sets of

$$(\delta_1, \delta_2 .... \delta_n)$$

With corresponding coordinates

$$(x(\delta_1), x(\delta_2) .... x(\delta_n))$$

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Let now

$$m(\lambda_*, 2) = n-1,\ m(\lambda_*, 2) < n-1$$

,

$$\gamma_{1*} = \inf\{\gamma_{n-m}, m < m(\lambda_*, 2)\},$$

,

$$\gamma_{2*} = \inf\{\gamma_{n-m}, m \geq m(\lambda_*, 2)\},$$

,

$$\gamma_1{}^* = \sup\{\gamma_{n-m}, m < m(\lambda_*, 2)\},$$

,

$$\gamma_2{}^* = \sup\{\gamma_{n-m}, m \geq m(\lambda_*, 2)\},$$

,

$$\theta = \frac{1+2^{2n\beta}}{(1+2^{2\beta})2^{2n\beta}}$$

,

$$\sigma_1 \leq \frac{\sigma_n}{2^{A(n-1)}} - \frac{\alpha^*}{\gamma_{1*}2^{A(n-1)}} \sum_{m=1}^{m=m(\lambda_*,i)} 2^{A(m)} +$$

$$\frac{\alpha^*}{\gamma_{2*}2^{A(n-1)}} \sum_{m=m(\lambda_*,i)}^{m=n-1} 2^{A(m)}$$

considering all mentioned above, we get the following estimate

$$\alpha_* \frac{1}{\gamma_{2*}\theta} \leq \sigma_1 \leq \alpha^* \frac{1}{\gamma_{2*}\theta}$$

$$\sigma_1 \leq 1.3$$

$$\sigma_2 \leq 1.3 * (1 - ln2/2)/2$$

From the last estimate, it follows that after a zero, no more than two ones can follow, after which at least one zero will appear, as these arguments can be sequentially applied to $\sigma_i 1, \sigma_i 2$.... where $\alpha_i 1 > 0, \alpha_i 2 > 0$..... Note that in our reasoning we used that the first one does not introduce any changes in the process of estimating σ_j as $\beta_j = 2^{\delta_j}$.
Let's now move to more precise estimates Consider the following equalities

$$\sigma_{j+1} = \alpha_j + \beta_j \gamma_j \sigma_{j-k}$$

$$\sigma_{j+1} = 2^{\delta_j}(-1 + \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + \gamma_j \sigma_{j-k})$$

Consider the variant $k = 0$ then

$$\sigma_{j+1} = 2^{\delta_j}(-1 + \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + \sigma_j)$$

from which it follows

$$\sigma_j \geq 1 - \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2}$$

Here we have σ_{j+1} can approach zero only if σ_j approaches 1. That is, we will get the number of ones balanced by a larger number of zeros !
Consider the variant $k = 1$ then

$$\sigma_{j+1} = \alpha_j + \beta_j \beta_{j-1} \sigma_{j-1}$$

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$$\sigma_{j+1} \geq 2^{\delta_j}(-1 + \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + \beta_{j-1}/2)$$

$$\sigma_{j+1} \geq \frac{1 - 2^{-\delta_j+1}}{\ln 2}$$

from which it follows that $\delta_j = 2$ as for $\delta_j > 2$ we get a contradiction.
Consider the variant $k = 2$ then

$$\sigma_{j+1} \geq 2^{\delta_j}(1 + \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2})$$

from which it follows that this is not possible.

$$\Rightarrow L \geq n/2 \Rightarrow$$

The statement of the theorem is true. □

Theorem 2. *Let*

$$M = 3^n = 2^{[\alpha]+\{\alpha\}} = \sum_{i=1}^{n^*} \gamma_i 2^i,$$

$$1 - \{\alpha\} > 0.5, \quad n^* = \left[n\frac{\ln(3)}{\ln(2)} \right], \tag{3}$$

then

$$\sum_{\gamma_i=0} 1 \geq \frac{n^*}{2}.$$

Proof. Let

$$3^n = 2^\alpha \Rightarrow \alpha = \frac{n}{\ln 3/\ln 2} \Rightarrow 3^n = 2^{[\alpha]+\{\alpha\}}.$$

Using Theorem 1, we create the sequence

$$\epsilon_i,\ m_i,\ \epsilon_1 = \{\alpha\},$$

$$2^{\epsilon_1} = \sum_{k=0}^{i-1} 2^{[\alpha_k]-\alpha_1} + 2^{\alpha_i - \alpha_1}.$$

Suppose the binary decomposition process, according to formula (1), stops at the j-th step. Then, it immediately follows that the remaining terms of the decomposition are zeros, and we immediately achieve the truth of the Theorem. Therefore, we will consider the case when the generation of the decomposition according to formula (1) does not stop, and j reaches n. This means that all $\sigma_j > 0, j < n$. Let's conduct a more detailed analysis of the number of zeros and ones in our binary representation. Introduce the following designations:

$$l - \text{the number of zeros in the binary representation,}$$

$$m - \text{the number of ones in the binary representation,}$$

$$n - \text{the dimensionality of the binary decomposition, where } n = l + m.$$

$$\delta_j = 1, \alpha_j = 0, \beta_j = \left((1 - \frac{\ln 2\sigma_{j+1}}{2})/2 + F_j\left(\frac{\sigma_{j+1}^2}{12}\right) \right)^{-1},$$

$$\delta_j > 1, \alpha_j = -2^{\delta_j}\left(1 - \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + 2^{-\delta_j} R_j \left(\frac{\ln^2 2\sigma_{j+1}^3}{8} + \frac{2^{-2\delta_j+1}}{\ln 2} \right) \right), \beta_j = 2^{\delta_j}.$$

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To solve the following equations

$$\sigma_{j+1} = \alpha_j + \beta_j \sigma_j,$$

let's introduce designations λ_k - the number of ones after the appearance of $\alpha_k > 0$ and before the next appearance of zero in the binary decomposition, and

$$\gamma_k = \prod_{m=k+1}^{k+1+\lambda_k} \beta_m, \quad \alpha_{k+1} > 0.$$

Consider the set $\lambda_1, \lambda_2, ..., \lambda_n$. By definition, $\lambda_1 \geq 1$. Let's define:

$$m(\lambda_*, i) = \inf_k \{k | \lambda_{k+l} \geq 1 + i, l \geq 0\}$$

if the set satisfying the condition is not empty. Let's perform a series of transformations to understand the next steps.

$$\sigma_{j+1} = \alpha_j + \beta_j \gamma_j \sigma_{j-k}.$$

Continuing transformations, we get

$$\sigma_{n+1} = \alpha_n + \beta_1 \gamma_1 \frac{\alpha_1}{\beta_1 \gamma_1} \prod_{k=0}^{n-2} \gamma_{n-k} \beta_{n-k} + \sum_{m=1}^{n-2} \gamma_{n-m} \beta_{n-m} \frac{\alpha_{n-m}}{\beta_{n-m} \gamma_{n-m}} \prod_{k=0}^{m-1} \beta_{n-k} \gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k}.$$

$$\sigma_{n+1} = \alpha_n + \frac{\alpha_1}{\beta_1 \gamma_1} \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k} + \sum_{m=1}^{n-2} \frac{\alpha_{n-m}}{\beta_{n-m} \gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k} \gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k}. \tag{4}$$

Taking into account the definition of $m(\lambda_*, i)$, in case of existence $\lambda_i > 1$

$$\sigma_{n+1} = \alpha_n + \frac{\alpha_1}{\beta_1 \gamma_1} \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k} + \sum_{m=1}^{m(\lambda_*, i)-1} \frac{\alpha_{n-m}}{\beta_{n-m} \gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k} \gamma_{n-k} +$$

$$\sum_{m=m(\lambda_*, i)}^{n-2} \frac{\alpha_{n-m}}{\beta_{n-m} \gamma_{n-m}} \prod_{k=0}^{m} \beta_{n-k} \gamma_{n-k} + \sigma_1 \prod_{k=0}^{n-1} \beta_{n-k} \gamma_{n-k}.$$

Let's introduce designations $\alpha_* = \inf_{0 \leq i \leq n} \frac{|\alpha_i|}{\beta_i}, \alpha^* = \sup_{0 \leq i \leq n} \frac{|\alpha_i|}{\beta_i}$, $\beta_* = \inf_{0 \leq i \leq n, \delta_i = 1} \beta_i$, $\beta^* = \sup_{0 \leq i \leq n, \delta_i = 1} \beta_i$

$$A(m) = \sum_{k=1, \delta_j = 1}^{m} \ln_2(\beta_j) + \sum_{k=1, \delta_j > 1}^{m} \ln_2(\beta_j) = A_1(m) + A_2(m).$$

Note that δ_k, σ_k occur at points with coordinates $x(\delta_k), x(\sigma_k), x(\delta_k) = x(\sigma_k)$ and by definition α_i, γ_i

$$1 < \alpha_* < \alpha^* < 1.3$$

$$2 < \beta* < \beta^* < 2/(1 - \ln 2/2)$$

So all possible options with L-zeros will be determined by all possible options of sets

$$(\delta_1, \delta_2, ...\delta_n)$$

With corresponding coordinates

$$(x(\delta_1), x(\delta_2)...x(\delta_n))$$

Now suppose that

$$m(\lambda_*, 2) = n - 1, m(\lambda_*, 2) < n - 1$$

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,

$$\gamma_{1*} = \inf\{\gamma_{n-m}, m < m(\lambda_*, 2)\},$$
$$\gamma_{2*} = \inf\{\gamma_{n-m}, m \geq m(\lambda_*, 2)\},$$
$$\gamma_1{}^* = \sup\{\gamma_{n-m}, m < m(\lambda_*, 2)\},$$
$$\gamma_2{}^* = \sup\{\gamma_{n-m}, m \geq m(\lambda_*, 2)\},$$
$$\theta = \frac{1 + 2^{2n\beta}}{(1 + 2^{2\beta})2^{2n\beta}},$$
$$\sigma_1 \leq \frac{\sigma_n}{2^{A(n-1)}} - \frac{\alpha^*}{\gamma_{1*}2^{A(n-1)}} \sum_{m=1}^{m=m(\lambda_*,i)} 2^{A(m)} +$$
$$\frac{\alpha^*}{\gamma_{2*}2^{A(n-1)}} \sum_{m=m(\lambda_*,i)}^{m=n-1} 2^{A(m)}.$$

Taking into account everything said above, we obtain the following estimate

$$\alpha_* \frac{1}{\gamma_{2*}\theta} \leq \sigma_1 \leq \alpha^* \frac{1}{\gamma_{2*}\theta}.$$

From the last estimate, it follows that after zero, only three ones can follow, after which there will be at least one zero, since these arguments can be sequentially applied to $\sigma_2, \sigma_3, ...$ Let's move on to more accurate estimates. Consider the following equalities

$$\sigma_{j+1} = \alpha_j + \beta_j \gamma_j \sigma_{j-k}$$

$$\sigma_{j+1} = 2^{\delta_j}(-1 + \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + \gamma_j \sigma_{j-k})$$

Consider the case $k = 0$ then

$$\sigma_{j+1} = 2^{\delta_j}(-1 + \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + \sigma_j)$$

from which it follows $\sigma_j \geq 1 - \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2}$ Here we have σ_{j+1} can approach zero only if σ_j approaches 1. That is, we get the number of ones is balanced by a large number of zeros in! Consider the case $k = 1$ then

$$\sigma_{j+1} = \alpha_j + \beta_j \beta_{j-1} \sigma_{j-1}$$
$$\sigma_{j+1} \geq 2^{\delta_j}(-1 + \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} + \beta_{j-1}/2)$$
$$\sigma_{j+1} \geq \frac{1 - 2^{-\delta_j+1}}{\ln 2}$$

from which it follows that $\delta_j = 2$ thus for $\delta_j > 2$ we get a contradiction. Consider the case $k = 2$ then

$$\sigma_{j+1} \geq 2^{\delta_j}(1 + \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2})$$

from which it follows that this is impossible.

$$\Rightarrow L \geq n/2 \Rightarrow \text{The statement of the theorem is true.}$$

□

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Theorem 3. *Let*

$$a_n = \sum_{i=0}^{n} \gamma_i 2^i, \quad n > 1000, \quad \gamma_i \in \{0,1\},$$

then

$$\exists j^* \in \{0,1\}, \quad \textit{and} \quad a_{4n-j^*} < a_n.$$

Proof. Introduce operators defined as follows:

$$Pf = \frac{f}{2}, \quad Tf = 3f + 1, \quad Zf = 3f,$$

$$T_i \in \{P, T\}, \quad R_i \in \{Z, P\}.$$

Consider all possible scenarios of Collatz sequence behavior, which can be written in the following form:

$$a_{n+n} = T_1 T_2 \dots T_n a_n,$$

We need to estimate each $2n$-th term of the Collatz sequence based on the number of applied operators P, T, Z during n steps.

$$a_{n+n} = T_n T_{n-1} \dots T_1 a_n,$$

Let a_n have m ones in its binary representation, then we count the number of applications of operator Z using the following formula:

$$m = \sum_{\substack{R_i = Z, \\ i \le n}} 1,$$

and the number of applications of operator P using the following formula:

$$\sum_{\substack{R_i = P, \\ i \le n}} 1 = m + n - m = n.$$

Since each application of Z is accompanied by operator P, and the number of applications of operator P corresponds to the number of zeros in a_n, which equals $n - m$. According to the rules of Collatz, after n steps we have:

$$a_{n+n} = \frac{3^m}{2^n} a_n + T_n T_{n-1} \dots T_1 1 = \frac{3^m}{2^n} a_n + B_n,$$

$$B_n \le 2^{-n+m} \sum_{j=1}^{m} \frac{3^j}{2^j} a_n < 2^{-n+m} \cdot 3^m / 2^m \cdot a_n \le 2^{-2n+1} \cdot 3^m \cdot a_n.$$

According to the last formula, we see that the growth of each term of the sequence depends on the number of ones in the binary representation. Next, we will show that a large number of ones at the $2n$-th step leads to an increase in the number of zeros at the $3n$-th step for binary representation according to the previous theorems, from which it follows that subsequent terms of the sequence decrease:

$$a_{2n} = 3^m a_n \cdot 2^{-n} + B_n = 3^m + 3^m (a_n - 2^n) + B_n,$$

Repeating the reasoning of Theorem 2, consider the equation

$$2^x = a_{2n} = 3^m a_n \cdot 2^{-n} + B_n = 3^m + 3^m (a_n - 2^n) \cdot 2^{-n} + B_n,$$

$$x \ln 2 = m \ln(3) + \ln\left(1 + (a_n - 2^n) \cdot 2^{-n} + B_n \cdot 3^{-m}\right),$$

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From the last equation, to apply the results of theorem 2, we need $\sigma_1 > \frac{1}{2\ln 2}$. To satisfy the last inequality, consider $m_j = m - j, \theta = (a_n - 2^n) \cdot 2^{-n}$,

$$\{x\} = \min_{j<10} \left\{ \frac{(m-j)\ln(3)}{\ln(2)} + \frac{\ln(1+\theta)}{\ln 2} + F_j \left(\frac{1}{2^n \ln 2} \right) \right\},$$

Consider $p = (m-j)\frac{\ln 3}{\ln 2} = (2k+l)1.5849625007\ldots, \epsilon = 1.5849625007\ldots - 1.5$, we get

$$p = (2k+l)(1.5 + \epsilon + \frac{\ln(1+\theta)}{\ln 2}) = 3k + (2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2},$$

$$\{p\} = \{1.5 \cdot l + (2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{1.5 \cdot l + \{(2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\}\},$$

Choosing l from even numbers less than 10, if inequalities $0 \leq \{(2k) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} \leq 0.5$, are true

$$\{p\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\},$$

Choosing l from odd numbers less than 10, if inequalities $0.5 < \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} < 1$, are true

$$\{p\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{0.5 + (2k+l) \cdot \epsilon\},$$

Using $\epsilon < 0.1$, also satisfy the condition $\sigma_1 = 1 - \{x\} > \frac{1}{2\ln 2}$.

$$m^* \text{ number of non-zero } \gamma_i,$$

According to theorem 2 we get

$$m^* \leq n/2 + (n - j^*) \cdot \ln 3 / \ln 2/2,$$

According to our application of Collatz rules, we have an element a_{4n-j^*}, and the order of its binary representation is

$$n_2 = n + (n - j^*) \cdot \ln 3 / \ln 2/2,$$

After $3n - j^*$ steps of applying Collatz rules we have

$$a_{4n-j^*} = \frac{3^{m^*}}{2^{2n-j^*}} a_{2n} + T_{3n-j^*} T_{3n-1-j^*} \ldots T_1 1 = \frac{3^{m^*}}{2^{2n}} a_{2n} + B_{3n},$$

$$a_{4n-j^*} = \frac{3^{m^*}}{2^{2n}} a_{2n} + T_{3n-j^*} T_{3n-j^*-1} \ldots T_1 1 = \frac{3^{m^*}}{2^{2n}} \left(\frac{3^m}{2^{n-j^*}} a_n + B_n \right) + B_{3n-j^*},$$

$$a_{4n-j^*} = 3^{m^*+m} \cdot 2^{-3n-j^*} a_n + 3^{m^*} \cdot 2^{-2n-j^*} B_n + B_{3n-j^*},$$

$$a_{4n-j^*} \leq q_1 \cdot a_n,$$

By definition of m^*, l^*, B_n we get

$$q_1 < 1,$$

Using $n > 1000$, it follows that $q_1 < 1 \Rightarrow a_{4n-j^*} < a_n$. □

Theorem 4. *Let*

$$a_n = \sum_{i=0}^{n} \gamma_i 2^i, \quad n > 1000, \quad \gamma_i \in \{0, 1\},$$

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then

$$\exists j^* < 0.1n, \quad \text{and} \quad a_{4n-j^*} < a_n.$$

Proof. Let's introduce operators defined by the formulas

$$Pf = \frac{f}{2}, \quad Tf = 3f + 1, \quad Zf = 3f,$$

$$T_i \in \{P, T\}, \quad R_i \in \{Z, P\}.$$

Consider all possible scenarios of the behavior of the Collatz sequence, which can be written in the following form:

$$a_{n+n} = T_1 T_2 \dots T_n a_n,$$

It is necessary to calculate an estimate for each $2n$-th member of the Collatz sequence based on the number of P, T, Z operators applied during n steps.

$$a_{n+n} = T_n T_{n-1} \dots T_1 a_n,$$

Let a_n have m units in its binary representation, then calculate the number of applications of the Z operator by the following formula:

$$m = \sum_{\substack{R_i = Z, \\ i \le n}} 1,$$

and calculate the number of applications of the P operator by the following formula:

$$\sum_{\substack{R_i = P, \\ i \le n}} 1 = m + n - m = n.$$

Since each application of Z is accompanied by the P operator, and the number of applications of the P operator corresponds to the number of zeros in a_n, which is equal to $n - m$. According to the rules of Collatz after n steps, we have:

$$a_{n+n} = \frac{3^m}{2^n} a_n + T_n T_{n-1} \dots T_1 1 = \frac{3^m}{2^n} a_n + B_n,$$

$$B_n \le 2^{-n+m} \sum_{j=1}^{m} \frac{3^j}{2^j} a_n < 2^{-n+m} \cdot 3^m / 2^m \cdot a_n \le 2^{-2n+1} \cdot 3^m \cdot a_n.$$

According to the last formula, we see that the growth of each member of the sequence depends on the number of units in the binary representation. Next, we will show that a large number of units on the $2n$-th step leads to an increase in the number of zeros in the $3n$-th step for the binary representation according to previous theorems, hence the reduction of subsequent members of the sequence:

$$a_{2n} = 3^m a_n \cdot 2^{-n} + B_n = 3^m + 3^m (a_n - 2^n) + B_n,$$

Repeating the reasoning of Theorem 2, consider the equation

$$2^x = a_{2n} = 3^m a_n \cdot 2^{-n} + B_n = 3^m + 3^m (a_n - 2^n) \cdot 2^{-n} + B_n,$$

$$x \ln 2 = m \ln(3) + \ln\left(1 + (a_n - 2^n) \cdot 2^{-n} + B_n \cdot 3^{-m}\right),$$

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From the last equation, in order to apply the results of theorem 2, we need $\sigma_1 = 1 - \{x\} > 0.5$. To fulfill the last inequality, consider $m_j = m - j, \theta = (a_n - 2^n) \cdot 2^{-n}$,

$$\{x\} = \min_{j \in \{0,1\}} \left\{ \frac{(m-j)\ln(3)}{\ln(2)} + \frac{\ln(1+\theta)}{\ln 2} + F_j\left(\frac{1}{2^n \ln 2}\right) \right\},$$

Consider $p = (m-j)\frac{\ln 3}{\ln 2} = (2k+l)1.5849625007\ldots, \epsilon = 1.5849625007\ldots - 1.5$, we get

$$p = (2k+l)(1.5 + \epsilon + \frac{\ln(1+\theta)}{\ln 2}) = 3k + (2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2},$$

$$\{p\} = \{1.5 \cdot l + (2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{1.5 \cdot l + \{(2k+l) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\}\},$$

Choosing $l = 0$, if the inequalities $0 \leq \{(2k) \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} \leq 0.5$ are true,

$$\{p\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\},$$

Choosing $l = 1$, if the inequalities $0.5 < \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} < 1$ are true,

$$\{p\} = \{2k \cdot \epsilon + \frac{\ln(1+\theta)}{\ln 2}\} = \{0.5 + (2k+l) \cdot \epsilon\},$$

Using $\epsilon < 0.1$, we also satisfy the condition $\sigma_1 = 1 - \{x\} > 0.51$.

m^* is the number of non-zero γ_i,

According to theorem 2 we get

$$m^* \leq n/2 + (n - j^*) \cdot \ln 3 / \ln 2/2,$$

According to our application of the Collatz rules, we have the element a_{4n-j^*}, and the order of its binary representation is

$$n_2 = n + (n - j^*) \cdot \ln 3 / \ln 2/2,$$

After $3n - j^*$ steps of applying the Collatz rules, we have

$$a_{4n-j^*} = \frac{3^{m^*}}{2^{2n-j^*}} a_{2n} + T_{3n-j^*} T_{3n-1-j^*} \ldots T_1 1 = \frac{3^{m^*}}{2^{2n}} a_{2n} + B_{3n},$$

$$a_{4n-j^*} = \frac{3^{m^*}}{2^{2n}} a_{2n} + T_{3n-j^*} T_{3n-j^*-1} \ldots T_1 1 = \frac{3^{m^*}}{2^{2n}} \left(\frac{3^m}{2^{n-j^*}} a_n + B_n \right) + B_{3n-j^*},$$

$$a_{4n-j^*} = 3^{m^*+m} \cdot 2^{-3n-j^*} a_n + 3^{m^*} \cdot 2^{-2n-j^*} B_n + B_{3n-j^*},$$

$$a_{4n-j^*} \leq q_1 \cdot a_n,$$

By definition of m^*, l^*, B_n we get

$$q_1 < 1,$$

Using $n > 1000$, implies $q_1 < 1 \Rightarrow a_{4n-j^*} < a_n$. □

Theorem 5. *Let*

$$a_n = \sum_{i=0}^{n} \gamma_i 2^i, \quad n > 1000, \quad \gamma_i \in \{0,1\},$$

then for a_n the Collatz conjecture is true.

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Proof. The proof follows from Theorems 1-3. □

Proof. Proof follows from theorem 1-3

6. Conclusions

Our assertion proves that after $3n$ steps, a sequence with an initial binary length of n arrives at a number strictly smaller than the initial one, from which the solution to the Collatz conjecture follows. This is because by applying this process n times, we are guaranteed to arrive at 1.

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