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A Natural Origin of the Beautiful Gamma Function from the WJ Distribution

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Abstract: We show succinctly a natural origin of the beautiful Gamma function in light of the WJ distribution, linking to quite distinct fields.

Keywords: gamma function; WJ distribution; integral function; bi-exponent; computation; statistics

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Introduction

The Gamma function, as initiated for factorial problems, is ingeniously set forth by the infinite integral function (IF) in single exponent (SE) of

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \tag{1}$$

for $z > 0$ ($\Re(z) > 0$ in the case that z is a complex number) [1–5]. This integral function is the most applied form of the factorial function though other extensions exist and extends to the entire complex plane via analytic continuation except non-positive integers which just result in simple poles.

In a variety of fields such as probability and statistics, quantum physics, string theory, artificial intelligence, and machine learning, this beautiful Gamma function finds a broad range of applications exemplar in the appearance of gamma distribution, beta function, Dirichlet's distribution, Stirling's asymptotic expansion, the zeta function and Riemann's hypothesis.

In contrast, the WJ distribution is a double exponential (or bi-exponent (BE)) probability distribution function [6], described by the following formula

$$f_{\mathrm{WJ}}(x, \alpha, \beta, \nu_{\alpha,\beta}) = \frac{\beta}{\Gamma(\alpha/\beta)} e^{-\alpha(x-\nu_{\alpha,\beta}) - e^{-\beta(x-\nu_{\alpha,\beta})}} \tag{2}$$

or in equivalence

$$f_{\mathrm{WJ}}(x, \alpha, \beta, \mu_{\mathrm{WJ}}) = \frac{\beta}{\Gamma(\alpha/\beta)} e^{-\alpha\left[x-\mu_{\mathrm{WJ}}-\frac{1}{\beta}\psi(\frac{\alpha}{\beta})\right] - e^{-\beta\left[x-\mu_{\mathrm{WJ}}-\frac{1}{\beta}\psi(\frac{\alpha}{\beta})\right]}} \tag{3}$$

with $\nu_{\alpha,\beta} = \mu_{\mathrm{WJ}} + \frac{1}{\beta}\psi(\frac{\alpha}{\beta})$ ($\psi(x)$ is the digamma function), defined over the domain of $(-\infty, +\infty)$. According to Equations (2) or (3), the WJ distribution function has three different free parameters α, β and μ_{WJ}: The parameter μ_{WJ} shifts the location of the curve but does not exert effect on the shape of the distribution. The parameters α and β taking positive values jointly govern the shape of the curve and make a contribution of $\psi(\alpha/\beta)/\beta$ to the horizontal setting of the curve.

The WJ distribution function, unifying a series of classical distributions, affords effectively a universal mechanism to tackle extreme events and critical phenomena, ranging from extraordinary occurrences to critical properties, enormously diverse in types and disparate in properties [6–9]. Still, the WJ distribution function has been corroborated to be an appropriate procedure in analyzing the Kohlrausch-Williams-Watts (KWW) relaxation [10] and well representing the Gaussian function with

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a specific set of the parameters (α, β, $\nu_{\alpha,\beta}$) [11], along with potential application in information theory [12].

Results

In this work, we shall establish a close linkage between the Gamma function and the WJ distribution. We define a bi-exponent integral function (BEIF), $\mathrm{G}(\alpha,\beta)$, in the vein of the WJ distribution $f_{\mathrm{WJ}}(x,\alpha,\beta,\nu_{\alpha,\beta})$

$$\mathrm{G}(\alpha,\beta)=\int_{-\infty}^{+\infty}\beta e^{-\alpha(x-\nu_{\alpha,\beta})-e^{-\beta(x-\nu_{\alpha,\beta})}}dx \tag{4}$$

Alternatively, the same definition applies to $f_{\mathrm{WJ}}(x,\alpha,\beta,\mu_{\mathrm{WJ}})$,

$$\mathrm{G}(\alpha,\beta)=\int_{-\infty}^{+\infty}\beta e^{-\alpha\left[x-\mu_{\mathrm{WJ}}-\frac{1}{\beta}\psi(\frac{\alpha}{\beta})\right]-e^{-\beta\left[x-\mu_{\mathrm{WJ}}-\frac{1}{\beta}\psi(\frac{\alpha}{\beta})\right]}}dx \tag{5}$$

which will be incurred in the process of deduction later.

By rearranging the terms in Equation (4), we obtain

$$\mathrm{G}(\alpha,\beta)=-\frac{\beta}{\alpha}\int_{-\infty}^{+\infty}e^{-e^{-\beta(x-\nu_{\alpha,\beta})}}de^{-\alpha(x-\nu_{\alpha,\beta})} \tag{6}$$

Conducting the partial integration, Equation (6) turns out to be

$$\mathrm{G}(\alpha,\beta)=\frac{\beta}{\alpha}\int_{-\infty}^{+\infty}e^{-\alpha(x-\nu_{\alpha,\beta})}de^{-e^{-\beta(x-\nu_{\alpha,\beta})}} \tag{7}$$

which leads to the expression as follows

$$\mathrm{G}(\alpha,\beta)=\frac{\beta}{\alpha}\left\{\beta\int_{-\infty}^{+\infty}e^{-(\alpha+\beta)(x-\nu_{\alpha,\beta})-e^{-\beta(x-\nu_{\alpha,\beta})}}dx\right\} \tag{8}$$

Considering the fact that the argument $(x-\nu_{\alpha+\beta,\beta})$ has the same outcome as that of $(x-\nu_{\alpha+\beta,\beta})$ for the infinite integral, thus Equation (8) adopts the formulation

$$\mathrm{G}(\alpha,\beta)=\frac{\beta}{\alpha}\left\{\beta\int_{-\infty}^{+\infty}e^{-(\alpha+\beta)(x-\nu_{\alpha+\beta,\beta})-e^{-\beta(x-\nu_{\alpha+\beta,\beta})}}dx\right\} \tag{9}$$

The part in the great bracket is analogous to Equation (2) with the variable set (α,β) being changed to $(\alpha+\beta,\beta)$, that is, the corresponding integral function assumes the form

$$\mathrm{G}(\alpha+\beta,\beta)=\int_{-\infty}^{+\infty}\beta e^{-(\alpha+\beta)(x-\nu_{\alpha+\beta,\beta})-e^{-\beta(x-\nu_{\alpha+\beta,\beta})}}dx \tag{10}$$

Evidently, we get a recurrence relation for the bi-exponent integral function

$$\mathrm{G}(\alpha+\beta,\beta)=\frac{\alpha}{\beta}\mathrm{G}(\alpha,\beta) \tag{11}$$

It is perceived that Equation (11) is the same form as the recurrence relation of the Gamma function in disguise

$$\Gamma(z+1)=z\Gamma(z) \tag{12}$$

In point of fact, Equation (4) is reducible to the following single-exponent infinite integral (SEIF)

$$\mathrm{G}(\alpha,\beta)=\int_{0}^{+\infty}t^{\frac{\alpha}{\beta}-1}e^{-t}dt \tag{13}$$

which yields in essence the same expression as Equation (1) for the Gamma function with $z=\frac{\alpha}{\beta}$, after we carry out the exchange of the variables

$$t=e^{-\beta(x-\nu_{\alpha,\beta})} \tag{14}$$

Moreover, it is recognized that $\mathrm{G}(\alpha,\beta)$ is scalable to $\mathrm{G}(\frac{\alpha}{\beta},1)$, viz.,

$$\mathrm{G}(\alpha,\beta)=\mathrm{G}(\frac{\alpha}{\beta},1) \tag{15}$$

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Nonetheless, the association of the Gamma function with the WJ distribution may be instructively scrutinized from the embedding of the former in the formula of the latter (cf. Equations (2) & (3)). Since the WJ distribution is a probability distribution function and has intrinsically the unitary property of the total integration

$$\int_{-\infty}^{+\infty} f_{\mathrm{WJ}}(x,\alpha,\beta,\nu_{\alpha,\beta})dx = 1 \tag{16}$$

we may directly acquire the Gamma function given in the argument of (α/β) by inserting the explicit formula of $f_{\mathrm{WJ}}(x,\alpha,\beta,\nu_{\alpha,\beta})$ in Equation (2) and rearranging

$$\Gamma\left(\frac{\alpha}{\beta}\right) = \int_{-\infty}^{+\infty} \beta e^{-\alpha(x-\nu_{\alpha,\beta})-e^{-\beta(x-\nu_{\alpha,\beta})}} dx \tag{17}$$

which is identical to Equation (4). Consequently, we have demonstrated that the Gamma function may be naturally inferred from the WJ distribution $f_{\mathrm{WJ}}(x,\alpha,\beta,\nu_{\alpha,\beta})$, or in other words, the bi-exponent integral function is equivalent to the Gamma function. Conclusively, it is appreciated that the BEIF of Equation (4) is correspondent to the Gamma function of Equation (1), which is particularly transparent in its reduced form of Equation (13). Figure 1 presents the amplitude of the Gamma function $\Gamma(z)$ or equivalently $\mathrm{G}(\alpha,\beta)$ in complex plane in 3D created with Matlab & Origin.

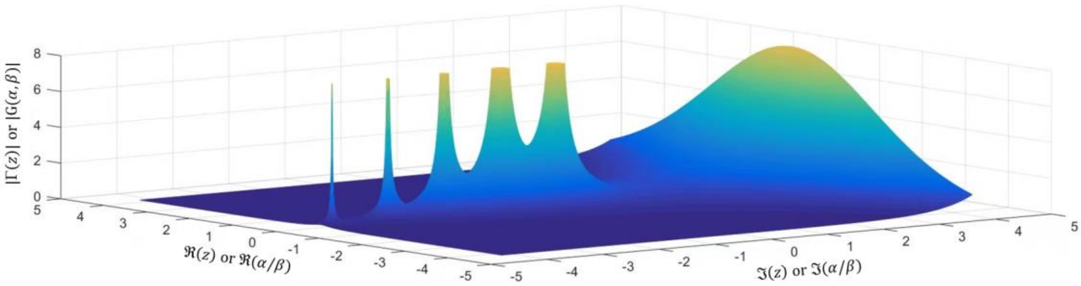

Figure 1. Plotting the absolute value of Gamma function $\Gamma(z)$ or equivalently $\mathrm{G}(\alpha,\beta)$ in complex plane in 3D created with Matlab & Origin.

Concerning the probability distribution function for the Gamma function, we come to two representations, one of which is designated by the original WJ distribution $f_{\mathrm{WJ}}(x,\alpha,\beta,\nu_{\alpha,\beta})$ of Equation (2) (or equivalent $f_{\mathrm{WJ}}(x,\alpha,\beta,\mu_{\mathrm{WJ}})$ of Equation (3)) and the other is, based on Equation (13), the classic form of the Gamma distribution specified by

$$f_{\Gamma}\left(t,\frac{\alpha}{\beta}\right) = \frac{1}{\Gamma(\alpha/\beta)} t^{\frac{\alpha}{\beta}-1} e^{-t} \tag{18}$$

Their first and second statistical quantities, the expectation E and the square deviation σ, respectively, read

$$E_{\mathrm{WJ}}(x) = \mu_{\mathrm{WJ}} + \frac{2}{\beta}\psi(\frac{\alpha}{\beta}) \tag{19}$$

and

$$\sigma_{\mathrm{WJ}}(x) = \frac{1}{\beta}\sqrt{\psi'(\alpha/\beta)} \tag{20}$$

for the original WJ distribution, as well as

$$E_{\Gamma}(t) = \frac{\alpha}{\beta} \tag{21}$$

and

$$\sigma_{\Gamma}(t) = \sqrt{\frac{\alpha}{\beta}} \tag{22}$$

for the classical expression of the Gamma distribution derivable from the original WJ distribution.

It is immediately noticed that the two kinds of the Gamma distribution, essentially resulting from the WJ distribution, have quite distinctive characteristics, as partially reflected in their statistical quantities as above-elucidated. Relating to the expectation of the original WJ distribution (cf. Equation (19)), Figure 2 plots the amplitude of the digamma function $\psi(z)$ or equivalently $|\beta[E_{\mathrm{WJ}}(x)-\mu_{\mathrm{WJ}}]/2|$ in complex plane in 3D created with Matlab & Origin for direct visualization.

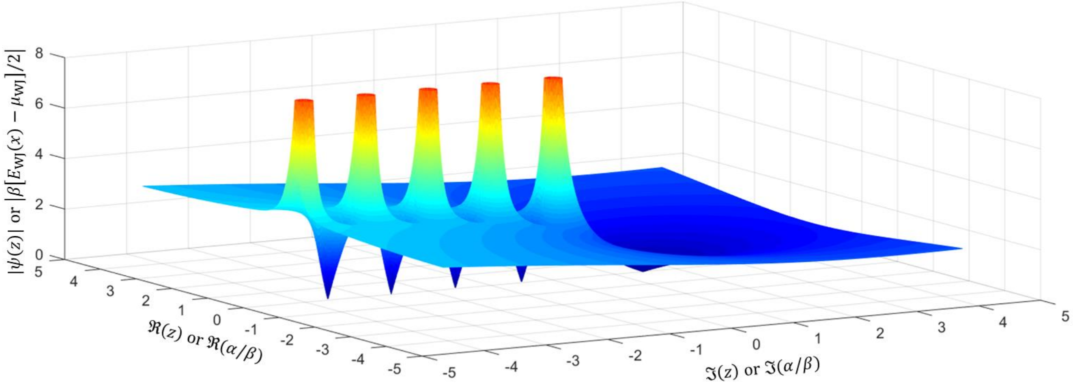

Figure 2. Plotting the absolute value of the digamma function $\psi(z)$ or equivalently $|\beta[E_{\mathrm{WJ}}(x)-\mu_{\mathrm{WJ}}]/2|$ in complex plane in 3D created with Matlab & Origin.

In addition, we can conduct Fourier or Laplace transformation on these two different representations, separately. As the WJ distribution is defined over the whole domain $(-\infty, +\infty)$, a Fourier transform is pertinent

$$F_{\mathrm{WJ}}(\omega)=\int_{-\infty}^{+\infty}e^{-\mathrm{i}\omega t}f_{\mathrm{WJ}}(t,\alpha,\beta,\nu_{\alpha,\beta})dt \tag{23}$$

By replacing the expression of Equation (2), we have the explicit formulation of

$$F_{\mathrm{WJ}}(\omega)=e^{-\mathrm{i}\omega\nu_{\alpha,\beta}}\Gamma\left(\frac{\alpha+\mathrm{i}\omega}{\beta}\right)/\Gamma\left(\frac{\alpha}{\beta}\right) \tag{24}$$

for the transform. Correspondingly, a Laplace transform can be performed on the classical form of the Gamma distribution in the form of Equation (18)

$$L_{\Gamma}(s)=\int_{0}^{+\infty}e^{-st}f_{\Gamma}\left(t,\frac{\alpha}{\beta}\right)dt \tag{25}$$

or explicitly

$$L_{\Gamma}(s)=(1+s)^{-\frac{\alpha}{\beta}} \tag{26}$$

Conclusion

We have shown that the WJ distribution may offer a concise definition of the beautiful Gamma function, prospectively casting a deeper insight into the natural origin of the latter.

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References

1. Davis, P.J. Leonhard Euler's Integral: A Historical Profile of the Gamma Function. *The American Mathematical Monthly* **1959**, *66*, 849-869.
2. Borwein, J.M.; Corless, R.M. Gamma and Factorial in the Monthly. *The American Mathematical Monthly* **2018**, *125*, 400-424,
3. Whittaker, E.T.; Watson, G.N.; Moll, V.H. *A course of modern analysis*. 5th ed.; Cambridge University Press: Cambridge, UK, 2021.
4. Oldham, K.B.; Myland, J.C.; Spanier, J. *An Atlas of Functions*. 2nd ed.; Springer-Verlag: New York, NY, USA, 2009.
5. Havil, J. *Gamma: Exploring Euler's Constant*. Princeton University Press: Princeton, NJ, USA, 2003.
6. Wu, J.H.; Jia, Q. A universal mechanism of extreme events and critical phenomena. *Sci. Rep*. **2016**, *6*, 21612.
7. Bramwell, S.T.; Holdsworth, P.C.W.; Pinton, J.F. Universality of rare fluctuations in turbulence and critical phenomena. *Nature* **1998**, *396*, 552–554.
8. Bramwell, S.T. The distribution of spatially averaged critical properties. *Nat. Phys.* **2009**, *5*, 443–447.
9. Albeverio, S.; Jentsch, V.; Kantz, H. (Eds.) *Extreme Events in Nature and Society*. Springer: Berlin/Heidelberg, Germany, 2006.
10. Wu, J.H.; Jia, Q. The heterogeneous energy landscape expression of KWW relaxation. *Sci. Rep*. **2016**, *6*, 20506.
11. Ge, S.R.; Wu, J.H. The Analysis of WJ Distribution as an Extended Gaussian Function: Case Study. *Appl. Sci*. **2022**, *12*, 7773.
12. Liang, G.Y.; Xue, H.; Jia, Q.; Wu, J.H. Some Properties of the WJ Distribution and Implication in Information Theory. *J. Phys. Conf. Ser*. **2019**, *1237*, 022081.