Article

Not peer-reviewed version

Existence and Properties of Solution of Nonlinear Differential Equations with Impulses at Variable Times

Huifu Xia * , Yunfei Peng , Peng Zhang

Posted Date: 3 January 2024

doi: 10.20944/preprints202401.0171.v1

Keywords: Differential equation; Impulses at variable times; Existence; Qualitative theory; Pulse phenomena

Article

Existence and Properties of Solution of Nonlinear Differential Equations with Impulses at Variable Times

Huifu Xia [1]*, Yunfei Peng [1] and Peng Zhang [2]

[1] School of Mathematics and Statistics, Guizhou University, Guiyang, 550025, China; xiahuifu0907@163.com; pengyf0803@163.com
[2] Department of Mathematics, Zunyi Normal University, Zunyi, 563006, China; 185972912@qq.com
* Correspondence: xiahuifu0907@163.com

Abstract: In this paper, a class of nonlinear ordinary differential equations with impulses at variable times is considered. The existence and uniqueness of solution are given. At the same time, modifying the classical definitions of continuous dependence and Gâteaux differentiability, some results on continuous dependence and Gâteaux differentiable of solution relative to the initial value also are presented in new topology sense. For the autonomous impulsive system, the periodicity of solution is given. As an application, properties of solution for a type of controlled nonlinear ordinary differential equation with impulses at variable times is obtained. These results are foundation to study optimal control problems of systems governed by the differential equations with impulses at variable times.

Keywords: differential equation; impulses at variable times; existence; qualitative theory; pulse phenomena

MSC: 34A37; 34A12

1. Introduction

We begin with introducing the problem studied. Let $\mathbb{R}^+ \triangleq [0, +\infty)$, $Y(t) = \{y_i(t) | i \in \Lambda \triangleq \{1, 2, \cdots, p\}\}$, $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $y_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ and $J_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $(i = 1, 2, \cdots, p)$ are given maps. Consider the following differential equations with impulses at variable times

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & \{x(t)\} \bigcap Y(t) = \emptyset, t \geq 0, \\ x(t+) = J_i(x(t)) + x(t), & \{x(t)\} \bigcap Y(t) = y_i(t), t \geq 0, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

The main purpose of this study are (i) to provide a sufficient condition to the existence and uniqueness of solution x for the impulsive system (1.1); and (ii) to give the necessary and sufficient condition to the exact times for the solution x meets the curve Y; (iii) to present properties of solution relative to the initial value.

There are some interesting phenomena for the impulsive system (1.1). All of first, it is clear that the system $\dot{x}(t) = x(t) + u(t)$ is controllable, but the following impulsive system

$$\begin{cases} \dot{x}(t) = x(t) + u(t), & x(t) \neq 1, \\ x(t+) = 0, & x(t) = 1 \end{cases}$$

is not controllable. Similarly, the system $\dot{x}(t) = -x(t)$ is stable, but the impulsive system

$$\begin{cases} \dot{x}(t) = -x(t), & x(t) \neq 1, \\ x(t+) = 2, & x(t) = 1 \end{cases}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

is not stable when the initial value $x(0) \geq 1$. Let us look at the third example. Denote by $x(\cdot;0,x_0)$ the solution of the following impulsive differential system

$$\begin{cases} \dot{x}(t) = 2t, & x(t) \neq 1, t > 0, \\ x(t+) = 0, & x(t) = 1, t \geq 0 \end{cases}$$

with the initial value $(0, x_0)$. Then, we have

$$\begin{cases} x\left(t;0,1+\frac{1}{n}\right) = t^2 + 1 + \frac{1}{n}, t \geq 0, \\ x(t;0,1) = \begin{cases} 1, & t = 0, \\ t^2 - m, & t \in \left(\sqrt{m}, \sqrt{m+1}\right], m \in \mathbb{N}. \end{cases} \end{cases}$$

This implies that the impulsive system (1.1) never has not continuous of solution to respect to the initial value in L^1. In addition to, we can also use simple cases to show that the implsive system (1.1) may has not global solution.

The motivation to study this problem is as follows. First of all, many physical phenomena and application models are characterized by (1.1). For example, Integrate-and-Fire models derived from the physical oscillation circuit [1,2] is widely used in neuroscience, neuroscience research concerns current-voltage relations at which the states can be reset once the voltage reaches a threshold level [3,4]. Again, in the application, it is crucial to choose appropriate threshold levels for making decisions to trigger or suspend an impulsive intervention: [5] use glucose threshold level guided injections of insulin; [6] use the time that is economic threshold is reached by the amount of pests as the time of impulsive intervention. Secondly, the theory of impulsive differential equations has been an object of increasing interest because of its wide applicability in biology, in medicine and in more and more fields (see [7] and the its references). Significant interest in the investigation of differential equations with impulse effect is explained by the development of equipment in which significant role is played by complex systems [8–10]. Particularly, the qualitative theory of the impulsive system (1.1) has not been systematically established. It is natural to ask for the present the qualitative theory of the impulsive system (1.1). We will discuss the existence and uniqueness of global solution and its properties for the nonlinear ordinary differential equations with impulses at variable times (1.1) under weaker conditions. It is worth pointing out that the solutions of differential systems with impulses may experience the pulse phenomena, namely the solutions may hit a given surface finite or infinite number of times causing rhythmical beating. This situation presents difficulties in the investigation of properties of solutions of such systems. In addition to, it is not suitable to the stronger conditions for control problem. Consequently, it is desirable to find weaker conditions that guarantee the absence or presence of pulse phenomena. More generally, it is significant to find conditions that the solution meets a given surface just has $k \in \mathbb{N}$ times ($\mathbb{N}$ denote the natural numbers set).

Before concluding this section, we review the previous literature to qualitative analysis on the impulsive differential equations. In fact, the qualitative analysis on the impulsive differential equations can at least be traced back to the works by N.M. Kruylov and N.N. Bogolyubov [11] in 1937 in their classical monograph to **NonlinearMechanics**. Mathematical formulation of the differential equation with impulses at fixed times was firstly presented by A.M. Samoilenko and N.A. Perestyuk [12] in 1974. Since then, the qualitative theory for differential equation with impulses at fixed times in finite (or infinite) dimensional spaces has been extensively studied (see [13–16] and the references therein). For the differential equations with impulses at variable times, A.M. Samoilenko and N.A. Perestyuk [17] in 1981 given the mathematical model

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \neq \tau_i(x(t)), \\ x(t+) = x(t) + J_i(x(t)), & t = \tau_i(x(t)). \end{cases} \tag{1.2}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

Later relevant works were made by D.D. Bainov and A.B. Dishliev [18] in 1984, S. Hu [19] in 1989, etc. For more details, one can see the monographs of V.Lakshmikantham [20] in 1989, A.M. Samoilenko [21] in 1995, D.D. Bainov [22] in 1995 and M. Benchohra [23] in 2006, and so on. In a word, these works established the qualitative theory of (1.2) under stronger conditions. Specially, it is not suitable to the stronger conditions for control problem and impulsive differential equation in infinite dimensional spaces. At the same time, when $y_i, (\forall i \in \Lambda)$ is a one to one mapping, $x(t) = y_i(t)$ be equal to $t = y_i^{-1}(x(t))$. Hence, (1.2) can be treated as a simplified case of (1.1). For the linear case of (1.1), Peng etc [24] obtained the existence and uniqueness of solution and its properties.

The rest of the paper is organized as follows. Section 2 presents the main results. In Sections 3, 4, 5, the proofs of the three main theorems are given in turns. Periodicity of autonomous impulsive system is presented in Section 6. As an application, variation of solution relative to control is presented in Section 7, which is foundation to study optimal control problems of systems governed by the differential equations with impulses at variable times. Finally, some new phenomena of impulsive differential system are summarized.

2. Main Results

We are going to present our main results in the section. To state the first one, some preliminaries will be introduced. Throughout this paper, we fix $T > 0$ and approve that $T = +\infty$. We firstly introduce several definitions. We define the function set: $PC_Y([0,T), \mathbb{R}^n) = \{x : [0,T) \longrightarrow \mathbb{R}^n | x$ is continuous at t when $x(t) \notin Y(t)$, x is left continuous at t and the right limit $x(t+)$ exists when $x(t) \in Y(t)\}$. For $x \in PC_Y([0,T), \mathbb{R}^n)$, $t \in [0,T)$ is called to be an irregular point if $x(t) \in Y(t)$. Otherwise, t is called to be a regular point. One can directly verify that the function set $PC_Y([0,T), \mathbb{R}^n)$ is not linear. Denoted by $B(z, \theta^2)$ the closed ball (in $\mathbb{R}^n$) centered at z and of radius $\theta^2 > 0$.

Definition 2.1. *A piecewise continuous function x_θ is said to be an approximate PC−solution of (1.1) if $x_\theta(\cdot) \equiv x_\theta(\cdot; 0, x_0)$ satisfies the following integral equation with impulses*

$$x_\theta(t) = x_0 + \int_0^t f(\tau, x_\theta(\tau))\, d\tau + \sum_{\substack{0 \le t_i < t, \\ x_\theta(t_i) \in B(y_j(t_i), \theta^2)}} J_j(x_\theta(t_i)). \tag{2.1}$$

Particularly, when $\theta = 0$, we call $x(\cdot) \equiv x^0(\cdot) \in PC_Y([0,T), \mathbb{R}^n)$ is a PC−solution of (1.1).

Meanwhile, we introduced the following basic assumptions.

[F] (1) $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is measurable in t on $\mathbb{R}^+$ and locally Lipschitz continuous in x i.e. for any $\rho > 0$, there exits $L(\rho) > 0$ such that for all $x, y \in \mathbb{R}^n$ with $|x|, |y| \le \rho$, we have

$$|f(t,x) - f(t,y)| \le L(\rho)|x - y| \text{for any} t \in \mathbb{R}^+.$$

(2) There exists a constant $\tilde{k} > 0$ such that

$$|f(t,x)| \le \tilde{k}(1 + |x|) \text{for any} t \in \mathbb{R}^+.$$

(3) f is continuous partially differentiable in x and $f_x(\cdot, x) \in L^1_{loc}(\mathbb{R}^+, \mathbb{R}^{n\times n})$.
[Y](1) $y_i \in C(\mathbb{R}^+, \mathbb{R}^n)$ and $y_i(t) \ne y_j(t)$ for all $t \in \mathbb{R}^+$ and $i \ne j$ $(i, j \in \Lambda)$.
(2) $y_i \in C^1([0,T], \mathbb{R}^n)$ and $f(t, y_i(t)) \ne \dot{y}_i(t)$ $(i \in \Lambda)$.
[J](1) $J_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and

$$Y_i(t) \equiv y_i(t) + J_i(y_i(t)) \ne y_j(t) \text{for all} t \in \mathbb{R}^+ \text{and} i, j \in \Lambda. \tag{2.2}$$

(2) $J_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous partially differentiable.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

It is clear that when the assumptions [F](1)(2) hold, for any fix $(s, z_s) \in \mathbb{R}^+ \times \mathbb{R}^n$, the differential equation

$$\begin{cases} \dot{z}(t) = f(t, z(t)), t > s, \\ z(s) = z_s, \end{cases} \tag{2.3}$$

has a unique solution $z(\cdot; s, z_s) \in C([s, +\infty), \mathbb{R}^n)$ given by

$$z(t; s, z_s) = z_s + \int_s^t f(\tau, z(\tau; s, z_s))d\tau. \tag{2.4}$$

We define several functions:

$$F_i(t; s, z_s) = \langle z(t; s, z_s) - y_i(t), z_s - y_i(s)\rangle (i = 1, 2, \cdots, p), t \geq s \tag{2.5}$$

and

$$F_{ij}(t; s, Y_i(s)) = \langle z(t; s, Y_i(s)) - y_j(t), Y_i(s) - y_j(s)\rangle (i, j = 1, 2, \cdots, p), t \geq s, \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

The first main result is presented as follows.

Theorem 2.2. *Suppose the assumptions [F](1)(2), [Y](1) and [J](1) hold.*

(1) The system (1.1) admits a unique PC−solution $x \in PC_Y(\mathbb{R}^+, \mathbb{R}^n)$.

(2) x has exactly irregular point set $\{t_i | 0 \leq t_1 < t_2 < \cdots < t_k < +\infty\}$ *over* $\mathbb{R}^+$ *if and only if there exists* $l_i \in \Lambda$ *(i = 1, 2, ⋯ , k) such that*

$$F_{l_1}(t_1; 0, x_0) = 0, F_{l_i l_{i+1}}(t_{i+1}; t_i, Y_{l_i}(t_i)) = 0 \mathit{for}\, i = 1, 2, \cdots, k-1, \tag{2.7}$$

and

$$F_{l_k j}(t; t_k, Y_{l_k}(t_k)) > 0 \mathit{for\ any} t \in [t_k, +\infty) \mathit{for\ all} j \in \Lambda. \tag{2.8}$$

We have to point out that the necessary and sufficient conditions of pulse phenomenon also is given in Theorem 2.2. Not only that, for the existence of solution of the system (1.2), in order to ensure $t_k = \tau_k(x)$ monotonous with respect to k in [20], it requires that $\tau_k(x)$ to be smooth and satisfy the corresponding inequality conditions. However, using Theorem 2.2, we can obtain immediately the following result.

Corollary 2.3. *Suppose the assumptions [F](1)(2), [Y](1) and [J](1) hold. If* y_i *is invertible and* $\tau_i = y_i^{-1}$ *for any* $i \in \Lambda$, *then the system (1.2) admits a unique PC−solution* $x \in PC_Y(\mathbb{R}^+, \mathbb{R}^n)$.

Now, we state second and third main results. It follows from Theorem 2.2 that for any fixed sufficient small $\theta > 0$, (1.1) has a unique approximate solution x_θ provided that the assumptions [F](1)(2), [Y](1) and [J](1) hold. Let $v \in \mathbb{R}^n$, $x_\theta(\cdot; \theta, x_0 + \theta v)$ be an approximate $PC-$solution of the equation (1.1) corresponding to $(\theta, x_0 + \theta v)$. We note that (1.1) is not well posed. Thus, we can never expect to have the continuity of solution to respect to the initial value. We have to modify the classical definition of continuous and differentiability, respectively.

Definition 2.4. *Let* $v \in \mathbb{R}^n$ *be fixed. The PC−solution* $x(\cdot; 0, x_0)$ *of (1.1) is said to have continuous dependence relative to the initial value* $(0, x_0)$ *if the following facts hold:*
(i) when $x(t; 0, x_0) \neq y_i(t)$ $(i \in \Lambda)$, $x_\theta(t; \theta, x_0 + \theta v) \longrightarrow x(t; 0, x_0)$ *as* $\theta \to 0$;
(ii) for any sufficient small $\varepsilon > 0$, *there exist* $\delta > 0$ *and* $I_\varepsilon \subseteq [0, T]$ *such that*

$$|x_\theta(t; \theta, x_0 + \theta v) - x(t; 0, x_0)| < \varepsilon \mathit{for\ any} t \in I_\varepsilon, \tag{2.9}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

when $\mu([0,T]\backslash I_\varepsilon) < \varepsilon, \theta < \delta$*, where* μ *denote the Lebesgue measure.*

Definition 2.5. *Let* $v \in \mathbb{R}^n$ *be fixed. The* $PC-$*solution* $x(\cdot;0,x_0)$ *of (1.1) is said to Gâteaux differentiable relative to the initial value* $(0,x_0)$ *if Gâteaux derivative* $\varphi(t)$ *of* $x(t;0,x_0)$ *exists at* $(0,x_0)$ *for all* $t \in [0,T]$ *with* $x(t;0,x_0) \neq y_i(t)$*, otherwise,*

$$\varphi(t) = \lim_{s \nearrow t} \varphi(s), \tag{2.10}$$

where

$$\varphi(t) = \lim_{\varepsilon \to 0} \frac{x_\varepsilon(t;\varepsilon,x_0+\varepsilon v) - x(t;0,x_0)}{\varepsilon} \text{when} x(t;0,x_0) \neq y_i(t). \tag{2.11}$$

Let we state the following main results.

Theorem 2.6. *Suppose the assumptions [F](1)(2), [Y](1) and [J](1) hold. Then* $PC-$*solution* $x(\cdot;0,x_0)$ *of (1.1) have continuous dependence relative to the initial value* $(0,x_0)$ *in the sense of Definition 2.4.*

Theorem 2.7. *Suppose the assumptions [F], [Y] and [J] hold. Then* $PC-$*solution* $x(\cdot;0,x_0)$ *of (1.1) is Gâteaux differentiable relative to the initial value* $(0,x_0)$ *in the sense of Definition 2.5. Moreover, its Gâteaux derivative* φ *is a* $PC-$*solution of the following differential equation with impulses*

$$\begin{cases} \dot{\varphi}(t) = f_x(t,x(t))\varphi(t), & t \in (0,T], x(t) \bigcap Y(t) = \emptyset, \\ \varphi(t+) = \varphi(t) + \nabla J_i(y_i(t))[\varphi(t) + \dot{h}_t(0) f(t,y_i(t))], & x(t) \bigcap Y(t) = y_i(t), \\ \varphi(0) = v - f(0,x_0). \end{cases} \tag{2.12}$$

Here, h_t *denote solution of the equation* $\{x_\varepsilon(t;\varepsilon,x_0+\varepsilon v)\} \bigcap \partial B(y_i(t),\varepsilon^2) \neq \emptyset$ *in* ε *for some* $i \in \Lambda$*.*

3. Proof of Theorem 2.2

Throughout this section, we define the function $r : (0,+\infty) \longrightarrow \mathbb{R}^+$ given by

$$r(T) \triangleq \frac{1}{2} \inf_{s,t \in [0,T]} \left\{ |y_i(s) - y_j(t)|, |y_i(s) - \mathrm{Y}_j(t)|, |y_i(s) - \mathrm{Y}_i(t)| \middle| i,j \in \Lambda \text{and} i \neq j \right\}, \tag{3.1}$$

where Y_j is defined by (2.1). It is easy from the assumptions [J](1) and [Y](1) to see $\mathrm{Y}_i \in C([0,T],\mathbb{R}^n)$ for all $i \in \Lambda$. Hence, there exists a constant $M(T)$ such that

$$|\mathrm{Y}_i(t)| \leq M(T) \text{for any} t \in [0,T] \text{and} i \in \Lambda \tag{3.2}$$

and

$$r(T) > 0 \text{for all} T > 0. \tag{3.3}$$

To claim the existence and uniqueness of solution of (1.1), we need the following Lemma.

Lemma 3.1. *If the assumptions [F](1)(2), [Y](1) and [J](1) hold, then for any* $(s,\xi) \in [0,T) \times \{\mathrm{Y}_i(t) | t \in [0,T], i \in \Lambda\}$*, there is a* $\delta > 0$ *which is independent of* (s,ξ) *such that the following differential equation*

$$\begin{cases} \dot{\phi}(t) = f(t,\phi(t)), t > s, \\ \phi(s) = \xi, \end{cases} \tag{3.4}$$

has a unique solution $\phi \in C([s,s+\delta],\mathbb{R}^n)$ *and*

$$|\phi(t) - y_i(t)| \geq \frac{r(T)}{2} \text{for any} t \in [s,s+\delta] \text{and} i \in \Lambda. \tag{3.5}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

Proof. It follows from the assumptions [F](1)(2) that (3.4) has a unique solution $\phi \in C([s,T],\mathbb{R}^n)$ and

$$|\phi(t)| \le |\xi| + \int_s^t \tilde{k}(1+|\phi(\tau)|)d\tau.$$

Using Gronwall's inequality, we have

$$|\phi(t)| \le \left(|\xi| + \tilde{k}T\right) e^{k(t-s)}.$$

Together with (3.2), this means that

$$|\phi(t)| \le \left(M(T) + \tilde{k}T\right) e^{kT} \equiv \tilde{M}(T;\tilde{k}) \text{for any} t \in [0,T].$$

Consequently, for any $t \in [0,T]$, we have

$$|\phi(t) - \xi| \le \int_s^t |f(\tau,0)|d\tau + L(\tilde{M}(T;\tilde{k}))\tilde{M}(T;\tilde{k})|t-s|.$$

Together with (3.3) and

$$|\phi(t) - y_i(t)| \ge |y_i(t) - \xi|\,| - |\phi(t) - \xi|\,|,$$

there exists a constant $\delta = \delta(T,\tilde{k}) > 0$ such that (3.5) holds. □

By Lemma 3.1, one can prove that for any $T > 0$, the equation (1.1) has a unique $PC-$solution. Thus, the equation (1.1) admits a unique $PC-$solution $x(\cdot;0,x_0)$ on $\mathbb{R}^+$.

Next, we discus the number of irregular point for the solution x of (1.1) over $\mathbb{R}^+$. It is easy that the solution x of (1.1) has no irregular point over $\mathbb{R}^+$ if and only if for all $i \in \Lambda$, $x(t;u,0,x_0) \neq y_i(t)$ for any $t \in \mathbb{R}^+$. Together with (2.4), we have following result.

Lemma 3.2. *If the assumptions [F](1)(2), [Y](1) and [J](1) hold, then the solution x of (1.1) has no irregular point over $\mathbb{R}^+$ if and only if the following algebraic equations*

$$F_i(t;0,x_0) = 0 \textit{has no solution on} \mathbb{R}^+ \textit{for all} i \in \Lambda. \tag{3.6}$$

Now, we prove the necessity on (2) of Theorem 2.2. For convenience, we let $x(\cdot) = x(\cdot;0,x_0)$ and $\{t_i | 0 \le t_1 < t_2 < \cdots < t_k < +\infty\}$ stand for the irregular point set of x over $\mathbb{R}^+$. Then there firstly exists $l_1 \in \Lambda$ such that

$$x(t_1) = y_{l_1}(t_1).$$

Together with (2.4), we can affirm

$$F_{l_1}(t_1;0,x_0) = 0.$$

For the second irregular point t_2 of x, there exists $l_2 \in \Lambda$ such that

$$x(t_2) = x(t_2;t_1,\Upsilon_{l_1}(y_{l_1}(t_1))) = y_{l_2}(t_2).$$

Together with (2.5), it follows

$$F_{l_1 l_2}(t_2;t_1,\Upsilon_{l_1}(y_{l_1}(t_1)) = 0.$$

Similarly, for the irregular point t_k of x, there is a $l_k \in \Lambda$ such that

$$F_{l_{k-1} l_k}(t_k;t_{k-1},\Upsilon_{l_{k-1}}(t_{k-1})) = 0.$$

Moreover, we can see from Lemma 3.2 that x has not irregular point on $(t_k,+\infty)$ if and only if

$$F_{l_k j}\left(t;t_k,\Upsilon_{l_k}(t_k)\right) = 0 \text{has no solution on} (t_k,+\infty) \text{for all} j \in \Lambda. \tag{3.7}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

Combined with (2.5), it is easy from the assumptions [J] and [Y] to see $F_{l_k j}\left(\cdot;t_k,\Upsilon_{l_k}(t_k)\right)\in C([t_k,+\infty))$ and

$$F_{l_k j}\left(t_k;t_k,\Upsilon_{l_k}(t_k)\right)=\langle z(t_k;t_k,\Upsilon_{k_k}(t_k))-y_j(t_k),\Upsilon_{l_k}(t_k)-y_j(t_k)\rangle>0\text{for all}j\in\Lambda.$$

Therefore, together with (3.7), this means (2.7) and (2.8) hold.

To the sufficient on (2) of Theorem 2.2, suppose $\{t_i|0\leq t_1<t_2<\cdots<t_k<+\infty\}$ satisfies (2.7) and (2.8). Then it is not difficult to claim that t_j is the irregular point x of the (1.1) over $\mathbb{R}^+$. Simultaneously, one can conclude for (2.8) and Lemma 3.1 that $\{t_i|0\leq t_1<t_2<\cdots<t_k<+\infty\}$ is the irregular point set of x over $\mathbb{R}^+$. This completes the proof.

4. Proof of Theorem 2.6

Throughout this section, we fix $T>0$. It follows from Theorem 2.2 that the irregular point to the $PC-$solution x of (1.1) at most a finite number of times on the interval $[0,T]$. There are only two possibilities which are Case (1): x has no irregular point on $[0,T]$ and Case (2): x has at least one irregular point on $[0,T]$.

In Case (1), the $PC-$solution x has continuous dependence relative to the initial value in the sense of classical definition, i.e.

$$||x_\theta\left(\cdot;\theta,x_0+\theta v\right)-x\left(\cdot;0,x_0\right)||_{C([0,T],\mathbb{R}^n)}\longrightarrow 0\text{as}\theta\to 0.$$

In Case (2), if $x_0=y_i(0)$ for some $i\in\Lambda$, we only study the $PC-$ solution $x\left(\cdot;0+,\Upsilon_i(0)\right)$. Consequently, we may assume that $x\left(\cdot;0,x_0\right)$ meets the warning wall $Y(\cdot)$, k times in $[0,T]$ and let $\bar{t}^i_j$ be the moments of $x\left(\cdot;0,x_0\right)$ hits the warning line $y_i(\cdot)$ $(i\in\Lambda,j=1,2,\cdots,k)$, $0<\bar{t}^i_1<\bar{t}^j_2<\cdots<\bar{t}^r_k<T$. By Theorem 2.2, we can prove that there exists $\bar{\delta}>0$ such that when $0\leq\theta<\bar{\delta}$, the impulsive differential equation (1.1) has a unique approximate $PC-$solution $x_\theta(\cdot;\theta,x_0+\theta v)$ corresponding to the initial value $(\theta,x_0+\theta v)$.

Let $t^i_j(\theta)$ be the moments of $x_\theta(\cdot;\theta,x_0+\theta v)$. For sufficient small $\varepsilon>0$, it is obvious from the definition of solution (see (2.1)) that there is an $\delta_1>0$ with $\delta_1<\bar{\delta}$ such that for any $\theta>0$ with $\theta<\delta_1$,

$$||x_\theta\left(t;\theta,x_0+\theta v\right)-x\left(t;0,x_0\right)||<\varepsilon\text{for any}t\in\left[0,\bar{t}^i_1-\frac{\varepsilon}{4k}\right]. \tag{4.1}$$

This means

$$\lim_{\theta\to 0}t^i_1(\theta)=\bar{t}^i_1. \tag{4.2}$$

Together with the continuity of J_i, we have

$$\lim_{\theta\to 0}J_i\left(x_\theta\left(t^i_1(\theta);\theta,x_0+\theta v\right)\right)=J_i\left(x\left(\bar{t}^i_1;0,x_0\right)\right). \tag{4.3}$$

It follows from (2.2) that

$$\lim_{\theta\to 0}\Upsilon_i\left(t^i_1(\theta)\right)=\Upsilon_i\left(\bar{t}^i_1\right), \tag{4.4}$$

where

$$\Upsilon_i\left(t^i_1(\theta)\right)=x_\theta\left(t^i_1(\theta);\theta,x_0+\theta v\right)+J_i\left(x_\theta\left(t^i_1(\theta);\theta,x_0+\theta v\right)\right). \tag{4.5}$$

Consequently, combine with (2.1), there is an $\delta_2>0$ with $\delta_2<\delta_1$ such that for any $\theta>0$ with $\theta<\delta_2$,

$$||x_\theta\left(t;\theta,x_0+\theta v\right)-x\left(t;u,0,x_0\right)|| \;=\; \left|\left|x_\theta\left(t;t^i_1(\theta),\Upsilon_i\left(t^i_1(\theta)\right)\right)-x\left(t;\bar{t}^i_1,\Upsilon_i\left(\bar{t}^i_1\right)\right)\right|\right|$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

$$< \quad \varepsilon \text{for any} t \in \left[\bar{t}_1^i + \frac{\varepsilon}{4k}, \bar{t}_2^j - \frac{\varepsilon}{4k}\right]. \tag{4.6}$$

Let

$$\Upsilon_i\left(t_j^i(\theta)\right) = x_\theta\left(t_j^i(\theta);\theta,x_0+\theta v\right) + J_i\left(x_\theta\left(t_j^i(\theta);\theta,x_0+\theta v\right)\right), j > 1, i \in \Lambda. \tag{4.7}$$

In general, repeat the above process, one can show that there is an $\delta_{j+1} > 0$ with $\delta_{j+1} < \delta_j$ such that for any $\theta > 0$ with $\theta < \delta_{j+1}$,

$$\begin{aligned} ||x_\theta\left(t;\theta,x_0+\theta v\right) - x\left(t;0,x_0\right)|| &= \left||x_\theta\left(t;t_j^i(\theta),\Upsilon_i\left(t_j^i(\theta)\right)\right) - x\left(t;\bar{t}_j^i,\Upsilon_i\left(\bar{t}_j^i\right)\right)\right|| \\ &< \varepsilon \text{for any} t \in \left[\bar{t}_j^i + \frac{\varepsilon}{4k}, \bar{t}_{j+1}^r - \frac{\varepsilon}{4k}\right] \end{aligned} \tag{4.8}$$

and

$$\lim_{\theta\to 0} t_{j+1}^r(\theta) = \bar{t}_{j+1}^r, \tag{4.9}$$

$$\lim_{\theta\to 0} J_r\left(x_\theta\left(t_{j+1}^r(\theta);\theta,x_0+\theta v\right)\right) = J_r\left(x\left(\bar{t}_{j+1}^r;0,x_0\right)\right), \tag{4.10}$$

$$\lim_{\theta\to 0} \Upsilon_r\left(t_{j+1}^r(\theta)\right) = \Upsilon_r\left(\bar{t}_{j+1}^r\right), \tag{4.11}$$

where

$$\Upsilon_r\left(t_{j+1}^r(\theta)\right) = x_\theta\left(t_{j+1}^r(\theta);\theta,x_0+\theta v\right) + J_r\left(x_\theta\left(t_j^r(\theta,);\theta,x_0+\theta v\right)\right). \tag{4.12}$$

In short, for any sufficient small $\varepsilon > 0$, there exists an $\delta > 0$ such that

$$|x_\theta\left(t;\theta,x_0+\theta v\right) - x\left(t;0,x_0\right)| < \varepsilon \text{for any} t \in I_\varepsilon \text{when} \theta < \delta, \tag{4.13}$$

where

$$I_\varepsilon = \left[0,\bar{t}_1^i - \frac{\varepsilon}{4k}\right] \bigcup \left(\bigcup_{j=1}^{k-1}\left[\bar{t}_j^i + \frac{\varepsilon}{4k}, \bar{t}_{j+1}^r - \frac{\varepsilon}{4k}\right]\right) \bigcup \left[\bar{t}_k^r + \frac{\varepsilon}{4k}, T\right].$$

This completes the proof.

5. Proof of Theorem 2.7

Throughout this section, we fix $T > 0$. It follows from Theorem 2.6 that there are only two possibilities which are Case (i): $x(\cdot;0,x_0)$ has no irregular point on $[0,T]$ and Case (ii): $x(\cdot;0,x_0)$ has at least one irregular point on $[0,T]$.

In Case (i), one can directly check that $x(\cdot;0,x_0)$ of (1.1) is Gâteaux differentiable and its Gâteaux derivative φ is a weak solution of the following differential equation

$$\begin{cases} \dot{\varphi}(t) = f_x(t,x(t;0,x_0))\varphi(t), t \in (0,T], \\ \varphi(0) = v - f(0,x_0). \end{cases} \tag{5.1}$$

To discuss Case (ii), we define function h_t given by

$$h_t(\varepsilon) \text{denote solution of the equation} H(\varepsilon,t) = 0. \tag{5.2}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

Here,

$$H(\varepsilon,t) = x_\varepsilon\left(t;\varepsilon,x_0+\varepsilon v\right) - \tilde{y}(t,\varepsilon), \tag{5.3}$$

where $\tilde{y}(t,\varepsilon) = \tilde{y}_i(t,\varepsilon)$ for some $i \in \Lambda$, $\tilde{y}_i(t,\varepsilon) \in \partial B\left(y_i(t),\varepsilon^2\right)$. By Theorem 2.6, when $x\left(t;0,x_0\right) = y_i(t)$, there is an $\delta > 0$ such that the definition (5.2) is well for all $\varepsilon \in [0,\delta]$, that is, $h_t : [0,\delta] \longrightarrow O(t)$ is a function and $h_t(0) = t$, where $O(t)$ denote some neighborhood of t. For convenience, let $\{t_j^i | 0 < t_1^i < t_2^j < \cdots < t_k^r < T\}$ denote the irregular point set of $x\left(\cdot;0,x_0\right)$ on $[0,T]$. If $y_i \in C^1([0,T],\mathbb{R}^n)$, it follows from Theorem 2.6 and (5.3) that there is an $\delta > 0$ such that

$$H \in C\left([0,\delta]\times[0,T]\right) \text{and} H\left(\varepsilon, h_{t_j^i}(\varepsilon)\right) = 0 \text{for any} \varepsilon \in [0,\delta], i \in \Lambda, j = 1,2,\cdots,k \tag{5.4}$$

and

$$H_t(\varepsilon,t) = f\left(t, x_\varepsilon\left(t;\varepsilon,x_0+\varepsilon v\right)\right) - \tilde{y}_t(t,\varepsilon).$$

Further, when $f\left(t_j^i, y_i\left(t_j^i\right)\right) \neq \dot{y}_i\left(t_j^i\right)$ $(j = 1,2,\cdots,k, i \in \Lambda)$, we have

$$H_t\left(\varepsilon, h_{t_j^i}(\varepsilon)\right) = f\left(h_{t_j^i}(\varepsilon), x_\varepsilon\left(h_{t_j^i}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)\right) - \dot{y}_i(h_{t_j^i}(\varepsilon)) \neq 0 \text{in} \mathbb{R}^n, \forall \varepsilon \in [0,\delta], \tag{5.5}$$

where $j = 1, 2, \cdots, k$. Let $f = \left(f^1, f^2, \cdots, f^n\right)^\top$, $y_i = \left(y_i^1, y_i^2, \cdots, y_i^n\right)^\top$ $(i \in \Lambda)$. Without loss of generality, we suppose

$$f^1\left(h_{t_j^i}(\varepsilon), x_\varepsilon\left(h_{t_j^i}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)\right) - \dot{y}_i^1(h_{t_j^i}(\varepsilon)) \neq 0 \text{in} \mathbb{R}, \forall \varepsilon \in [0,\delta], j = 1,2,\cdots,k, \tag{5.6}$$

We introduce the following functions

$$\Phi_\varepsilon(t,s) = \exp\left(\int_s^t f_x(\tau, x_\varepsilon(\tau;\varepsilon,x_0+\varepsilon v))d\tau\right), \tag{5.7}$$

then

$$\Phi(t,s) = \lim_{\varepsilon\to 0}\Phi_\varepsilon(t,s) = \exp\left(\int_s^t f_x(\tau, x(\tau;0,x_0))d\tau\right). \tag{5.8}$$

We let

$$\Phi_\varepsilon^1(t,s) \text{and} \Phi^1(t,s) \text{denote the first line vector of} \Phi_\varepsilon(t,s) \text{and} \Phi(t,s) \text{respectively}. \tag{5.9}$$

We firstly claim the following lemma.

Lemma 5.1. *Suppose the assumption [F](3) holds. Then h_t is differentiable over $[0,\delta]$ for some $\delta > 0$ and its derivative given by*

$$\dot{h}_{t_j^i}(0) = \begin{cases} \frac{\Phi^1\left(t_1^i,0\right)\left(f\left(0,x_0\right)-v\right)}{f^1\left(t_1^i,y_i\left(t_1^i\right)\right)-\dot{y}_i^1\left(t_1^i\right)}, & j = 1, \\ \frac{\dot{h}_{t_{j-1}^r}(0)\Phi^1\left(t_j^i,t_{j-1}^r\right)\left[f\left(t_{j-1}^r,y_r\left(t_{j-1}^r\right)\right)-\left(I+\nabla J_r\left(y_r\left(t_{j-1}^r\right)\right)\right)\dot{y}_r\left(t_{j-1}^r\right)\right]}{f^1\left(t_j^i,y_i\left(t_j^i\right)\right)-\dot{y}_i^1\left(t_j^i\right)}, & j > 1. \end{cases} \tag{5.10}$$

Here, I is unit matrix.

Proof. When $t \in \left(0, h_{t_1^i}(\varepsilon)\right)$, it follows from the assumption [F](3) and (2.9), (1.2) that

$$\begin{aligned} H_\varepsilon(\varepsilon, t) &= \lim_{\xi \to 0} \frac{x_{\varepsilon+\xi}(t; \varepsilon+\xi, x_0 + (\varepsilon+\xi)v) - x_\varepsilon(t; \varepsilon, x_0 + \varepsilon v)}{\xi} + \frac{\partial}{\partial \varepsilon}\tilde{y}_i(t, \varepsilon) \\ &= \lim_{\xi \to 0} \int_{\varepsilon+\xi}^{t} \int_0^1 f_x(s, x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v) + \theta(x_{\varepsilon+\xi}(s; \varepsilon+\xi, x_0 + (\varepsilon+\xi)v) \\ &\quad -x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v))) \frac{x_{\varepsilon+\xi}(s; \varepsilon+\xi, x_0 + (\varepsilon+\xi)v) - x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v)}{\xi} d\theta ds \\ &\quad v - f(\varepsilon, x_0 + \varepsilon v) + \frac{\partial}{\partial \varepsilon}\tilde{y}_i(t, \varepsilon). \end{aligned}$$

One can see from (5.7) and the above equality that

$$H_\varepsilon(\varepsilon, t) = \Phi_\varepsilon(t, \varepsilon)(v - f(\varepsilon, x_0 + \varepsilon v)) + \frac{\partial}{\partial \varepsilon}\tilde{y}_i(t, \varepsilon).$$

Combine with (5.6), (5.7) and (5.9), we have

$$\dot{h}_{t_1^i}(\varepsilon) = -\frac{\Phi_\varepsilon^1\left(h_{t_j^i}(\varepsilon), \varepsilon\right)(v - f(\varepsilon, x_0 + \varepsilon v)) + \frac{\partial}{\partial \varepsilon}\tilde{y}_i^1(t, \varepsilon)}{f^1\left(h_{t_1^i}(\varepsilon), x_\varepsilon\left(h_{t_1^i}(\varepsilon); \varepsilon, x_0 + \varepsilon v\right)\right) - \dot{y}_i^1\left(h_{t_1^i}(\varepsilon)\right)} \tag{5.11}$$

and

$$\dot{h}_{t_1^i}(0) = \frac{\Phi^1\left(t_1^i, 0\right)(v - f(0, x_0))}{\dot{y}_i^1\left(t_1^i\right) - f^1\left(t_1^i, y_i\left(t_1^i\right)\right)}, \tag{5.12}$$

In general, when $t \in \left(h_{t_{j-1}^r}(\varepsilon), h_{t_j^i}(\varepsilon)\right)$, it follows from the assumption [F](3) and (2.9), (1.2), (2.1) that

$$\begin{aligned} H_\varepsilon(\varepsilon, t) &= \lim_{\xi \to 0} \frac{x_{\varepsilon+\xi}(t; \varepsilon+\xi, x_0 + (\varepsilon+\xi)v) - x_\varepsilon(t; \varepsilon, x_0 + \varepsilon v)}{\xi} + \frac{\partial}{\partial \varepsilon}\tilde{y}_i(t, \varepsilon) \\ &= \lim_{\xi \to 0} \frac{x_{\varepsilon+\xi}\left(t; h_{t_{j-1}^r}(\varepsilon+\xi), \Upsilon_r\left(h_{t_{j-1}^r}(\varepsilon+\xi)\right)\right) - x_\varepsilon\left(t; h_{t_{j-1}^r}(\varepsilon), \Upsilon_r\left(h_{t_{j-1}^r}(\varepsilon)\right)\right)}{\xi} \\ &\quad + \frac{\partial}{\partial \varepsilon}\tilde{y}_i(t, \varepsilon) \\ &= \lim_{\xi \to 0} \int_{h_{t_{j-1}^r}(\varepsilon+\xi)}^{t} \int_0^1 f_x(s, x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v) + \theta(x_{\varepsilon+\xi}(s; \varepsilon+\xi, x_0 + (\varepsilon+\xi)v) \\ &\quad -x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v))) \frac{x_{\varepsilon+\xi}(s; \varepsilon+\xi, x_0 + (\varepsilon+\xi)v) - x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v)}{\xi} d\theta ds \\ &\quad + \lim_{\xi \to 0} \frac{\Upsilon_r\left(h_{t_{j-1}^r}(\varepsilon+\xi), \varepsilon+\xi\right) - \Upsilon_r\left(h_{t_{j-1}^r}(\varepsilon), \varepsilon\right)}{\xi} \\ &\quad - \lim_{\xi \to 0} \frac{\int_{h_{t_{j-1}^r}(\varepsilon)}^{h_{t_{j-1}^r}(\varepsilon+\xi)} f(s, x(s; \varepsilon, x_0 + \varepsilon v))\, ds}{\xi} + \frac{\partial}{\partial \varepsilon}\tilde{y}_r(t, \varepsilon) \\ &= \lim_{\xi \to 0} \int_{h_{t_{j-1}^r}(\varepsilon+\xi)}^{t} \int_0^1 f_x(s, x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v) + \theta(x_{\varepsilon+\xi}(s; \varepsilon+\xi, x_0 + (\varepsilon+\xi)v) \\ &\quad -x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v))) \frac{x_{\varepsilon+\xi}(s; \varepsilon+\xi, x_0 + (\varepsilon+\xi)v) - x_\varepsilon(s; \varepsilon, x_0 + \varepsilon v)}{\xi} d\theta ds \end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

$$
\begin{aligned}
&+\left(I+\nabla J_r\left(\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)\right)\left[\dot{h}_{t^r_{j-1}}(\varepsilon)\frac{\partial}{\partial t}\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right.\\
&\left.+\frac{\partial}{\partial\varepsilon}\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right]-\dot{h}_{t^r_{j-1}}(\varepsilon)f\left(h_{t^r_{j-1}}(\varepsilon),\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)+\frac{\partial}{\partial\varepsilon}\tilde{y}_r(t,\varepsilon).
\end{aligned}
$$

We also can infer from (5.7) and the above the equality that

$$
\begin{aligned}
H_\varepsilon(\varepsilon,t) &= \frac{\partial}{\partial\varepsilon}\tilde{y}_r(t,\varepsilon)+\Phi_\varepsilon\left(t,h_{t^r_{j-1}}(\varepsilon)\right)\left(I+\nabla J_r\left(\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)\right)\left[\dot{h}_{t^r_{j-1}}(\varepsilon)\frac{\partial}{\partial t}\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right.\\
&\left.+\frac{\partial}{\partial\varepsilon}\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right]-\dot{h}_{t^r_{j-1}}(\varepsilon)\Phi_\varepsilon\left(t,h_{t^r_{j-1}}(\varepsilon)\right)f\left(h_{t^r_{j-1}}(\varepsilon),\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right). \qquad (5.13)
\end{aligned}
$$

Together with (5.6) and (5.9), by implicit function theorem, we have

$$
\begin{aligned}
\dot{h}_{t^i_j}(\varepsilon) &= -\frac{\Phi^1_\varepsilon\left(h_{t^i_j}(\varepsilon),h_{t^r_{j-1}}(\varepsilon)\right)\left(I+\nabla J_r\left(\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)\right)}{f^1\left(h_{t^i_j}(\varepsilon),x_\varepsilon\left(h_{t^i_j}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)\right)-\dot{y}^1_i\left(h_{t^j_2}(\varepsilon)\right)}\\
&\cdot\left[\dot{h}_{t^r_{j-1}}(\varepsilon)\frac{\partial}{\partial t}\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)+\frac{\partial}{\partial\varepsilon}\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right] \qquad (5.14)\\
&-\frac{\frac{\partial}{\partial\varepsilon}\tilde{y}^1_r(t,\varepsilon)-\dot{h}_{t^r_{j-1}}(\varepsilon)\Phi^1_\varepsilon\left(h_{t^i_j}(\varepsilon),h_{t^r_{j-1}}(\varepsilon)\right)f\left(h_{t^r_{j-1}}(\varepsilon),\tilde{y}_r\left(h_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)}{f^1\left(h_{t^i_j}(\varepsilon),x_\varepsilon\left(h_{t^i_j}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)\right)-\dot{y}^1_i\left(h_{t^i_j}(\varepsilon)\right)}.
\end{aligned}
$$

Further, this means that

$$
\dot{h}_{t^i_j}(0)=\frac{\dot{h}_{t^r_{j-1}}(0)\Phi^1\left(t^i_j,t^r_{j-1}\right)\left[f\left(t^r_{j-1},y_r\left(t^r_{j-1}\right)\right)-\left(I+\nabla J_r\left(y_r\left(t^r_{j-1}\right)\right)\right)\dot{y}_r\left(t^r_{j-1}\right)\right]}{f^1\left(t^i_j,y_i\left(t^i_j\right)\right)-\dot{y}^1_i\left(t^i_j\right)}. \qquad (5.15)
$$

This completes the proof. □

Now, we claim the Case (ii). For $t\in\left(0,t^i_1\right)$, similarly the Case (i), it is not difficult to check the following result

$$
\begin{cases}
\dot{\varphi}(t)=f_x(t,x(t;0,x_0))\varphi(t), & t\in\left(0,t^i_1\right],\\
\varphi(0)=v-f(0,x_0),
\end{cases} \qquad (5.16)
$$

Combine with Lemma 5.1, we firstly note that

$$
\begin{aligned}
&\lim_{\varepsilon\to0}\frac{x_\varepsilon\left(h_{t^i_j}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)-x\left(t^i_j;0,x_0\right)}{\varepsilon}\\
=&\lim_{\varepsilon\to0}\frac{x_\varepsilon\left(h_{t^i_j}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)-x_\varepsilon\left(t^i_j;\varepsilon,x_0+\varepsilon v\right)}{\varepsilon}+\lim_{\varepsilon\to0}\frac{x_\varepsilon\left(t^i_j;\varepsilon,x_0+\varepsilon v\right)-x\left(t^i_j;0,x_0\right)}{\varepsilon}\\
=&\ \varphi\left(t^i_j\right)+\dot{h}_{t^i_j}(0)f\left(t^i_j,y_i\left(t^i_j\right)\right). \qquad (5.17)
\end{aligned}
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

Together with the assumption [J](2), it follows from $h_{t_j^i}(\varepsilon) > t_j^i$ that

$$\begin{aligned}
\varphi\left(t_j^i+\right) &= \lim_{\varepsilon\to 0} \frac{x_\varepsilon\left(h_{t_j^i}(\varepsilon)+;\varepsilon, x_0+\varepsilon v\right) - x\left(h_{t_j^i}(\varepsilon);0,x_0\right)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[x_\varepsilon\left(h_{t_j^i}(\varepsilon);\varepsilon,x_0+\varepsilon v\right) + J_i\left(x_\varepsilon\left(h_{t_j^i}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)\right) \\
&\quad -x\left(h_{t_j^i}(\varepsilon);t_j^i, x\left(t_j^i;0,x_0\right) + J_i\left(x\left(t_j^i;0,x_0\right)\right)\right)\Big] \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[x_\varepsilon\left(h_{t_j^i}(\varepsilon);\varepsilon,x_0+\varepsilon v\right) + J_i\left(x_\varepsilon\left(h_{t_j^i}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)\right) \\
&\quad -x\left(t_j^i;0,x_0\right) - J_i\left(x\left(t_j^i;0,x_0\right)\right) - \int_{t_j^i}^{h_{t_j^i}(\varepsilon)} f\left(s,x\left(s;0,x_0\right)\right)ds\Big] \\
&= \left(I+\nabla J_i\left(y_i\left(t_j^i\right)\right)\right)\left[\varphi\left(t_j^i-\right) + \dot h_{t_j^i}(0) f\left(t_j^i, y_i\left(t_j^i\right)\right)\right] - \dot h_{t_j^i}(0) f\left(t_j^i, y_i\left(t_j^i\right)\right) \\
&= \varphi\left(t_j^i-\right) + \nabla J_i\left(y_i\left(t_j^i\right)\right)\left[\varphi\left(t_j^i\right) + \dot h_{t_j^i}(0) f\left(t_j^i, y_i\left(t_j^i\right)\right)\right].
\end{aligned}$$

When $h_{t_j^i}(\varepsilon) < t_j^i$, we also have

$$\begin{aligned}
\varphi\left(t_j^i+\right) &= \lim_{\varepsilon\to 0} \frac{x_\varepsilon\left(t_j^i;\varepsilon, x_0+\varepsilon v\right) - x\left(t_j^i+;0,x_0\right)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[x_\varepsilon\left(h_{t_j^i}(\varepsilon);\varepsilon,x_0+\varepsilon v\right) + J_i\left(x_\varepsilon\left(h_{t_j^i}(\varepsilon);\varepsilon,x_0+\varepsilon v\right)\right) \\
&\quad -x\left(t_j^i;0,x_0\right) - J_i\left(x\left(t_j^i;0,x_0\right)\right) - \int_{t_j^i}^{h_{t_j^i}(\varepsilon)} f\left(s,x_\varepsilon\left(s;\varepsilon,x_0+\varepsilon v\right)\right)ds\Big] \\
&= \varphi\left(t_j^i\right) + \nabla J_i\left(y_i\left(t_j^i\right)\right)\left[\varphi\left(t_j^i-\right) + \dot h_{t_j^i}(0) f\left(t_j^i, y_i\left(t_j^i\right)\right)\right].
\end{aligned}$$

Consequently, we have

$$\varphi\left(t_j^i+\right) = \varphi\left(t_j^i\right) + \nabla J_i\left(y_i\left(t_j^i\right)\right)\left[\varphi\left(t_j^i\right) + \dot h_{t_j^i}(0) f\left(t_j^i, y_i\left(t_j^i\right)\right)\right], i\in\Lambda, j=1,2,\cdots,k. \tag{5.18}$$

Therefore, when $t\in\left(t_j^i, t_{j+1}^r\right)$ $(j=1,2,\cdots,k-1)$ or $t\in\left(t_k^r, T\right]$, it follows from the assumption [F](3) and (2.9), (1.2), (2.1), (4.7), (5.9), (5.17) that

$$\begin{aligned}
\varphi(t) &= \lim_{\theta\to 0}\frac{x_\theta(t;\theta,x_0+\theta v) - x(t;0,x_0)}{\theta} \\
&= \lim_{\theta\to 0}\frac{x_\theta(t;h_{t_j^i}(\theta), \Upsilon_r(h_{t_j^i}(\theta))) - x(t;t_j^i,\Upsilon_r(t_j^i))}{\theta} \\
&= \lim_{\theta\to 0}\frac{\Upsilon_i(h_{t_j^i}(\theta)) - \Upsilon_i(t_j^i))}{\theta} + \lim_{\theta\to 0}\int_{h_{t_j^i}(\theta)}^t\int_0^1 f_x(s,x(s;0,x_0)+\xi(x_\theta(s;\theta,x_0+\theta v) \\
&\quad -x(s;0,x_0)))\frac{x_\theta(s;\theta,x_0+\theta v) - x(s;0,x_0)}{\theta}d\xi ds - \lim_{\theta\to 0}\frac{1}{\theta}\int_{t_j^i}^{h_{t_j^i}(\theta)} f(s,x(s;0,x_0))ds \\
&= -\dot h_{t_j^i}(0) f\left(t_j^i, y_i\left(t_j^i\right)\right) + \lim_{\theta\to 0}\int_{h_{t_j^i}(\theta)}^t\int_0^1 f_x(s,x(s;0,x_0)+\xi(x_\theta(s;\theta,x_0+\theta v)
\end{aligned} \tag{5.19}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

$$-x(s;0,x_0)))\frac{x_\theta(s;\theta,x_0+\theta v)-x(s;0,x_0)}{\theta}d\xi ds$$
$$+\left(I+\nabla J_i\left(y_i\left(t_j^i\right)\right)\right)\left(\varphi\left(t_j^i\right)+\dot{h}_{t_j^i}(0)f\left(t_j^i,y_i\left(t_j^i\right)\right)\right).$$

Thus, combine with (5.16) and (5.18), we obtain from the above equality that

$$\begin{cases} \dot{\varphi}(t)=f_x(t,x(t;0,x_0))\varphi(t), t\in(0,T]\text{and}t\neq t_j^i, i\in\Lambda, j=1,2,\cdots,k, \\ \varphi(0)=v-f(0,x_0), \\ \varphi\left(t_j^i+\right)=\varphi\left(t_j^i\right)+\nabla J_i\left(y_i\left(t_j^i\right)\right)\left(\varphi\left(t_j^i\right)+\dot{h}_{t_j^i}(0)f\left(t_j^i,y_i\left(t_j^i\right)\right)\right), j=1,2,\cdots,k. \end{cases} \tag{5.20}$$

This completes the proof of Theorem 2.7.

6. Periodicity of Autonomous Impulsive System

As an application, in this section, we discus the periodicity of solution of the following impulsive differential equation

$$\begin{cases} \dot{x}(t)=g(x(t)), & x(t)\neq y_1, t\geq 0, \\ x(t+)=y_2, & x(t)=y_1, t\geq 0, \\ x(0)=x_0, \end{cases} \tag{6.1}$$

where $y_1,y_2\in\mathbb{R}^n$ and $y_1\neq y_2$. We introduce functions:

$$G(t;s,z_s)=\langle z(t,s,z_s)-y_1,y_2-y_1\rangle \text{ for any} t\geq s\geq 0. \tag{6.2}$$

Here,

$$z(t,s,z_s)=z_s+\int_s^t g(z(\tau,s,z_s))d\tau, \text{for any} t\geq s\geq 0. \tag{6.3}$$

For the function $G(\cdot;0,x_0)$, it is clear that

$$G(t;0,x_0)=0 \text{has no solution on} \mathbb{R}^+ \tag{6.4}$$

or

$$t_1 \text{is minimum solution of} G(t;0,x_0)=0 \text{on} \mathbb{R}^+. \tag{6.5}$$

Similarly, it is obvious that

$$G(t;t_1,y_2)=0 \text{has no solution on} (t_1,+\infty) \tag{6.6}$$

or

$$t_2 \text{is minimum solution of} G(t;t_1,y_2)=0 \text{on} (t_1,+\infty). \tag{6.7}$$

Let $PC_{y_1y_2}(\mathbb{R}^+,\mathbb{R}^n)=\{x:[0,+\infty)\longrightarrow\mathbb{R}^n|x$ is continuous at t when $x(t)\neq y_1$, x is left continuous at t and the right limit $x(t+)$ exists when $x(t)=y_1\}$. We will check the following main result for the autonomous impulsive system (6.1).

Theorem 6.1. *Suppose $g:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is locally Lipschitz continuous in x and there exists a constant $\tilde{k}>0$ such that*

$$|g(x)|\leq\tilde{k}(1+|x|) \textit{for any} t\geq 0.$$

(1) If (6.4) holds, then (6.1) has a unique solution $x\in C(\mathbb{R}^+,\mathbb{R}^n)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

(2) If (6.5) and (6.6) holds, then the solution of (6.1) has a unique irregular point t_1.
(3) If (6.6) and (6.7) holds, then the solution of (6.1) is a periodic function on $[t_1, +\infty)$.

Proof. Using Theorem 2.2, we directly check that the autonomous impulsive system (6.1) has a unique solution $x \in PC_{y_1 y_2}(\mathbb{R}^+, \mathbb{R}^n)$. Further, the solution only three possibilities which are Case (i): x has not irregular point on $\mathbb{R}^+$; Case (ii): x has a unique irregular point on $\mathbb{R}^+$ and Case (iii): x has two irregular points on $\mathbb{R}^+$ at least.

For the Case (i), it follows from (2) of Theorem 2.2 that x has not irregular point on $\mathbb{R}^+$ if and only if (6.4) holds. This means (6.1) has a unique solution $x \in C(\mathbb{R}^+, \mathbb{R}^n)$. Similarly, for the Case (ii), together with (6.5) and (6.6), we can infer also that x has only a unique irregular point t_1.

For the Case (iii), let t_1 and t_2 denote the smallest two the irregular points of solution x on $\mathbb{R}^+$ and $T = t_2 - t_1$. We claim

$$x(t+T) = x(t) \text{for any} t \in [t_1, +\infty). \tag{6.8}$$

By the definitions of t_1 and t_2 (see (6.5) and (6.7)), the solution x of (6.1) has not irregular point on (t_1, t_2) and satisfies

$$x(t) = y_2 + \int_{t_1}^{t} g(x(s))ds \text{for any} t \in (t_1, t_2] \text{and} x(t_2) = x(t_1) = y_1. \tag{6.9}$$

When $t \in (t_1, t_2]$, we have $t + T \in (t_2, t_2 + T]$ and

$$x(t+T) = y_2 + \int_{t_1+T}^{t+T} g(x(s))ds = y_2 + \int_{t_1}^{t} g(x(s+T))ds. \tag{6.10}$$

It is easy to see that by assumption conditions of g, there exists $\rho > 0$ such that $|x(t)|$, $|x(T+t)| \leq \rho$ for every $t \in (t_1, t_2]$. Furthermore, it asserts from (6.9) and (6.10) that

$$|x(t+T) - x(t)| \leq \int_{t_1}^{t} |g(x(s+T)) - g(x(s))|ds \tag{6.11}$$

$$\leq L(\rho) \int_{t_1}^{t} |x(s+T) - x(s)|ds. \tag{6.12}$$

Together with Gronwall inequality, one can verify that

$$x(t+T) = x(t) \text{for any} t \in (t_1, t_2]. \tag{6.13}$$

Consequently, we can infer (6.8) holds. Thus, this means that the solution x of (6.1) is a periodic function on $[t_1, +\infty)$ with period T. The proof is completed. □

7. Application

As an application, in this section, we will discus the variation of solution relative to control for the following control impulsive differential equation

$$\begin{cases} \dot{x}(t) = f(t, x(t)) + B(t)u(t), & \{x(t)\} \bigcap Y(t) = \emptyset, t \geq 0, \\ x(t+) = J_i(x(t)) + x(t), & \{x(t)\} \bigcap Y(t) = y_i(t), t \geq 0, \\ x(0) = x_0, \end{cases} \tag{7.1}$$

where, control function $u \in L^1_{loc}(\mathbb{R}^+, \mathbb{R}^m)$, $B \in L^\infty_{loc}(\mathbb{R}^+, \mathbb{R}^{n\times m})$.

Using the idea of Theorem 2.2 and Theorem 2.6, for any $T > 0$ and $u \in L^1((0,T), \mathbb{R}^m)$, one can prove the following result.

Theorem 7.1. *Suppose the assumptions [F](1)(2), [Y](1) and [J] hold. Then the system (7.1) has a unique PC−solution* $x(\cdot;u) \equiv x(\cdot;u,0,x_0) \in PC_Y([0,T],\mathbb{R}^n)$ *given by*

$$x(t;u) = x_0 + \int_0^t [f(\tau, x(\tau;u)) + B(\tau)u(\tau)]d\tau + \sum_{\substack{0 \le t_i < t, \\ x(t_i;u) = y_j(t_i))}} J_j(x(t_i;u)). \tag{7.2}$$

Moreover, the solution $x(\cdot;u)$ *have continuous dependence relative to the control* u *in the sense of Definition 2.4.*

Not only that, for any fixed sufficient small $\theta > 0$ and fixed $v \in L^1([0,T],\mathbb{R}^m)$, (7.1) has a unique approximate solution $x_\theta(\cdot) \equiv x_\theta(\cdot;u+\theta v,0,x_0)$ which satisfies

$$x_\theta(t) = x_0 + \int_0^t [f(\tau, x_\theta(\tau)) + B(\tau)(u(\tau) + \theta v(\tau))]d\tau + \sum_{\substack{0 \le t_i < t, \\ x_\theta(t_i) \in B(y_j(t_i),\theta^2)}} J_j(x_\theta(t_i)). \tag{7.3}$$

To discus variation of solution relative to control, we introduce the following definitions.

Definition 7.2. *The PC−solution* $x(\cdot;u,0,x_0)$ *of (7.1) is said to Gâteaux differentiable relative to the control* u *if Gâteaux derivative* $\psi(\cdot)$ *of* $x(t;u)$ *exists at* u *for all* $t \in [0,T]$ *with* $x(t;u,0,x_0) \neq y_i(t)$*, otherwise,*

$$\psi(t) = \lim_{s \nearrow t} \psi(s), \tag{7.4}$$

where

$$\psi(t) = \lim_{\varepsilon \to 0} \frac{x_\varepsilon(t;u+\varepsilon v,0,x_0) - x(t;u,0,x_0)}{\varepsilon} \text{when} x(t;u,0,x_0) \neq y_i(t). \tag{7.5}$$

Theorem 7.3. *Suppose the assumptions [F], [Y] and [J] hold and* $u \in C([0,T],\mathbb{R}^m)$, $B \in C([0,T],\mathbb{R}^{n\times m})$*. The PC−solution* $x(\cdot) = x(\cdot;u,0,x_0)$ *of (7.1) is Gâteaux differentiable relative to the control* u *in the sense of Definition 7.2. Moreover, its Gâteaux derivative* ψ *is a PC−solution of the following differential equation with impulses*

$$\begin{cases} \dot{\psi}(t) = f_x(t,x(t))\psi(t) + B(t)v(t), & t \in (0,T], x(t) \neq y_i(t), i \in \Lambda, \\ \psi(0) = 0, \\ \psi(t+) = \psi(t) + \nabla J_i(y_i(t))\left[\psi(t) + \dot{g}_t(0)(f(t,y_i(t)) + B(t)u(t)\right], x(t) = y_i(t), i \in \Lambda. \end{cases} \tag{7.6}$$

Proof. There are only two possibilities which are Case (I): $x(\cdot;u,0,x_0)$ has no irregular point on $[0,T]$ and Case (II): $x(\cdot;u,0,x_0)$ has at least one irregular point on $[0,T]$.

In the Case (I), one can directly check that $x(\cdot;u,0,x_0)$ of (7.1) is Gâteaux differentiable and its Gâteaux derivative ψ is a weak solution of the following differential equation

$$\begin{cases} \dot{\psi}(t) = f_x(t,x(t;u))\psi(t) + B(t)v(t), t \in (0,T], \\ \psi(0) = 0. \end{cases} \tag{7.7}$$

To discuss the Case (II), we define function g_t given by

$$g_t(\varepsilon)\text{denote solution of the equation} G(\varepsilon,t) = 0. \tag{7.8}$$

Here,

$$G(\varepsilon,t) = x_\varepsilon(t;u+\varepsilon v,0,x_0) - \tilde{y}(t,\varepsilon). \tag{7.9}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

By Theorem 7.1, when $x(t;u,0,x_0)=y_i(t)$, there is an $\delta>0$ such that for all $\varepsilon\in[0,\delta]$, $g_t:[0,\delta]\longrightarrow O(t)$ is a function and $g_t(0)=t$, where $O(t)$ denote some neighborhood of t. For convenience, let $\{t^i_j|0<t^i_1<t^j_2<\cdots<t^r_k<T\}$ denote the irregular point set of $x(\cdot;u,0,x_0)$ on $[0,T]$. If $y_i\in C^1([0,T],\mathbb{R}^n)$, it follows that there is a $\delta>0$ such that

$$G_t(\varepsilon,t)=f\left(t,x_\varepsilon\left(t;u+\varepsilon v,0,x_0\right)\right)+B(t)[u(t)+\varepsilon v(t)]-\tilde{y}_t(t,\varepsilon).$$

Further, when $f\left(t^i_j,y_i\left(t^i_j\right)\right)+B\left(t^i_j\right)u\left(t^i_j\right)\neq\dot{y}_i\left(t^i_j\right)$ ($j=1,2,\cdots,k,i\in\Lambda$), without loss of generality, we suppose

$$\begin{aligned}&f^1\left(g_{t^i_j}(\varepsilon),x_\varepsilon\left(g_{t^i_j}(\varepsilon);u+\varepsilon v,0,x_0\right)\right)+B^1\left(g_{t^i_j}(\varepsilon)\right)u\left(g_{t^i_j}(\varepsilon)\right)-\dot{y}^1_i(g_{t^i_j}(\varepsilon))\neq 0\text{in}\mathbb{R},\\&i\in\Lambda,\forall\varepsilon\in[0,\delta],j=1,2,\cdots,k,\end{aligned}\tag{7.10}$$

where B^1 denote the first line vector of B. We introduce the following functions given by

$$\Psi_\varepsilon(t,s)=\exp\left(\int_s^t f_x(\tau,x_\varepsilon(\tau;u+\varepsilon v,0,x_0))d\tau\right),\tag{7.11}$$

then

$$\Psi(t,s)=\lim_{\varepsilon\to 0}\Psi_\varepsilon(t,s)=\exp\left(\int_s^t f_x(\tau,x(\tau;u,0,x_0))d\tau\right).\tag{7.12}$$

We let

$$\Psi^1_\varepsilon(t,s)\text{and}\Psi^1(t,s)\text{denote the first line vector of}\Psi_\varepsilon(t,s)\text{and}\Psi(t,s)\text{respectively.}\tag{7.13}$$

Now we calculate the variation of solution relative to control in the Case (II). For $t\in[0,t^i_1]$, similarly the Case (I), it is not difficult to check the following result

$$\begin{cases}\dot{\psi}(t)=f_x(t,x(t;u,0,x_0))\psi(t)+B(t)v(t), & t\in\left(0,t^i_1\right],\\ \psi(0)=0.\end{cases}\tag{7.14}$$

When $t\in\left(0,g_{t^i_1}(\varepsilon)\right)$, it follows from the assumption [F](3), (7.3)and (2.9) that

$$\begin{aligned}G_\varepsilon(\varepsilon,t)&=\lim_{\xi\to 0}\frac{x_{\varepsilon+\xi}(t;u+(\varepsilon+\xi)v,0,x_0)-x_\varepsilon(t;u+\varepsilon v,0,x_0)}{\xi}+\frac{\partial}{\partial\varepsilon}\tilde{y}_i(t,\varepsilon)\\&=\lim_{\xi\to 0}\int_0^t\int_0^1 f_x(s,x_\varepsilon(s;u+\varepsilon v,0,x_0)+\theta(x_{\varepsilon+\xi}(s;u+(\varepsilon+\xi)v,0,x_0)\\&\quad -x_\varepsilon(s;u+\varepsilon v,0,x_0)))\frac{x_{\varepsilon+\xi}(s;u+(\varepsilon+\xi)v,0,x_0)-x_\varepsilon(s;u+\varepsilon v,0,x_0)}{\xi}d\theta ds\\&\quad\int_0^t B(s)v(s)ds+\frac{\partial}{\partial\varepsilon}\tilde{y}_i(t,\varepsilon).\end{aligned}$$

It follows from (7.11) and the above that

$$G_\varepsilon(\varepsilon,t)=\int_0^t\Psi_\varepsilon(t,s)B(s)v(s)ds+\frac{\partial}{\partial\varepsilon}\tilde{y}_i(t,\varepsilon).$$

Using implicit function theorem, combine with (7.10), we have

$$\dot{g}_{t_1^i}(\varepsilon) = -\frac{\int_0^{g_{t_1^i}(\varepsilon)} \Psi_\varepsilon^1(g_{t_1^i}(\varepsilon), s) B(s) v(s) ds + \frac{\partial}{\partial \varepsilon} \tilde{y}_i^1(g_{t_1^i}(\varepsilon), \varepsilon)}{f^1\left(g_{t_1^i}(\varepsilon), x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right)\right) + B^1\left(g_{t_1^i}(\varepsilon)\right) u\left(g_{t_1^i}(\varepsilon)\right) - \dot{y}_i^1(g_{t_1^i}(\varepsilon))}. \tag{7.15}$$

In the above equation, vector product is the inner product operation. In the following operations, the vector product is also the inner product operation. Together with Theorem 7.1, we obtain

$$\dot{g}_{t_1^i}(0) = -\frac{\int_0^{t_1^i} \Psi^1\left(t_1^i, s\right) B(s) v(s) ds}{f^1\left(t_1^i, x\left(t_1^i; u, 0, x_0\right)\right) + B^1\left(t_1^i\right) u\left(t_1^i\right) - \dot{y}_i^1\left(t_1^i\right)}. \tag{7.16}$$

Further,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right) - x\left(t_1^i; u, 0, x_0\right)}{\varepsilon} \\
= &\lim_{\varepsilon\to 0} \frac{x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right) - x_\varepsilon\left(t_1^i; u+\varepsilon v, 0, x_0\right)}{\varepsilon} \\
&+ \lim_{\varepsilon\to 0} \frac{x_\varepsilon\left(t_1^i; u+\varepsilon v, 0, x_0\right) - x\left(t_1^i; u, 0, x_0\right)}{\varepsilon} \\
= &\psi\left(t_1^i\right) + \dot{g}_{t_1^i}(0)\left[f\left(t_1^i, y_i\left(t_1^i\right)\right) + B\left(t_1^i\right) u\left(t_1^i\right)\right].
\end{aligned} \tag{7.17}$$

Together with the assumption [J](2), it follows from (7.16) and (7.17) that when $g_{t_1^i}(\varepsilon) > t_1^i$,

$$\begin{aligned}
\psi\left(t_1^i+\right) &= \lim_{\varepsilon\to 0} \frac{x_\varepsilon\left(g_{t_1^i}(\varepsilon)+; u+\varepsilon v, 0, x_0\right) - x\left(g_{t_1^i}(\varepsilon); u, 0, x_0\right)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right) + J_i\left(x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right)\right) \\
&\quad - x\left(g_{t_1^i}(\varepsilon); u, t_1^i, x\left(t_1^i; u, 0, x_0\right) + J_i\left(x\left(t_1^i; u, 0, x_0\right)\right)\right)\Big] \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right) + J_i\left(x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right)\right) \\
&\quad - x\left(t_1^i; u, 0, x_0\right) - J_i\left(x\left(t_1^i; u, 0, x_0\right)\right) - \int_{t_1^i}^{g_{t_1^i}(\varepsilon)} \left[f\left(s, x\left(s; u, 0, x_0\right)\right) + B(s) u(s)\right] ds\Big] \\
&= \left(I + \nabla J_i\left(y_i\left(t_1^i\right)\right)\right)\left[\psi\left(t_1^i-\right) + \dot{g}_{t_1^i}(0)\left(f\left(t_1^i, y_i\left(t_1^i\right)\right) + B\left(t_1^i\right) u\left(t_1^i\right)\right)\right] \\
&\quad - \dot{g}_{t_1^i}(0)\left(f\left(t_1^i, y_i\left(t_1^i\right)\right) + B\left(t_1^i\right) u\left(t_1^i\right)\right) \\
&= \psi\left(t_1^i\right) + \nabla J_i\left(y_i\left(t_j^i\right)\right)\left[\psi\left(t_j^i\right) + \dot{g}_{t_1^i}(0)\left(f\left(t_1^i, y_i\left(t_1^i\right)\right) + B\left(t_1^i\right) u\left(t_1^i\right)\right)\right],
\end{aligned}$$

and when $g_{t_1^i}(\varepsilon) < t_1^i$, we also have

$$\begin{aligned}
\varphi\left(t_1^i+\right) &= \lim_{\varepsilon\to 0} \frac{x_\varepsilon\left(t_1^i; u+\varepsilon v, 0, x_0\right) - x\left(t_1^i+; u, 0, x_0\right)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right) + J_i\left(x_\varepsilon\left(g_{t_1^i}(\varepsilon); u+\varepsilon v, 0, x_0\right)\right) \\
&\quad - x\left(t_1^i; u, 0, x_0\right) - J_i\left(x\left(t_1^i; u, 0, x_0\right)\right)\Big]
\end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

$$
\begin{aligned}
&-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_1^i}^{g_{t_1^i}(\varepsilon)}\left[f\left(s,x_\varepsilon\left(s;u+\varepsilon v,0,x_0\right)\right)+B(s)(u(s)+\varepsilon v(s))\right]ds\\
=\;&\varphi\left(t_1^i-\right)+\nabla J_i\left(y_i\left(t_1^i\right)\right)\left[\varphi\left(t_1^i-\right)+\dot g_{t_1^i}(0)\left(f\left(t_1^i,y_i\left(t_1^i\right)\right)+B\left(t_1^i\right)u\left(t_1^i\right)\right)\right].
\end{aligned}
$$

Consequently, we have

$$
\varphi\left(t_1^i+\right)=\varphi\left(t_1^i\right)+\nabla J_i\left(y_i\left(t_1^i\right)\right)\left[\varphi\left(t_1^i\right)+\dot g_{t_1^i}(0)f\left(t_1^i,y_i\left(t_1^i\right)\right)\right],i\in\Lambda. \tag{7.18}
$$

Generally speaking, we firstly note that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{x_\varepsilon\left(g_{t_{j-1}^r}(\varepsilon);u+\varepsilon v,0,x_0\right)-x\left(t_{j-1}^r;u,0,x_0\right)}{\varepsilon}\\
=\;&\lim_{\varepsilon\to 0}\frac{x_\varepsilon\left(g_{t_{j-1}^r}(\varepsilon);u+\varepsilon v,0,x_0\right)-x_\varepsilon\left(t_{j-1}^r;u+\varepsilon v,0,x_0\right)}{\varepsilon}\\
&+\lim_{\varepsilon\to 0}\frac{x_\varepsilon\left(t_{j-1}^r;u+\varepsilon v,0,x_0\right)-x\left(t_{j-1}^r;u,0,x_0\right)}{\varepsilon}\\
=\;&\varphi\left(t_{j-1}^r\right)+\dot g_{t_{j-1}^r}(0)\left[f\left(t_{j-1}^r,y_r\left(t_{j-1}^r\right)\right)+B\left(t_{j-1}^r\right)u\left(t_{j-1}^r\right)\right].
\end{aligned} \tag{7.19}
$$

Further, when $t\in\left(g_{t_{j-1}^r}(\varepsilon),g_{t_j^i}(\varepsilon)\right)$, one can infers from the assumption [F](3) and (7.3), (2.9), (7.19) that

$$
\begin{aligned}
G_\varepsilon(\varepsilon,t)\;=\;&\lim_{\xi\to 0}\frac{x_{\varepsilon+\xi}(t;u+(\varepsilon+\xi)v,0,x_0)-x_\varepsilon(t;u+\varepsilon v,0,x_0)}{\xi}+\frac{\partial}{\partial\varepsilon}\tilde y_i(t,\varepsilon)\\
=\;&\lim_{\xi\to 0}\frac{1}{\xi}\Big[x_{\varepsilon+\xi}\left(t;u+(\varepsilon+\xi)v,g_{t_{j-1}^r}(\varepsilon+\xi),\Upsilon_r\left(g_{t_{j-1}^r}(\varepsilon+\xi)\right)\right)\\
&-x_\varepsilon\left(t;u+\varepsilon v,g_{t_{j-1}^r}(\varepsilon),\Upsilon_r\left(g_{t_{j-1}^r}(\varepsilon)\right)\right)\Big]+\frac{\partial}{\partial\varepsilon}\tilde y_i(t,\varepsilon)\\
=\;&\lim_{\xi\to 0}\int_{g_{t_{j-1}^r}(\varepsilon+\xi)}^{t}\int_0^1 f_x(s,x_\varepsilon(s;u+\varepsilon v,0,x_0)+\theta(x_{\varepsilon+\xi}(s;u+(\varepsilon+\xi)v,0,x_0)\\
&-x_\varepsilon(s;u+\varepsilon v,0,x_0)))\frac{x_{\varepsilon+\xi}(s;u+(\varepsilon+\xi)v,0,x_0)-x_\varepsilon(s;u+\varepsilon v,0,x_0)}{\xi}d\theta ds\\
&+\lim_{\xi\to 0}\frac{\Upsilon_r\left(g_{t_{j-1}^r}(\varepsilon+\xi),\varepsilon+\xi\right)-\Upsilon_r\left(g_{t_{j-1}^r}(\varepsilon),\varepsilon\right)}{\xi}+\lim_{\xi\to 0}\int_{g_{t_{j-1}^r}(\varepsilon+\xi)}^{t}B(s)v(s)ds\\
&-\lim_{\xi\to 0}\frac{\int_{g_{t_{j-1}^r}(\varepsilon)}^{g_{t_{j-1}^r}(\varepsilon+\xi)}\left[f\left(s,x(s;u+\varepsilon v,0,x_0)\right)+B(s)(u(s)+\varepsilon v(s))\right]ds}{\xi}+\frac{\partial}{\partial\varepsilon}\tilde y_r(t,\varepsilon)\\
=\;&\lim_{\xi\to 0}\int_{g_{t_{j-1}^r}(\varepsilon+\xi)}^{t}\int_0^1 f_x(s,x_\varepsilon(s;u+\varepsilon v,0,x_0)+\theta(x_{\varepsilon+\xi}(s;u+(\varepsilon+\xi)v,0,x_0)\\
&-x_\varepsilon(s;u+\varepsilon v,0,x_0)))\frac{x_{\varepsilon+\xi}(s;u+(\varepsilon+\xi)v,0,x_0)-x_\varepsilon(s;u+\varepsilon v,0,x_0)}{\xi}d\theta ds\\
&+\int_{g_{t_{j-1}^r}(\varepsilon)}^{t}B(s)v(s)ds+\psi\left(g_{t_{j-1}^r}(\varepsilon)\right)+\frac{\partial}{\partial\varepsilon}\tilde y_r(t,\varepsilon)\\
&+\nabla J_r\left(\tilde y_r\left(g_{t_{j-1}^r}(\varepsilon),\varepsilon\right)\right)\left[\psi\left(g_{t_{j-1}^r}(\varepsilon)\right)\right.
\end{aligned}
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

$$+\dot{g}_{t^r_{j-1}}(\varepsilon)\left(f\left(g_{t^r_{j-1}}(\varepsilon),\tilde{y}_r\left(g_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)+B\left(g_{t^r_{j-1}}(\varepsilon)\right)u\left(g_{t^r_{j-1}}(\varepsilon)\right)\right)\Bigg].$$

Moreover, one can see from (7.11) and the above equality that

$$\begin{aligned}G_\varepsilon(\varepsilon,t) &= \frac{\partial}{\partial\varepsilon}\tilde{y}_r(t,\varepsilon)+\Psi_\varepsilon\left(t,g_{t^r_{j-1}}(\varepsilon)\right)\nabla J_r\left(\tilde{y}_r\left(g_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)\Big[\psi\left(g_{t^r_{j-1}}(\varepsilon)\right)\\ &\quad+\dot{g}_{t^r_{j-1}}(\varepsilon)\left(f\left(g_{t^r_{j-1}}(\varepsilon),\tilde{y}_r\left(g_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)+B\left(g_{t^r_{j-1}}(\varepsilon)\right)u\left(g_{t^r_{j-1}}(\varepsilon)\right)\right)\Big]\\ &\quad+\Psi_\varepsilon\left(t,g_{t^r_{j-1}}(\varepsilon)\right)\psi\left(g_{t^r_{j-1}}(\varepsilon)\right)+\int_{g_{t^r_{j-1}}(\varepsilon)}^{t}\Psi_\varepsilon(t,s)B(s)v(s)ds. \end{aligned}\tag{7.20}$$

Together with (7.10), by implicit function theorem, we have

$$\begin{aligned}\dot{g}_{t^i_j}(\varepsilon) &= -\frac{\Psi^1_\varepsilon\left(g_{t^i_j}(\varepsilon),g_{t^r_{j-1}}(\varepsilon)\right)\nabla J_r\left(\tilde{y}_r\left(g_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)}{f^1\left(g_{t^i_j}(\varepsilon),x_\varepsilon\left(g_{t^i_j}(\varepsilon);u+\varepsilon v,0,x_0\right)\right)+B^1\left(g_{t^i_j}(\varepsilon)\right)u\left(g_{t^i_j}(\varepsilon)\right)-\dot{y}^1_i(g_{t^i_j}(\varepsilon))}\\ &\quad\cdot\left[\psi\left(g_{t^r_{j-1}}(\varepsilon)\right)+\dot{g}_{t^r_{j-1}}(\varepsilon)\left(f\left(g_{t^r_{j-1}}(\varepsilon),\tilde{y}_r\left(g_{t^r_{j-1}}(\varepsilon),\varepsilon\right)\right)+B\left(g_{t^r_{j-1}}(\varepsilon)\right)u\left(g_{t^r_{j-1}}(\varepsilon)\right)\right)\right]\\ &\quad-\frac{\frac{\partial}{\partial\varepsilon}\tilde{y}^1_r\left(g_{t^i_j}(\varepsilon),\varepsilon\right)+\Psi^1_\varepsilon\left(g_{t^i_j}(\varepsilon),g_{t^r_{j-1}}(\varepsilon)\right)\psi\left(g_{t^r_{j-1}}(\varepsilon)\right)+\int_{g_{t^r_{j-1}}(\varepsilon)}^{g_{t^i_j}(\varepsilon)}\Psi^1_\varepsilon(t,s)B(s)v(s)ds}{f^1\left(g_{t^i_j}(\varepsilon),x_\varepsilon\left(g_{t^i_j}(\varepsilon);u+\varepsilon v,0,x_0\right)\right)+B^1\left(g_{t^i_j}(\varepsilon)\right)u\left(g_{t^i_j}(\varepsilon)\right)-\dot{y}^1_i(g_{t^i_j}(\varepsilon))}.\end{aligned}$$

Further, it follows from the above expression and (7.12), Theorem 7.1 that

$$\begin{aligned}\dot{g}_{t^i_j}(0) &= -\frac{\Psi^1\left(t^i_j,t^r_{j-1}\right)\nabla J_r\left(y_r\left(t^r_{j-1}\right)\right)}{f^1\left(t^i_j,x\left(t^i_j;u,0,x_0\right)\right)+B^1\left(t^i_j\right)u\left(t^i_j\right)-\dot{y}^1_i\left(t^i_j\right)}\\ &\quad\cdot\left[\psi\left(t^r_{j-1}\right)+\dot{g}_{t^r_{j-1}}(0)\left(f\left(t^r_{j-1},y_r\left(t^r_{j-1}\right)\right)+B\left(t^r_{j-1}\right)u\left(t^r_{j-1}\right)\right)\right]\\ &\quad-\frac{\Psi^1\left(t^i_j,t^r_{j-1}\right)\psi\left(t^r_{j-1}\right)+\int_{t^r_{j-1}}^{t^i_j}\Psi^1\left(t^i_j,s\right)B(s)v(s)ds}{f^1\left(t^i_j,x\left(t^i_j;u,0,x_0\right)\right)+B^1\left(t^i_j\right)u\left(t^i_j\right)-\dot{y}^1_i\left(t^i_j\right)},i\in\Lambda,j=1,2,\cdots,k.\end{aligned}\tag{7.21}$$

Similar to (7.19), we can obtain

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{x_\varepsilon\left(g_{t^i_j}(\varepsilon);u+\varepsilon v,0,x_0\right)-x\left(t^i_j;u,0,x_0\right)}{\varepsilon}\\ &=\psi\left(t^i_j\right)+\dot{g}_{t^i_j}(0)\left[f\left(t^i_j,y_i\left(t^i_j\right)\right)+B\left(t^i_j\right)u\left(t^i_j\right)\right],i\in\Lambda,j=1,2,\cdots,k.\end{aligned}\tag{7.22}$$

Together with the assumption [J](2) and (7.22), (7.21), it follows that when $g_{t^i_j}(\varepsilon)>t^i_j$,

$$\begin{aligned}\varphi\left(t^i_j+\right) &= \lim_{\varepsilon\to0}\frac{x_\varepsilon\left(g_{t^i_j}(\varepsilon)+;u+\varepsilon v,,x_0\right)-x\left(g_{t^i_j}(\varepsilon);u,0,x_0\right)}{\varepsilon}\\ &= \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left[x_\varepsilon\left(g_{t^i_j}(\varepsilon);u+\varepsilon v,0,x_0\right)+J_i\left(x_\varepsilon\left(g_{t^i_j}(\varepsilon);u+\varepsilon v,0,x_0\right)\right)\right.\end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

$$
\begin{aligned}
& -x\left(g_{t_j^i}(\varepsilon); u, t_j^i, x\left(t_j^i; u, 0, x_0\right) + J_i\left(x\left(t_j^i; u, 0, x_0\right)\right)\right)\Big] \\
= \; & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big[x_\varepsilon\left(g_{t_j^i}(\varepsilon); u+\varepsilon v, 0, x_0\right) + J_i\left(x_\varepsilon\left(g_{t_j^i}(\varepsilon); u+\varepsilon v, 0, x_0\right)\right) \\
& -x\left(t_j^i; u, 0, x_0\right) - J_i\left(x\left(t_j^i; u, 0, x_0\right)\right) - \int_{t_j^i}^{g_{t_j^i}(\varepsilon)} [f\left(s, x\left(s; u, 0, x_0\right)\right) + B(s)u(s)] ds\Big] \\
= \; & \left(I + \nabla J_i\left(y_i\left(t_j^i\right)\right)\right)\left[\psi\left(t_j^i\right) + \dot{g}_{t_j^i}(0)\left[f\left(t_j^i, y_i\left(t_j^i\right)\right) + B\left(t_j^i\right)u\left(t_j^i\right)\right]\right] \\
& -\dot{g}_{t_j^i}(0)\left[f\left(t_j^i, y_i\left(t_j^i\right)\right) + B\left(t_j^i\right)u\left(t_j^i\right)\right] \\
= \; & \psi\left(t_j^i\right) + \nabla J_i\left(y_i\left(t_j^i\right)\right)\left[\psi\left(t_j^i\right) + \dot{g}_{t_j^i}(0)\left[f\left(t_j^i, y_i\left(t_j^i\right)\right) + B\left(t_j^i\right)u\left(t_j^i\right)\right]\right],
\end{aligned}
$$

and when $g_{t_j^i}(\varepsilon) < t_j^i$,

$$
\begin{aligned}
\varphi\left(t_j^i+\right) = \; & \lim_{\varepsilon \to 0} \frac{x_\varepsilon\left(t_j^i; u+\varepsilon v, 0, x_0\right) - x\left(t_j^i+; u, 0, x_0\right)}{\varepsilon} \\
= \; & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big[x_\varepsilon\left(g_{t_j^i}(\varepsilon); u+\varepsilon v, 0, x_0\right) + J_i\left(x_\varepsilon\left(g_{t_j^i}(\varepsilon); u+\varepsilon v, 0, x_0\right)\right) \\
& -x\left(t_j^i; u, 0, x_0\right) - J_i\left(x\left(t_j^i; u, 0, x_0\right)\right) \\
& - \int_{t_j^i}^{g_{t_j^i}(\varepsilon)} \left[f\left(s, x_\varepsilon\left(s; u+\varepsilon v, 0, x_0\right)\right) + B(s)\left(u(s) + \varepsilon v(s)\right)\right] ds\Big] \\
= \; & \psi\left(t_j^i\right) + \nabla J_i\left(y_i\left(t_j^i\right)\right)\left[\psi\left(t_j^i\right) + \dot{g}_{t_j^i}(0)\left[f\left(t_j^i, y_i\left(t_j^i\right)\right) + B\left(t_j^i\right)u\left(t_j^i\right)\right]\right].
\end{aligned}
$$

Consequently, we have

$$
\psi\left(t_j^i+\right) = \psi\left(t_j^i\right) + \nabla J_i\left(y_i\left(t_j^i\right)\right)\left[\psi\left(t_j^i\right) + \dot{g}_{t_j^i}(0)\left[f\left(t_j^i, y_i\left(t_j^i\right)\right) + B\left(t_j^i\right)u\left(t_j^i\right)\right]\right] \tag{7.23}
$$

for $i \in \Lambda$, $j = 1, 2, \cdots, k$. Therefore, when $t \in \left(t_j^r, t_{j+1}^r\right)$ $(j = 1, 2, \cdots, k-1)$ or $t \in (t_k^r, T]$, it follows from the assumption [F](3) and (2.9), (7.3), (2.1), (4.7), (7.21), (7.22) that

$$
\begin{aligned}
\psi(t) = \; & \lim_{\theta \to 0} \frac{x_\theta(t; u+\theta v, 0, x_0) - x(t; u, 0, x_0)}{\theta} \\
= \; & \lim_{\theta \to 0} \frac{x_\theta(t; u+\theta v, g_{t_j^i}(\theta), \mathrm{Y}_i(g_{t_j^i}(\theta))) - x(t; u, t_j^i, \mathrm{Y}_i(t_j^i))}{\theta} \\
= \; & \lim_{\theta \to 0} \frac{\mathrm{Y}_i(g_{t_j^i}(\theta)) - \mathrm{Y}_i(t_j^i))}{\theta} + \lim_{\theta \to 0} \int_{g_{t_j^i}(\theta)}^{t} \int_0^1 f_x(s, x(s; u, 0, x_0) + \xi(x_\theta(s; u+\theta v, 0, x_0) \\
& -x(s; u, 0, x_0))) \frac{x_\theta(s; u+\theta v, 0, x_0) - x(s; u, 0, x_0)}{\theta} d\xi ds + \lim_{\theta \to 0} \int_{g_{t_j^i}(\theta)}^{t} B(s)v(s) ds \\
& - \lim_{\theta \to 0} \frac{1}{\theta} \int_{t_j^i}^{g_{t_j^i}(\theta)} [f(s, x(s; u, 0, x_0)) + B(s)u(s)] ds \\
= \; & \psi\left(t_j^i\right) + \nabla J_i\left(y_i\left(t_j^i\right)\right)\left[\psi\left(t_j^i\right) + \dot{g}_{t_j^i}(0)\left(f\left(t_j^i, y_i\left(t_j^i\right)\right) + B\left(t_j^i\right)u\left(t_j^i\right)\right)\right] \\
& + \int_{t_j^i}^{t} B(s)v(s) ds + \lim_{\theta \to 0} \int_{g_{t_j^i}(\theta)}^{t} \int_0^1 f_x(s, x(s; u, 0, x_0) + \xi(x_\theta(s; u+\theta v, 0, x_0)
\end{aligned}
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 January 2024 doi:10.20944/preprints202401.0171.v1

$$-x(s;u,0,x_0)))\frac{x_\theta(s;u+\theta v,0,x_0)-x(s;u,0,x_0)}{\theta}d\xi ds. \tag{7.24}$$

Thus, it follows from (7.14), (7.18), (7.23) that

$$\begin{cases} \dot{\psi}(t)=f_x(t,x(t;u,0,x_0))\psi(t)+B(t)v(t), t\in(0,T]\text{and}t\neq t_j^i, i\in\Lambda, j=1,2,\cdots,k, \\ \psi(0)=0, \\ \psi\left(t_j^i+\right)=\left(I+\nabla J_i\left(y_i\left(t_j^i\right)\right)\right)\psi\left(t_j^i\right) \\ \qquad +\dot{g}_{t_j^i}(0)\nabla J_i\left(y_i\left(t_j^i\right)\right)\left[f\left(t_j^i,y_i\left(t_j^i\right)\right)+B\left(t_j^i\right)u\left(t_j^i\right)\right], j=1,2,\cdots,k. \end{cases} \tag{7.25}$$

This completes the proof of Theorem 7.3. □

8. Conclusions

In this paper, we proposed a class of widely applied impulsive differential systems and gave it's qualitative theory under some weaker conditions, including the existence and uniqueness, periodicity of the solution, the continuous dependence and differentiability of the solution on the initial value. For the pulse phenomena of solution, it is significant to give the sufficient and necessary conditions. It is very interesting that the pulse may destroy the intrinsical properties of the system, such as the existence, the continuous dependence and differentiability of solution. Moreover, these results given in this paper also lay a theoretical foundation for the optimal control problem given by the impulsive different systems with impulses at variable times and the application of such systems. **Acknowledgments:** The authors are grateful to the referees for their helpful comments. This work was partially supported by the National Natural Science Foundation of China (No. 12061021 and No. 11161009).

Conflicts of Interest: The authors declare there is no conflicts of interest.

References

1. Bainov, D.; Simeonov, P. *Impulsive differential equations: periodic solutions and applications*, 1rd ed.; Longman Scientific and Technical: New York, USA, 1993; PP. 39–58.
2. Kobayashi, Y.; Nakano, H.; Saito, T. A simple chaotic circuit with impulsive switch depending on time and state. *Nonlinear Dynam.* **2006**, *44*, 73–79.
3. Touboul, J.; Brette, R. Spiking dynamics of Bidimensional integrate-and-fire neurons. *SIAM J. Appl. Dyn. Syst.* **2009**, *8*, 1462—1506.
4. Izhikevich, E. M. *Dynamical systems in neuroscience*, 1rd ed.; MIT Press: Massachusetts, USA, 2007.
5. Huang, M.; Li, J.; Song X.; Guo, H. Modeling impulsive injections of insulin: towards artificial pancreas. *SIAM J. Appl. Math.* **2012**, *72*, 1524–1548.
6. Zhang, Q.; Tang, B.; Cheng T.; Tang, S. Bifurcation analysis of a generalized impulsive Kolmogorov model with applitions to pest and disease control. *SIAM J. Appl. Math.* **2020**, *80*, 1796-1819.
7. Catll*á*, A. J.; Schaeffer, D. G.; Witelski, T. P.; Monson, E. E. On spiking models for synaptic activity and impulsive differential equations. *SIAM Rev.* **2008**, *50*, 553–569.
8. Akhmet, M.; Yilmaz, E. impulsive differential equations. In *Neural networks with discontinuous/impact activations*; Akhmet, M., Yilmaz, E., Eds.; Springer: New York, USA, 2014; pp. 67–83.
9. Lions, P. L.; Perthame, B. Quasi-variational inequalities and ergodic impulse control. *SIAM J. Control Optim.* **1986**, *24*, 604–615.
10. Wang, Y.; Lu, J. Some recent results of analysis and control for impulsive systems. *Commun Nonlinear Sci Numer Simulat* **2019**, *80*, 1–15.
11. Krylov, M. M.; Bogolyubov, N. N. *Introduction to nonlinear mechanics*, Academiya Nauk Ukrin: SSR, Kiev, 1937(in Russian).
12. Samoilenko, A. M.; Perestyuk, N. A. The method of averaging in systems with an impulsive action. *Ukrainian Mathematical Journal* **1974**, *26*, 342–347.
13. Ahmed, N. U.Existence of optimal controls for a general class of impulsive systems on Banach spaces. *SIAM J. Control Optim.***2003**, *42*, 669–685.
14. Peng, Y.; Xiang, X. A class of nonlinear impulsive differential equation and optimal controls on time scales. *Discrete Cont. Dyn.-B***2011**, *16*, 1137–1155.

 doi:10.20944/preprints202401.0171.v1

15. Nain, A. K.; Vats, R. K.; Kumar, A. Caputo-Hadamard fractional differential equation with impulsive boundary conditions. *J. Math. Model.*2021, *9*, 93–106.
16. Malik, M.; Kumar, A. Existence and controllability results to second order neutral differential equation with non-instantaneous impulses. *Journal of Control and Decision*2020, *7*, 286–308.
17. Samoilenko, A. M.; Perestyuk, N. A. On stability of solutions of impulsive systems. *Differentsial'nye Uravneniya*1981, *17*, 1995-2002.
18. Bainov, D. D.; Dishliev, A. B. Sutlicient conditions for absence of beating in systems of differential equations with impulses. *Appl. Anal.*1984, *18*, 67-73.
19. Hu, S.; Lakshmikantham, V. Periodic boundary value problem for second order impulsive differential systems. *Nonlinear Analysis TMA*1989, *13*, 75-85.
20. Lakshmikantham, V.; Bainov, D. D.; Simenonov P. S. *Theory of Impulsive Differential Equences*, 1rd ed.; Wold Scientific: Hong Kong, china, 1989.
21. Samoilenko, A. M. Application of the method of averaging for the investigation of vibrations excited by instantaneous impulses in self-vibrating systems of the second order with a small parameter. *Ukrainian Mathematical Journal*1961, *13*, 103–108.
22. Bainov, D.; Simeonov, P. *Impulsive differential equations*, 1rd ed.; Wold Scientific: Singapore, 1995.
23. Benchohra, M.; Henderson, J.; Ntouyas, S. *Impulsive differential equations and inclusions*, 1rd ed.; Hindawi Publishing Corporation: New York, USA, 2006.
24. Peng, Y.; Wu, K.; Qin, S.; Kang, Y. Properties of solution of linear controlled systems with impulses at variable times, In Proceedings of the 36th Chinese Control Conference, Dalian, China, 26-28 July 2017.