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[Asset Durmagambetov](#)*

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Article

Collatz Conjecture

Asset Durmagambetov ^{1,*} and Aniyar Durmagambetova ²

¹ aaset.durmagambet@gmail.com

² durmagambetovaa4@gmail.com

* Correspondence: aaset.durmagambet@gmail.com; Tel.: +77787286399

Abstract: This paper presents an analysis of the number of zeros in the binary representation of natural numbers. The primary method of analysis involves the use of the concept of the fractional part of a number, which naturally emerges in the determination of binary representation. This idea is grounded in the fundamental property of the Riemann zeta function, constructed using the fractional part of a number. Understanding that the ratio between the fractional and integer parts of a number, analogous to the Riemann zeta function, reflects the profound laws of numbers becomes the key insight of this work. The findings suggest a new perspective on the interrelation between elementary properties of numbers and more complex mathematical concepts, potentially opening new directions in number theory and analysis.

Keywords: binary representation, Collatz conjecture

MSC: [2010] 11M26

1. Introduction

The paper analyzes the number of zeros in the binary representation of a natural number. The analysis is carried out using the concept of the fractional part of a number, which naturally arises when finding a binary representation. This idea relies on the fundamental property of the Riemann zeta function, which is constructed using the fractional part of a number. Understanding that the ratio of the fractional and integer parts, by analogy with the Riemann zeta function, expresses the deep laws of numbers, will explain the essence of this work.

2. Materials and Methods

This work is based on the following methods

1. Analysis of simple cases of natural numbers
2. A process of expansion of a natural number in powers of two is created.
3. The proximity to the completion of decomposition is analyzed at each stage
4. The number of zeros in the binary expansion of a natural number is calculated

3. Results

This document reveals a comprehensive solution to the Collatz Conjecture, as first proposed in [1]. The Collatz Conjecture, a well-known unsolved problem in mathematics, questions whether iterative application of two basic arithmetic operations can invariably convert any positive integer into 1. It deals with integer sequences generated by the following rule: if a term is even, the subsequent term is half of it; if odd, the next term is the previous term tripled plus one. The conjecture posits that all such sequences culminate in 1, regardless of the initial positive integer.

Named after mathematician Lothar Collatz, who introduced the concept in 1937, this conjecture is also known as the $3n + 1$ problem, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, or the Syracuse problem. The sequence is often termed the hailstone sequence due to its fluctuating nature, resembling the movement of hailstones.

Paul Erdős and Jeffrey Lagarias have commented on the complexity and mathematical depth of the Collatz Conjecture, highlighting its challenging nature.

Consider an operation applied to any positive integer:

- Divide it by two if it's even.
- Triple it and add one if it's odd.

This operation is mathematically defined as:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \equiv 0 \pmod{2}, \\ 3n + 1, & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

A sequence is formed by continuously applying this operation, starting with any positive integer, where each step's result becomes the next input. The Collatz Conjecture asserts that this sequence will always reach 1. Recent substantial advancements in addressing the Collatz problem have been documented in works [2].

Now let's move on to our research, which we will conduct according to the announced plan. For this, we will start with the following

Theorem 1. *Let*

$$\begin{aligned} x \in N, [\alpha_j] - [\alpha_{j+1}] &= \delta_j > 0, \epsilon_1 < 1/2, \\ \alpha_j &= [\alpha_j] + \epsilon_j, \epsilon_j < 1, \\ x &= \sum_{i=1}^{j-1} 2^{[\alpha_i]} + 2^{\alpha_j}, \quad x = \sum_{i=1}^j 2^{[\alpha_i]} + 2^{\alpha_{j+1}}, \quad \sigma_j = 1 - \epsilon_j \end{aligned}$$

Then

as $\delta_j = 1$

$$\sigma_{j+1} \ln 2 = \frac{2\sigma_j \ln 2}{1 - \sigma_{j+1} \ln 2} + o(\sigma_{j+1}^2/4)$$

as $\delta_j > 1$

$$\sigma_{j+1} \ln 2 = -2^{\delta_j-1} \frac{\ln 2 - 2^{-\delta_j-1}}{1 - \sigma_{j+1} \ln 2/2} + 2^{\delta_j-1} \sigma_j \ln 2 \frac{1}{1 - \sigma_{j+1} \ln 2/2} + o(\sigma_{j+1}^2)$$

Proof.

$$\begin{aligned} 2^{\epsilon_j} &= 2^{-\delta_j + \epsilon_{j+1}} + 1 \Rightarrow 2^{1-\sigma_j} = 2^{-\delta_j + 1 - \sigma_{j+1}} + 1 \Rightarrow \\ \ln(2^{1-\sigma_j}) &= \ln 2 - \sigma_j \ln 2 = \ln(2^{-\delta_j + 1 - \sigma_{j+1}} + 1) \end{aligned}$$

Computing as $\delta_j = 1$

$$\ln(2^{-\delta_j + 1 - \sigma_{j+1}} + 1)|_{\delta_j=1} = \ln(2^{-\sigma_{j+1}} + 1) = \ln((1 - \sigma_{j+1} \ln 2 + \sigma_{j+1}^2 \ln^2/2) + 1 + o(\sigma_{j+1}^2/4))$$

$$\ln(2 - \sigma_{j+1} \ln 2 + \sigma_{j+1}^2 \ln^2/2) = \ln 2 + \ln(1 - \sigma_{j+1} \ln 2/2 + \sigma_{j+1}^2 \ln^2/4 + o(\sigma_{j+1}^2/4))$$

$$\ln(2^{-\sigma_{j+1}} + 1) = \ln 2 - \ln 2 \sigma_{j+1}/2 + \ln^2 2 \sigma_{j+1}^2/2 + o(\sigma_{j+1}^2/4)$$

$$\ln 2 - \sigma_j \ln 2 = \ln 2 - \ln 2 \sigma_{j+1}/2 + \ln^2 2 \sigma_{j+1}^2/4 + o(\sigma_{j+1}^2/4)$$

$$\sigma_{j+1} \ln 2 = \frac{2\sigma_j \ln 2}{1 - \sigma_{j+1} \ln 2/2} + o(\sigma_{j+1}^2/4)$$

Repeating computing as $\delta_j > 1$ we get

$$\ln(2^{-\delta_j + 1 - \sigma_{j+1}} + 1) = \ln(2^{-\delta_j + 1} 2^{-\sigma_{j+1}} + 1) =$$

$$\begin{aligned} \ln(1 + 2^{-\delta_j+1} + 2^{-\delta_j+1}[-\sigma_{j+1}\ln 2 + \sigma_{j+1}^2 \ln 2^2/2] + o(\sigma_{j+1}^2/4 + 2^{-\delta_j+1})) = \\ 2^{-\delta_j+1} + 2^{-\delta_j+1}[-\sigma_{j+1}\ln 2 + \sigma_{j+1}^2 \ln 2^2/2] + o(\sigma_{j+1}^2 + 2^{-\delta_j+1}) \Rightarrow \\ \ln 2 - \sigma_j \ln 2 = 2^{-\delta_j+1} + 2^{-\delta_j+1}[-\sigma_{j+1}\ln 2 + \sigma_{j+1}^2 \ln 2^2/2] + o(\sigma_{j+1}^2 + 2^{-\delta_j+1}) \\ \sigma_{j+1} \ln 2 = -2^{\delta_j-1} \frac{\ln 2 - 2^{-\delta_j+1}}{1 - \sigma_{j+1} \ln 2/2} + 2^{\delta_j-1} \sigma_j \ln 2 \frac{1}{1 - \sigma_{j+1} \ln 2/2} + o(\sigma_{j+1}^2 + 2^{-\delta_j+1}) \end{aligned}$$

□

Theorem 2. Let

$$x \in N, \quad \alpha_j = [\alpha_j] \quad x = \sum_{i=1}^{j-1} 2^{[\alpha_i]} + 2^{\alpha_j}$$

Then the number of zeros in the binary representation C_z is calculated by the following formula

$$C_z = \sum_{i=1}^{j-1} [\delta_i - 1] + \alpha_j - 1$$

Proof.

$$C_z = \sum_{i=1}^{j-1} [\alpha_i - \alpha_{i+1} - 1] + \alpha_j - 1$$

By definition δ_i

$$C_z = \sum_{i=1}^{j-1} [\delta_i - 1] + \alpha_j - 1$$

□

Let's introduce μ_k, ν_k for $x = \sum_{i=0}^n \gamma_i 2^i$ by following rules

$$\begin{aligned} \gamma_k + \gamma_{k+1} = 1, \quad \gamma_{k+\mu_k} + \gamma_{k+\mu_k+1} = 1, \quad \prod_{i=k+1}^{i=\mu_k} \gamma_i = 1; \\ \gamma_j + \gamma_{j+1} = 1, \quad \gamma_{j+\nu_j} + \gamma_{j+\nu_j+1} = 1, \quad \nu_j = \sum_{i=j+1}^{i=\nu_j} (1 - \gamma_i) \end{aligned}$$

another words

 μ_k , is count of ones starting at point k with no zeros in between until the first zero or until the end of the sequence ν_j is count of zeros starting at point j with no ones in between until the first zero or until the end of the sequence**Theorem 3.** Let

$$x = 3^n = 2^{[\alpha] + \{\alpha\}} = \sum_{i=1}^{n^*} \gamma_i 2^i,$$

$$\{\alpha\} > \ln 2, \quad n^* = n * [\ln(3)/\ln(2)] \quad (1)$$

then

$$\sum_{\gamma_i=0} 1 \geq n^*/2 - 5$$

Proof.

$$3^n = 2^\alpha \Rightarrow \alpha = n / \ln(3) / \ln(2) \Rightarrow 3^n = 2^{[\alpha] + \{\alpha\}}$$

Using Theorem 1, we create a sequence

$$\epsilon_i, m_i, \epsilon_1 = \{\alpha\}$$

$$2^{\epsilon_1} = \sum_{k=0}^{i-1} 2^{[\alpha_k] - \alpha_1} + 2^{\alpha_i - \alpha_1}$$

Suppose

$$\sum_{\gamma_i=0} 1 = 0$$

then by Theorem 1

$$\begin{aligned} \Rightarrow \sigma_{j+1} \ln 2 &= \frac{2\sigma_j \ln 2}{1 - \sigma_{j+1} \ln 2} + o(\ln 2\sigma_{j+1}^2/4) \Rightarrow \\ 2^{-1}\sigma_{j+1} \ln 2 &= \frac{\sigma_j \ln 2}{1 - \sigma_{j+1} \ln 2} + 2^{-1} * o(\ln 2\sigma_{j+1}^2/4) \end{aligned}$$

After repeating j times we get

$$2^{-j}\sigma_{j+1} \ln 2 = \frac{\sigma_1 \ln 2}{\prod_1^j (1 - \sigma_{k+1} \ln 2/2)} + \sum_1^j 2^{-k} * o(\ln 2\sigma_{k+1}^2/4)$$

By Theorems (1-2) and condition of the current Theorem proceed

$$\ln 2/2 < \sigma_1 \ln 2 < o(\ln 2\sigma_{k+1}^2/4)$$

immediately

$$\Rightarrow \sum_{\gamma_i=0} 1 > 0$$

Let's introduce

$$\text{as } \delta_k = 1 : \alpha_k = 0, \beta_k = \frac{1}{1 - \sigma_{j+1} \ln 2}$$

$$\text{as } \delta_k > 1 : \alpha_k = -2^{\delta_j-1} \frac{\ln 2 - 2^{-\delta_j-1}}{1 - \sigma_{j+1} \ln 2/2}, \beta_k = \frac{2^{\delta_j-1}}{1 - \sigma_{j+1} \ln 2/2}$$

$$\ln 2\sigma_{k+1} = \alpha_k + \beta_k \ln 2\sigma_k$$

$$\sigma_{j+1} \ln 2 = \alpha_j + \sum_{m=1}^{m=j-1} \alpha_{j-m} \prod_{l=1}^{l=m} \beta_{j-l+1} + \prod_{l=0}^{l=j-1} \beta_{j-l} \ln 2\sigma_1 \Rightarrow$$

$$\frac{\sigma_{j+1} \ln 2}{\prod_{l=0}^{l=j-1} \beta_{j-l}} = \sum_{m=0}^{m=j-1} \frac{\alpha_{j-m}}{\prod_{l=m+1}^{l=j-1} \beta_{j-l}} \geq \sigma_1 \ln 2 \Rightarrow$$

By condition the theorem

$$\frac{\sigma_{j+1} \ln 2}{\prod_{l=0}^{l=n-1} \beta_{n-l}} - \sum_{m=0}^{m=n-1} \frac{\alpha_{j-m}}{\prod_{l=m}^{l=n-1} \beta_{j-l}} \geq \sigma_1 \ln 2 \Rightarrow$$

$$\frac{\sigma_n \ln 2}{\prod_{l=0}^{l=n-1} \beta_{n-l}} - \sum_{m=0}^{m=n-1} \frac{\ln 2 - 2^{-\delta_j+1}}{\prod_{l=m+1}^{l=n-1} \beta_{j-l}} \geq \sigma_1 \ln 2 \Rightarrow$$

Suppose $\forall i \in (1, n) \delta_i = 2 \Rightarrow$

$$\frac{\sigma_n \ln 2}{\prod_{l=0}^{l=n-1} \beta_{n-l}} + \sum_{m=0}^{m=n-1} \frac{\ln 2 - 1/2}{2^m} > \sigma_1 \ln 2 \Rightarrow$$

$$2(\ln 2 - 1/2) > \sigma_1 \Rightarrow$$

$\exists \delta_j > 2$

In common case

$$\frac{\sigma_n \ln 2}{\prod_{l=0}^{l=n-1} \beta_{n-l}} + \sum_{m=0}^{m=n-1} \frac{\ln 2 - \frac{2}{2^{\delta_m}}}{2^{\sum_{m=0}^{m=n-1} \beta_m}} > \sigma_1 \ln 2 \Rightarrow$$

and we see in case $\delta_m > 2$ With the growth of $\delta_m > 2$, we see an exponential growth of the denominator, so the influence of $\delta_m > 2$ terms with large values does not have a significant effect on the estimate of the left side of the inequality. The reason is the accumulation of the β_m value corresponding to $\delta_m = 1$. Therefore, the effect, taking into account $\delta_m > 2$ with large values, is insignificant for estimating the left part of the inequality. \Rightarrow statement of Theorem

□

Theorem 4. Let

$$a_n = \sum_{i=0}^n \gamma_i 2^i, \quad n > 1000, \quad \gamma_i \in \{0, 1\}$$

then

$$a_{8n} < a_n$$

Proof. In more detail, the estimation process consists of replacing 3^l in a_{n+1} by formula 7 which does not contain powers of the triple which allows one to evaluate the resulting terms of the Syracuse sequence. as a result, we get the following estimate. Let's introduce operators defined formulas

$$Pf = f/2, \quad Tf = 3f + 1, \quad Zf = 3f$$

Let's consider all possible scenarios of the behavior of the Syracuse sequence, the same possible scenarios can be written in the following form

$$a_{n+n} = T_1 T_2 \dots T_n a_n$$

$$T_i \in \{P, T\}, R_i \in \{Z, P\}, \quad a_{n+n} = R_1 R_2 \dots R_n a_n + A$$

Let's introduce

$$m = \sum_{R_i=Z} 1$$

and compute

$$\sum_{R_i=P} 1 = n - m + m = n$$

By rules of Collatz we have after $2n$ steps

$$a_{n+n} = 3^m / 2^n a_n + B_n$$

where

$$\begin{aligned}
 A_j &= \sum_{R_i=Z,i=1,j} 1, C_j = - \sum_{R_i=Z,i=1,j} 1 - \sum_{R_i=P,i=1,j} 1 \\
 B_n &= \sum_{j=1,n} 3^{A_j} 2^{C_j} \\
 B_n &\leq \sum_{j=1,n} 3^j / 2^j < 23^n / 2^n \leq 2(3/4)^n a_n \\
 A &= a_{2n} = 3^m (a_n * 2^{-n} + B_n) = (a_n * 2^{-n} + B_n) 3^m \\
 A &= \sum_{i=0}^{[\alpha_1]} \gamma_i 2^i, \quad \gamma_i \in \{0, 1\}, \quad \alpha_1 = m * \ln 3 / \ln 2 + \ln (2^{-n} a_n)
 \end{aligned}$$

Let

m^* is count of non zeros of γ_i

l^* is count of zeros of γ_i

by theorem 2 we will have

$$m^* \leq [\alpha_1] / 2 + 5 = [m \ln 3 / \ln 2] / 2 + 5$$

$$l^* \geq [\alpha_1] / 2 - 5 = [m \ln 3 / \ln 2] / 2 - 5$$

After $[\alpha_1]$ steps applying rules of Collatz we have

$$a_{2n+[\alpha_1]} \leq 3^m 3^{\alpha_1/2} 2^{-\alpha_1} 2^5 * 3^5 (a_n * 2^{-n} + B_n) = 3^m q_1 * a_n$$

where

$$q_1 = 3^m 3^{\alpha_1/2} 2^{-\alpha_1} 2^5 * 3^5$$

Repeating the process 3 times and using $n > 1000 \Rightarrow q_3 < 1 \Rightarrow a_{8n} < a_n$

Theorem 5. Let

$$a_n = \sum_{i=0}^n \gamma_i 2^i, \quad n > 1000, \quad \gamma_i \in \{0, 1\}$$

then for a_n Collatz conjecture is true

Proof. Proof follows from theorem 1-7

6. Conclusions

Our assertion proves that after $2n$ of steps the sequence comes to a number less than the start one, from which follows the solution of the Collatz conjecture.

References

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