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Tractability of Approximation of Functions Defined over Weighted Hilbert Spaces

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Abstract: We investigate L_2-approximation problems in the worst case setting in the weighted Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weights $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ under parameters $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and $1 < \alpha_1 \leq \alpha_2 \leq \cdots$. Several interesting weighted Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ appear in this paper. We consider the worst case error of algorithms that use finitely many arbitrary continuous linear functionals. We discuss tractability of L_2-approximation problems for the involved Hilbert spaces, which describes how the information complexity depends on d and ε^{-1}. As a consequence we study the strongly polynomial tractability (SPT), polynomial tractability (PT), weak tractability (WT), and (t_1, t_2)-weak tractability ((t_1, t_2)-WT) for all $t_1 > 1$ and $t_2 > 0$ in terms of the introduced weights under the absolute error criterion or the normalized error criterion.

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1. Introduction

We investigate multivariate approximation problems S_d with large or even huge d. Examples include these problems in statistics, computational finance and physics. In order to solve these problems we usually consider algorithms using finitely many evaluations of arbitrary continuous linear functionals. We use either the absolute error criterion (ABS) or the normalized error criterion (NOR). For $\mathrm{X} \in \{\mathrm{ABS}, \mathrm{NOR}\}$ we define the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ to be the minimal number of linear functionals which are needed to find an algorithm whose worst case error is at most ε. The behavior of the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ is the major concern when the accuracy ε of approximation goes to zero and the number d of variables goes to infinity. For small ε and large d, tractability is aimed at studying how the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ behaves as a function of d and ε^{-1}, while the exponential convergence-tractability (EC-tractability) is aimed at studying how the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ behaves as a function of d and $(1 + \ln(\varepsilon^{-1}))$. Recently the study of tractability and EC-tractability in the worst case setting has attracted much interest in analytic Korobov spaces, weighted Korobov spaces and weighted Gaussian ANOVA spaces; see [1–12] and the references therein.

Weighted multivariate approximation of functions on space $[0,1]^d$ are studied in many problems. We are interested in weighted Hilbert spaces of functions in this paper. We present three examples of weighted Hilbert spaces, which are similar but also different. We devote to discussing worst case tractability of L_2-approximation problem

$$\mathrm{APP} = \{\mathrm{APP}_d : H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \to L_2([0,1]^d)\}_{d\in\mathbb{N}}$$

with $\mathrm{APP}_d(f) = f$ for all $f \in H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ in weighted Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with three weights $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ under positive parameter sequences $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$. The tractability and EC-tractability of such problem APP in weighted Korobov spaces with parameters $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ and

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$1 < \alpha_1 = \alpha_2 = \cdots$ were discussed in [2,4,6,11] and in [13], respectively. Additionally, [4] considered the tractability of the L_2-approximation in several weighted Hilbert spaces for permissible information class consisting of arbitrary continuous linear functionals and consisting of functions evaluations.

In this paper we study SPT, PT, WT and (t_1, t_2)-WT for all $t_1 > 1$ and $t_2 > 0$ of the above problem APP with parameters

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0,$$

and

$$1 < \alpha_1 \leq \alpha_2 \leq \cdots$$

for the ABS or the NOR under the information class consisting of arbitrary continuous linear functionals. Especially, we get a compete sufficient and necessary condition for SPT, PT and WT, respectively, and the exponent of SPT.

The paper is organized as follows. In Section 2 we give preliminaries about multivariate approximation problems in Hilbert spaces for information class consisting of arbitrary continuous linear functionals in the worst case setting, and definitions of tractability. In Section 3 we present several examples of weighted Hilbert spaces and study some facts and relations between them. In Section 4 we discuss the tractability properties of L_2-approximation problems in the above weighted Hilbert spaces, then state out main result Theorem 4.1.

2. Approximation and tractability in Hilbert spaces

2.1. Approximation in Hilbert spaces

Let F_d and G_d be two sequences of Hilbert spaces. Consider a sequence of compact linear operators

$$S_d : F_d \to G_d$$

for all $d \in \mathbb{N}$. We approximation S_d by algorithm $A_{n,d}$ of the form

$$A_{n,d}(f) = \sum_{i=1}^{n} T_i(f) g_i, \text{ for } f \in S_d, \tag{2.1}$$

where functions $g_i \in G_d$ and continuous linear functionals $T_i \in F_d^*$ for $i = 1, \cdots, n$. The worst case error for the algorithm $A_{n,d}$ of the form (2.1) is defined as

$$e(A_{n,d}) := \sup_{f \in F_d, ||f||_{F_d} \leq 1} ||S_d(f) - A_{n,d}(f)||_{G_d}.$$

The nth minimal worst-case error, for $n \geq 1$, is defined by

$$e(n, S_d) := \inf_{A_{n,d}} e(A_{n,d}),$$

where the infimum is taken over all linear algorithms of the form (2.1). For $n = 0$, we use $A_{0,d} = 0$. We call

$$e(0, S_d) = \sup_{f \in F_d, ||f||_{F_d} \leq 1} ||S_d(f)||_{G_d}$$

the initial error of the problem S_d.

The information complexity for S_d can be studied using either the absolute error criterion (ABS), or the normalized error criterion (NOR). The information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ for $\mathrm{X} \in \{\mathrm{ABS}, \mathrm{NOR}\}$ is defined by

$$n^{\mathrm{X}}(\varepsilon, S_d) := \min\{n \in \mathbb{N}_0 : e(n, S_d) \leq \varepsilon \mathrm{CRI}_d\},$$

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where

$$\mathrm{CRI}_d := \begin{cases} 1, & \text{for } \mathrm{X} = \mathrm{ABS}, \\ e(0, S_d), & \text{for } \mathrm{X} = \mathrm{NOR}. \end{cases}$$

Here, $\mathbb{N}_0 = \{0, 1, \cdots\}$ and $\mathbb{N} = \{1, 2, \cdots\}$.

It is well known, see e.g., [7,14], that the nth minimal worst case errors $e(n, S_d)$ and the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ depend on the eigenvalues of the continuously linear operator $W_d = S_d^* S_d : F_d \to F_d$. Let $(\lambda_{d,j}, \eta_{d,j})$ be the eigenpairs of W_d, i.e.,

$$W_d \eta_{d,j} = \lambda_{d,j} \eta_{d,j} \text{ for all } j \in \mathbb{N},$$

where the eigenvalues $\lambda_{d,j}$ are ordered,

$$\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq 0,$$

and the eigenvectors $\eta_{d,j}$ are orthonormal,

$$\langle \eta_{d,i}, \eta_{d,j} \rangle_{F_d} = \delta_{i,j} \text{ for all } i, j \in \mathbb{N}.$$

Then the nth minimal error is obtained for the algorithm

$$A_{n,d}^{\diamond} f = \sum_{j=1}^{n} \langle f, \eta_{d,j} \rangle_{F_d} \eta_{d,j} \text{ for all } f \in F_d,$$

and

$$e(n, S_d) = e(A_{n,d}^{\diamond}) = \sqrt{\lambda_{d,n+1}} \text{ for all } n \in \mathbb{N}_0.$$

Hence the information complexity is equal to

$$\begin{aligned} n^{\mathrm{X}}(\varepsilon, S_d) &= \min\{n \in \mathbb{N}_0 : \sqrt{\lambda_{d,n+1}} \leq \varepsilon \mathrm{CRI}_d\} \\ &= \min\{n \in \mathbb{N}_0 : \lambda_{d,n+1} \leq \varepsilon^2 \mathrm{CRI}_d^2\} \\ &= |\{n \in \mathbb{N} : \lambda_{d,n} > \varepsilon^2 \mathrm{CRI}_d^2\}|, \end{aligned} \tag{2.2}$$

with $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$. We focus on the rate of the information complexity when the error threshold ε tends to 0 and the problem dimension d grows to infinity.

2.2. *Tractability*

In order to characterize the dependency of the information complexity $n^{\mathrm{X}}(\varepsilon, S_d)$ for the absolute error criterion and the normalized error criterion on the dimension d and the error threshold ε, we will briefly recall some of the basic tractability and exponential convergence-tractability (EC-tractability) notions.

Let $S = \{S_d\}_{d \in \mathbb{N}}$. For $\mathrm{X} \in \{\mathrm{ABS}, \mathrm{NOR}\}$, we say S is

• strongly polynomially tractable (SPT) iff there exist non-negative numbers C and p such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,

$$n^{\mathrm{X}}(\varepsilon, S_d) \leq C(\varepsilon^{-1})^p.$$

The exponent p^{str} of SPT is defined to be the infimum of all p for which the above inequality holds.

• polynomially tractable (PT) iff there exist non-negative numbers C, p and q such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,

$$n^{\mathrm{X}}(\varepsilon, S_d) \leq C d^q (\varepsilon^{-1})^p.$$

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• quasi-polynomially tractable (QPT) iff there exist two constants $C, t > 0$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0,1)$,

$$n^{\mathrm{X}}(\varepsilon, S_d) \le C \exp(t(1 + \ln \varepsilon^{-1})(1 + \ln d)).$$

The exponent t^{pol} of QPT is defined to be the infimum of all t for which the above inequality holds.

• uniformly weakly tractable (UWT) iff for all $t_1, t_2 > 0$,

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, S_d)}{d^{t_1} + (\varepsilon^{-1})^{t_2}} = 0;$$

• weakly tractable (WT) iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, S_d)}{d + \varepsilon^{-1}} = 0.$$

• (t_1, t_2)-weakly tractable ((t_1, t_2)-WT) for fixed positive t_1 and t_2 iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, S_d)}{d^{t_1} + (\varepsilon^{-1})^{t_2}} = 0.$$

We call that S suffers from the curse of dimensionality if there exist positive numbers C_1, C_2, ε_0 such that for all $0 < \varepsilon \le \varepsilon_0$ and infinitely many $d \in \mathbb{N}$,

$$n(\varepsilon, d) \ge C_1(1 + C_2)^d.$$

• Exponential convergence-strongly polynomially tractable (EC-SPT) iff there exist non-negative numbers C and p such that for all $d \in \mathbb{N}$, $\varepsilon \in (0,1)$,

$$n^{\mathrm{X}}(\varepsilon, S_d) \le C\big(1 + \ln(\varepsilon^{-1})\big)^p.$$

The exponent of SPT is defined to be the infimum of all p for which the above inequality holds.

• Exponential convergence-polynomially tractable (EC-PT) iff there exist non-negative numbers C, p and q such that for all $d \in \mathbb{N}$, $\varepsilon \in (0,1)$,

$$n^{\mathrm{X}}(\varepsilon, S_d) \le C d^q \big(1 + \ln(\varepsilon^{-1})\big)^p.$$

• Exponential convergence-uniformly weakly tractable (EC-UWT) iff for all $t_1, t_2 > 0$

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, S_d)}{d^{t_1} + \big(1 + \ln(\varepsilon^{-1})\big)^{t_2}} = 0.$$

• Exponential convergence-weakly tractable (EC-WT) iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, \mathrm{APP}_d)}{d + \ln(\varepsilon^{-1})} = 0.$$

• Exponential convergence-(t_1, t_2)-weakly tractable (EC-(t_1, t_2)-WT) for fixed positive t_1 and t_2 iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n^{\mathrm{X}}(\varepsilon, S_d)}{d^{t_1} + \big(1 + \ln(\varepsilon^{-1})\big)^{t_2}} = 0.$$

Clearly, (1,1)-WT is the same as WT, and EC-(1,1)-WT is the same as EC-WT. Obviously, in the definitions of SPT, PT, QPT, UWT, WT and (t_1, t_2)-WT, if we replace ε^{-1} by $(1 + \ln(\varepsilon^{-1}))$, we get the

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definitions of EC-SPT, EC-PT, EC-QPT, EC-UWT, EC-WT and EC-(t_1, t_2)-WT, respectively. We also have

$$\text{SPT} \Longrightarrow \text{PT} \Longrightarrow \text{QPT} \Longrightarrow \text{UWT} \Longrightarrow \text{WT},$$

$$\text{EC-SPT} \Longrightarrow \text{EC-PT} \Longrightarrow \text{EC-QPT} \Longrightarrow \text{EC-UWT} \Longrightarrow \text{EC-WT},$$

$$\text{EC-SPT} \Longrightarrow \text{SPT}, \quad \text{EC-PT} \Longrightarrow \text{PT}, \quad \text{EC-QPT} \Longrightarrow \text{QPT},$$

and

$$\text{EC-}(t_1, t_2)\text{-WT} \Longrightarrow (t_1, t_2)\text{-WT}, \quad \text{EC-UWT} \Longrightarrow \text{UWT}, \quad \text{EC-WT} \Longrightarrow \text{WT}.$$

We can learn more information about tractability of multivariate problems in the volumes [7–9] by Novak and Woźniakowski.

Lemma 2.1. *([7] Theorem 5.2) Consider the non-zero problem $S = \{S_d\}$ for compact linear problems S_d defined over Hilbert spaces. Then S is PT for NOR iff there exist $q \geq 0$ and $\tau > 0$ such that*

$$C_{\tau,q} := \sup_{d\in\mathbb{N}} \Big(\sum_{j=1}^{\infty} \big(\frac{\lambda_{d,j}}{\lambda_{d,1}}\big)^{\tau} \Big)^{\frac{1}{\tau}} d^{-q} < \infty. \tag{2.3}$$

Expecially, S is SPT for NOR iff (2.3) holds with q=0. The exponent of SPT is

$$p^{str} = \inf\{2\tau | \tau \text{ satisfies (2.3) with } q = 0\}.$$

3. Weighted Hilbert spaces

Let the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weight $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ under positive parameter sequences $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfying

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0, \tag{3.1}$$

and

$$1 < \alpha_1 \leq \alpha_2 \leq \cdots . \tag{3.2}$$

be a reproducing kernel Hilbert space. The reproducing kernel function $K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}} : [0,1]^d \times [0,1]^d \to \mathbb{C}$ of the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ is given by

$$K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x}, \boldsymbol{y}) := \prod_{k=1}^{d} K_{R_{\alpha_k,\gamma_k}}(x_k, y_k),$$

$\boldsymbol{x} = (x_1, x_2, \cdots, x_d),\ \boldsymbol{y} = (y_1, y_2, \cdots, y_d) \in [0,1]^d$, where

$$K_{R_{\alpha,\gamma}}(x, y) := \sum_{k\in\mathbb{N}_0} R_{\alpha,\gamma}(k) \exp(2\pi i k \cdot (x - y)), \ \ x,\ y \in [0,1]$$

is a universal weighted function. Here Fourier weight $R_{\alpha,\gamma} : \mathbb{N}_0 \to \mathbb{R}^+$ be a summable function, i.e., $\sum_{k\in\mathbb{N}_0} R_{\alpha,\gamma}(k) < \infty$. We will consider weight $R_{\alpha,\gamma}$ later on in some examples.

Then we have

$$K_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) \exp(2\pi i \boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{y})),\ \boldsymbol{x},\ \boldsymbol{y} \in [0,1]^d, \tag{3.3}$$

and the corresponding inner product

$$\langle f, g \rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sum_{\boldsymbol{k}\in\in\mathbb{N}_0^d} \frac{1}{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})} \widehat{f}(\boldsymbol{k}) \overline{\widehat{g}}(\boldsymbol{k}) \tag{3.4}$$

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and

$$||f||_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \sqrt{\langle f, f \rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})}},$$

where

$$R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(k_j), \ \boldsymbol{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{N}_0^d,$$

$$\boldsymbol{x} \cdot \boldsymbol{y} := \sum_{k=1}^{d} x_k \cdot y_k, \ \ \boldsymbol{x} = (x_1, x_2, \ldots, x_d), \ \boldsymbol{y} = (y_1, y_2, \ldots, y_d) \in [0,1]^d,$$

and

$$\widehat{f}(\boldsymbol{k}) = \int_{[0,1]^d} f(\boldsymbol{x}) \exp(-2\pi i \boldsymbol{k} \cdot \boldsymbol{x}) d\boldsymbol{x}.$$

We note that the kernel $K_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y})$ is well defined for $1 < \alpha_1 \le \alpha_2 \le \cdots$ and for all $\boldsymbol{x}, \ \boldsymbol{y} \in [0,1]^d$, since $|K_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y})| \le \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^{d} (\sum_{k \in \mathbb{N}_0} R_{\alpha_j,\gamma_j}(k)) < \infty$. If $\gamma_1 = \gamma_2 = \cdots = 1$ and $\alpha_1 = \alpha_2 = \cdots > 1$ then the space is called unweighted space.

The weights are introduced to model the importance of the functions from the space. The idea can be seen in the reference [15] by Sloan and Woźniakowski. There are various ways to introduce weighted Hilbert spaces. We consider possible choices for Fourier weights $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ on three examples.

3.1. A Korobov space

Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j \in \mathbb{N}}$ and $\boldsymbol{\gamma} = \{\gamma_j\}_{j \in \mathbb{N}}$ satisfy (3.1) and (3.2), respectively. We are interesting in the weighted Korobov space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ defined by Irrgeher and Leobacher (see [16]) with kernel (3.3) and corresponding inner product (3.4), where weight $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{d} r_{\alpha_j,\gamma_j}(k_j)$ with

$$r_{\alpha,\gamma}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \frac{\gamma}{k^{\lceil \alpha \rceil}}, & \text{for } k \ge 1, \end{cases}$$

for $\alpha > 1$ and $\gamma \in (0,1]$. Note that we have $r_{\alpha,\gamma}(k) \in (0,1]$ for all $k \in \mathbb{N}_0$.

The space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) := H(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ is a reproducing kernel Hilbert space with parameter sequences $\boldsymbol{\alpha} = \{\alpha_j\}_{j \in \mathbb{N}}$ and $\boldsymbol{\gamma} = \{\gamma_j\}_{j \in \mathbb{N}}$.

3.2. A first variant of the Korobov space

Let $\boldsymbol{\alpha} = \{\alpha_j\}_{j \in \mathbb{N}}$ and $\boldsymbol{\gamma} = \{\gamma_j\}_{j \in \mathbb{N}}$ satisfy (3.1) and (3.2), respectively. We consider the reproducing kernel Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with kernel (3.3) and corresponding inner product (3.4) determined by $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{d} \psi_{\alpha_j,\gamma_j}(k_j)$ with

$$\psi_{\alpha,\gamma}(k) := \begin{cases} 1, & \text{for } k = 0, \\ \frac{\gamma}{k!}, & \text{for } 1 \le k < \lceil \alpha \rceil, \\ \frac{\gamma (k - \lceil \alpha \rceil)!}{k!}, & \text{for } k \ge \lceil \alpha \rceil, \end{cases}$$

for $\alpha > 1$ and $\gamma \in (0,1]$.

The following lemma gives the upper bound and the lower bound of the weight $\psi_{\alpha,\gamma}(k)$, which shows that $\psi_{\alpha,\gamma}(k)$ has the same decay rate as the weight $r_{\alpha,\gamma}(k)$ of the Korobov space $H(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ under the same parameter sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$.

Lemma 3.1. *For all j, $k \in \mathbb{N}$ we have*

$$r_{\alpha_j,\gamma_j}(k) \le \psi_{\alpha_j,\gamma_j}(k) \le \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k).$$

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Proof. First for all $j, k \in \mathbb{N}$ we want to prove

$$\psi_{\alpha_j,\gamma_j}(k) \leq \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k).$$

For $1 \leq k < \lceil \alpha_j \rceil$ we have

$$\psi_{\alpha_j,\gamma_j}(k) = \frac{\gamma_j}{k!} \leq \gamma_j \leq \gamma_j \left(\frac{\lceil \alpha_j \rceil}{k} \right)^{\lceil \alpha_j \rceil}.$$

For $k \geq \lceil \alpha_j \rceil$ we have

$$\begin{aligned}
\psi_{\alpha_j,\gamma_j}(k) &= \frac{\gamma_j (k - \lceil \alpha_j \rceil)!}{k!} = \frac{\gamma_j}{k(k-1)\cdots(k - \lceil \alpha_j \rceil + 1)} \\
&\leq \frac{\gamma_j}{(k - \lceil \alpha_j \rceil + 1)^{\lceil \alpha_j \rceil}} = \frac{\gamma_j}{k^{\lceil \alpha_j \rceil}(1 - \frac{\lceil \alpha_j \rceil - 1}{k})^{\lceil \alpha_j \rceil}} \\
&\leq \frac{\gamma_j}{k^{\lceil \alpha_j \rceil}(1 - \frac{\lceil \alpha_j \rceil - 1}{\lceil \alpha_j \rceil})^{\lceil \alpha_j \rceil}} = \frac{\lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} \gamma_j}{k^{\lceil \alpha_j \rceil}}.
\end{aligned}$$

We find for all $k \in \mathbb{N}$ that

$$\psi_{\alpha_j,\gamma_j}(k) \leq \frac{\lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} \gamma_j}{k^{\lceil \alpha_j \rceil}} = \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k).$$

Next, for all $j, k \in \mathbb{N}$ we need to prove

$$\psi_{\alpha_j,\gamma_j}(k) \geq r_{\alpha_j,\gamma_j}(k).$$

For $1 \leq k < \lceil \alpha_j \rceil$ we have

$$\psi_{\alpha_j,\gamma_j}(k) = \frac{\gamma_j}{k!} \geq \frac{\gamma_j}{k^k} \geq \frac{\gamma_j}{k^{\lceil \alpha_j \rceil}}.$$

For $k \geq \lceil \alpha_j \rceil$ we have

$$\psi_{\alpha_j,\gamma_j}(k) = \frac{\gamma_j (k - \lceil \alpha_j \rceil)!}{k!} = \frac{\gamma_j}{k(k-1)\cdots(k - \lceil \alpha_j \rceil + 1)} \geq \frac{\gamma_j}{k^{\lceil \alpha_j \rceil}}.$$

Hence for all $j, k \in \mathbb{N}$ we obtain

$$\psi_{\alpha_j,\gamma_j}(k) \geq \frac{\gamma_j}{k^{\lceil \alpha_j \rceil}} = r_{\alpha_j,\gamma_j}(k).$$

This finishes the proof. □

3.3. A second variant of the Korobov space

In [17], the reproducing kernel Hilbert space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ was considered with kernel (3.3) and corresponding inner product (3.4). Here $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^d \omega_{\alpha_j,\gamma_j}(k_j)$ was defined as

$$\omega_{\alpha,\gamma}(k) := \left(1 + \frac{1}{\gamma} \sum_{l=1}^{\lceil \alpha \rceil} \theta_l(k) \right)^{-1},$$

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for $\alpha > 1$ and $\gamma \in (0,1]$, where

$$\theta_l(k) := \begin{cases} \frac{k!}{(k-l)!}, & \text{for } k \geq l, \\ 0, & \text{for } 0 \leq k < l. \end{cases}$$

Note that for $k \in \mathbb{N}$ we have

$$\sum_{l=1}^{\lceil\alpha\rceil} \theta_l(k) \leq 2k^{\lceil\alpha\rceil}. \tag{3.5}$$

Indeed, for $k = 1$ we have

$$\sum_{l=1}^{\lceil\alpha\rceil} \theta_l(k) = 1 \leq 2k^{\lceil\alpha\rceil},$$

for $2 \leq k \leq \lceil\alpha\rceil$ we have

$$\sum_{l=1}^{\lceil\alpha\rceil} \theta_l(k) = \sum_{l=1}^{k} \frac{k!}{(k-l)!} = k! \sum_{l=0}^{k-1} \frac{1}{l!} \leq k! \sum_{l=0}^{\infty} \frac{1}{l!} \leq k!e \leq 2k^k \leq 2k^{\lceil\alpha\rceil},$$

and for $k > \lceil\alpha\rceil$ we have

$$\sum_{l=1}^{\lceil\alpha\rceil} \theta_l(k) = \sum_{l=1}^{\lceil\alpha\rceil} \frac{k!}{(k-l)!} \leq \sum_{l=1}^{\lceil\alpha\rceil} k^l = k^{\lceil\alpha\rceil} + \frac{k^{\lceil\alpha\rceil} - k}{k-1} \leq 2k^{\lceil\alpha\rceil}.$$

Lemma 3.2. *For all $j, k \in \mathbb{N}$ we have*

$$\frac{1}{3} r_{\alpha_j,\gamma_j}(k) \leq \omega_{\alpha_j,\gamma_j}(k) \leq \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k).$$

Proof. First for all $j, k \in \mathbb{N}$ we want to prove

$$\omega_{\alpha_j,\gamma_j}(k) \leq \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k).$$

For $1 \leq k < \lceil\alpha_j\rceil$ we have

$$\omega_{\alpha_j,\gamma_j}(k) = \left(1 + \frac{1}{\gamma_j} \sum_{l=1}^{\lceil\alpha_j\rceil} \theta_l(k)\right)^{-1} = \left(1 + \frac{1}{\gamma_j} \sum_{l=1}^{k} \theta_l(k)\right)^{-1} \leq \left(\frac{1}{\gamma_j} \theta_k(k)\right)^{-1} = \frac{\gamma_j}{k!}.$$

For $k \geq \lceil\alpha_j\rceil$ we have

$$\omega_{\alpha_j,\gamma_j}(k) = \left(1 + \frac{1}{\gamma_j} \sum_{l=1}^{\lceil\alpha_j\rceil} \theta_l(k)\right)^{-1} \leq \left(\frac{1}{\gamma_j} \theta_{\lceil\alpha_j\rceil}(k)\right)^{-1} = \frac{\gamma_j (k - \lceil\alpha_j\rceil)!}{k!}.$$

Hence for all $j, k \in \mathbb{N}$ we get

$$\omega_{\alpha_j,\gamma_j}(k) \leq \psi_{\alpha_j,\gamma_j}(k),$$

and thus by Lemma 3.1

$$\omega_{\alpha_j,\gamma_j}(k) \leq \psi_{\alpha_j,\gamma_j}(k) \leq \lceil\alpha_j\rceil^{\lceil\alpha_j\rceil} r_{\alpha_j,\gamma_j}(k)$$

holds.

Next, for all $j, k \in \mathbb{N}$ we need to prove

$$\omega_{\alpha_j,\gamma_j}(k) \geq \frac{1}{3} r_{\alpha_j,\gamma_j}(k).$$

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It follows from (3.5) that for all $j,\, k \in \mathbb{N}$ we have

$$\omega_{\alpha_j,\gamma_j}(k) = \left(1 + \frac{1}{\gamma_j}\sum_{l=1}^{\lceil \alpha_j \rceil} \theta_l(k)\right)^{-1} \geq \left(1 + \frac{2k^{\lceil \alpha_j \rceil}}{\gamma_j}\right)^{-1} \geq \left(\frac{3k^{\lceil \alpha_j \rceil}}{\gamma_j}\right)^{-1} = \frac{1}{3} r_{\alpha_j,\gamma_j}(k).$$

This proof is complete. □

Remark 3.3. Set $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \varphi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$ for all $j,\, k \in \mathbb{N}$. From Lemma 3.1 and Lemma 3.2 we have for all $j,\, k \in \mathbb{N}$,

$$\frac{1}{3} r_{\alpha_j,\gamma_j}(k) \leq R_{\alpha_j,\gamma_j}(k) \leq \lceil \alpha_j \rceil^{\lceil \alpha_j \rceil} r_{\alpha_j,\gamma_j}(k). \tag{3.6}$$

Note that for all $j,\, k \in \mathbb{N}$ we have $\psi_{\alpha_j,\gamma_j}(k) \leq \psi_{\alpha_1,\gamma_j}(k)$, $r_{\alpha_j,\gamma_j}(k) \leq r_{\alpha_1,\gamma_j}(k)$, and $\omega_{\alpha_j,\gamma_j}(k) \leq \omega_{\alpha_1,\gamma_j}(k)$, which means that

$$R_{\alpha_j,\gamma_j}(k) \leq R_{\alpha_1,\gamma_j}(k), \quad \text{for all } \; j,\, k \in \mathbb{N}. \tag{3.7}$$

Combining with (3.6) and (3.7), we conclude

$$\frac{1}{3} r_{\alpha_j,\gamma_j}(k) \leq R_{\alpha_j,\gamma_j}(k) \leq R_{\alpha_1,\gamma_j}(k) \leq \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} r_{\alpha_1,\gamma_j}(k), \tag{3.8}$$

for all $j,\, k \in \mathbb{N}$.

Remark 3.4. The weight $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ are used to describe the importance of the different coordinates for the functions from the space $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$. According to (3.6) we have the weight $\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ and the weight $\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ have the same decay rate as the weight $r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$ of the Korobov space $H(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$. Hence the above reproducing kernel Hilbert spaces $H(K_{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$, $H(K_{\psi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ and $H(K_{\omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ are different but also similar.

4. L_2-approximation in weighted Hilbert spaces and main results

In this section we consider L_2-approximation

$$\mathrm{APP}_d : H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \to L_2([0,1]^d)$$

with $\mathrm{APP}_d(f) = f$ for all $f \in H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ in Hilbert spaces $H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$ with weights $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \varphi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$. It is well known from [6] that this embedding APP_d is compact with $1 < \alpha_1 \leq \alpha_2 \leq \cdots$. The kernel $K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})$ is well defined for $\alpha_1 > 1$ and for all $\boldsymbol{x},\boldsymbol{y} \in [0,1]^d$, since by (3.7)

$$|K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y})}| \leq \sum_{\boldsymbol{k}\in\mathbb{N}^d} R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^{d}(1 + \lceil \alpha_1 \rceil^{\lceil \alpha_1 \rceil} \zeta(\lceil \alpha_1 \rceil)\gamma_j) < \infty,$$

where $\zeta(\cdot)$ is the Riemann zeta function.

In the worst case setting the tractability and EC-tractability of L_2-approximation problems S_d with $G_d = L_2([0,1]^d)$ were investigated in analytic Korobov spaces and weighted Korobov spaces; see [1–3,6,10–13]. Additionally, [2,6,11,13] discussed tractability and EC-tractability in weighted Korobov spaces.

From subsection 2.1 the information complexity of APP_d depends on the eigenvalues of the operator $W_d = \mathrm{APP}_d^*\mathrm{APP}_d : H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}}) \to H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})$. Let $(\lambda_{d,j}, \eta_{d,j})$ be the eigenpairs of W_d,

$$W_d \eta_{d,j} = \lambda_{d,j} \eta_{d,j} \text{ for all } j \in \mathbb{N},$$

where the eigenvalues $\lambda_{d,j}$ are ordered,

$$\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq 0,$$

and the eigenvectors $\eta_{d,j}$ are orthonormal,

$$\langle \eta_{d,i}, \eta_{d,j} \rangle_{H(K_{R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}})} = \delta_{i,j} \text{ for all } i, j \in \mathbb{N}.$$

Obviously, we have $e(0, \text{APP}_d) = 1$ (or see [6]). Hence the NOR and the ABS for the problem APP_d coincide in the worst case setting. We abbreviate $n^{\text{X}}(\varepsilon, \text{APP}_d)$ as $n(\varepsilon, \text{APP}_d)$, i.e.,

$$n(\varepsilon, \text{APP}_d) := n^{\text{X}}(\varepsilon, \text{APP}_d).$$

It is well known that the eigenvalues of the operator W_d are $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k})$ with $\boldsymbol{k} \in \mathbb{N}^d$; see, e.g., [7, p. 215]. Hence by (2.2) we have

$$\begin{aligned} n(\varepsilon, \text{APP}_d) &= |\{n \in \mathbb{N} : \lambda_{d,n} > \varepsilon^2\}| = |\{\boldsymbol{k} \in \mathbb{N}_0^d : R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) > \varepsilon^2\}| \\ &= |\{\boldsymbol{k} \in \mathbb{N}_0^d : \prod_{j=1}^{d} R_{\alpha_j,\gamma_j}(k_j) > \varepsilon^2\}|. \end{aligned}$$

Tractability such as SPT, PT, WT, and (t_1, t_2)-WT for $t_1 > 1$, and EC-tractability such as EC-WT and EC-$(t_1, 1)$-WT for $t_1 < 1$ of the above problem $\text{APP} = \{\text{APP}_d\}$ with the parameter sequences $\boldsymbol{\gamma} = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfying

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$$

and

$$1 < \alpha = \alpha_1 = \alpha_2 = \cdots$$

have been solved by [2,4,11] and [13], respectively. The following conditions have been obtained therein:

- For $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} \in \{r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \varphi_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}, \omega_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}\}$, PT holds iff SPT holds iff

$$s_{\boldsymbol{\gamma}} := \inf\left\{\kappa > 0 : \sum_{j=1}^{\infty} \gamma_j^{\kappa} < \infty\right\} < \infty,$$

and the exponent of SPT is

$$p^{\text{str}} = 2\max\left(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha}\right).$$

- For $R_{d,\alpha,\boldsymbol{\gamma}} = r_{d,\alpha,\boldsymbol{\gamma}}$, QPT, UWT and WT are equivalent and hold iff

$$\gamma_I := \inf_{j\in\mathbb{N}} \gamma_j < 1.$$

For $R_{d,\alpha,\boldsymbol{\gamma}} \in \{\varphi_{d,\alpha,\boldsymbol{\gamma}}, \omega_{d,\alpha,\boldsymbol{\gamma}}\}$,

$$\gamma_I < \infty$$

implies QPT.

In those cases the exponent of QPT is

$$t^{\text{pol}} := \begin{cases} 2\max\left(\frac{1}{\alpha}, \frac{1}{\log \gamma_I^{-1}}\right), & \text{for } \gamma_I \neq 0, \\ \frac{2}{\alpha}, & \text{for } \gamma_I = 0. \end{cases}$$

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• For $R_{d,\boldsymbol{\alpha},\gamma} \in \{r_{d,\boldsymbol{\alpha},\gamma}, \varphi_{d,\boldsymbol{\alpha},\gamma}, \omega_{d,\boldsymbol{\alpha},\gamma}\}$ and $t_1 > 1$, (t_1, t_2)-WT holds for all $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$.
• For $R_{d,\alpha,\gamma} = r_{d,\alpha,\gamma}$, EC-WT holds iff

$$\lim_{j\to\infty} \gamma_j = 0.$$

• For $R_{d,\alpha,\gamma} = r_{d,\alpha,\gamma}$ and $t_1 < 1$, EC-$(t_1, 1)$-WT holds iff

$$\lim_{j\to\infty} \frac{\ln j}{\ln(\gamma_j^{-1})} = 0.$$

We will research the worst case tractability of the problem APP with sequences satisfying (3.1) and (3.2).

Theorem 4.1. *Let the sequences $\gamma = \{\gamma_j\}_{j\in\mathbb{N}}$ and $\boldsymbol{\alpha} = \{\alpha_j\}_{j\in\mathbb{N}}$ satisfy (3.1) and (3.2). Consider the L_2-approximation APP for the weighted Hilbert spaces $H_{R_{d,\alpha,\gamma}}$, $R_{d,\boldsymbol{\alpha},\gamma} \in \{r_{d,\boldsymbol{\alpha},\gamma}, \varphi_{d,\boldsymbol{\alpha},\gamma}, \omega_{d,\boldsymbol{\alpha},\gamma}\}$. Then we have the following tractability results:*

(1) SPT and PT are equivalent and hold iff

$$\delta := \liminf_{j\to\infty} \frac{\ln \gamma_j^{-1}}{\ln j} > 0. \tag{4.1}$$

The exponent of SPT is

$$p^{str} = 2 \max\Big\{\frac{1}{\delta}, \frac{1}{\lceil \alpha_1 \rceil}\Big\}.$$

(2) For $R_{d,\boldsymbol{\alpha},\gamma} = r_{d,\boldsymbol{\alpha},\gamma}$, WT holds iff

$$\lim_{j\to\infty} \gamma_j < 1.$$

(3) For $t_1 > 1$, (t_1, t_2)-WT holds.

Proof. (1) For the problem APP we have $\lambda_{d,1} = 1$. Assume that APP is PT. From Lemma 2.1 there exist $q \geq 0$ and $\tau > 0$ such that

$$C_{\tau,q} := \sup_{d\in\mathbb{N}} \Big(\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau}\Big)^{\frac{1}{\tau}} d^{-q} < \infty.$$

It follows from

$$\sum_{j=1}^{\infty} \lambda_{d,j}^{\tau} = \prod_{j=1}^{d} \Big(\sum_{k=0}^{\infty} (R_{\alpha_j,\gamma_j}(k))^{\tau} \Big) = \prod_{j=1}^{d} \Big(1 + \sum_{k=1}^{\infty} (R_{\alpha_j,\gamma_j}(k))^{\tau} \Big), \tag{4.2}$$

and (3.8) that

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$$
\begin{aligned}
\infty > C_{\tau,q} &\geq \Big(\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau}\Big)^{\frac{1}{\tau}} d^{-q}\\
&\geq \prod_{j=1}^{d}\Big(1+\sum_{k=1}^{\infty}\big(\frac{1}{3}r_{\alpha_j,\gamma_j}(k)\big)^{\tau}\Big)^{\frac{1}{\tau}} d^{-q}\\
&= \Big(\prod_{j=1}^{d}\big(1+\frac{1}{3^{\tau}}\gamma_j^{\tau}\zeta(\lceil\alpha_j\rceil\tau)\big)\Big)^{\frac{1}{\tau}} d^{-q}\\
&\geq \Big(\prod_{j=1}^{d}\big(1+\frac{\gamma_j^{\tau}}{3^{\tau}}\big)^{\frac{1}{\tau}}\Big) d^{-q}\\
&\geq \big(1+\frac{\gamma_d^{\tau}}{3^{\tau}}\big)^{\frac{d}{\tau}} d^{-q}.
\end{aligned}
$$

We conclude that

$$\ln C_{\tau,q} + q\ln d \geq \frac{d}{\tau}\ln\big(1+\frac{\gamma_d^{\tau}}{3^{\tau}}\big) \geq \frac{d}{2\tau}\cdot\frac{\gamma_d^{\tau}}{3^{\tau}},$$

where we used $\ln(1+x) \geq \frac{x}{2}$ for all $x \in [0,1]$. We further get

$$\ln(\ln C_{\tau,q} + q\ln d) \geq \ln d - \tau\ln\gamma_d^{-1} - \ln(2\tau\cdot 3^{\tau}),$$

i.e.,

$$\frac{\ln\gamma_d^{-1}}{\ln d} \geq \frac{\ln d - \ln(\ln C_{\tau,q}+q\ln d) - \ln(2\tau\cdot 3^{\tau})}{\tau\cdot\ln d}.$$

Hence we obtain

$$\delta = \liminf_{d\to\infty}\frac{\ln\gamma_d^{-1}}{\ln d} \geq \frac{1}{\tau} > 0. \tag{4.3}$$

Note that if APP is SPT, then it is PT. It implies that if APP is SPT, then (4.3) holds and the exponent

$$p^{\text{str}} \geq 2\max\{\frac{1}{\delta},\frac{1}{\lceil\alpha_1\rceil}\}.$$

On the other hand, assume that (4.1) holds. For an arbitrary $\varepsilon \in (0,\frac{\delta}{2})$, there exists an integer $N > 0$ such that for all $j \geq N$ we have

$$\frac{\ln\gamma_j^{-1}}{\ln j} \geq \delta - \varepsilon.$$

It means that for all $j \geq N$

$$\gamma_j \leq j^{-(\delta-\varepsilon)}.$$

Choosing $\tau = \frac{1}{\delta-2\varepsilon}$ and noting that $\frac{\delta-\varepsilon}{\delta-2\varepsilon} > 1$, we have

$$\sum_{j=N}^{\infty}\gamma_j^{\tau} \leq \sum_{j=N}^{\infty} j^{-(\delta-\varepsilon)\tau} = \sum_{j=N}^{\infty} j^{-\frac{\delta-\varepsilon}{\delta-2\varepsilon}} < \infty,$$

which yields that

$$\sum_{j=1}^{\infty}\gamma_j^{\tau} < \infty. \tag{4.4}$$

From (3.8) we get

$$
\begin{aligned}
\Big(\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau}\Big)^{\frac{1}{\tau}}d^{-q} &= \prod_{j=1}^{d}\Big(1+\sum_{k=1}^{\infty}(R_{\alpha_j,\gamma_j}(k))^{\tau}\Big)^{\frac{1}{\tau}}d^{-q} \\
&\le \prod_{j=1}^{d}\Big(1+\sum_{k=1}^{\infty}(\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil}r_{\alpha_1,\gamma_j}(k))^{\tau}\Big)^{\frac{1}{\tau}}d^{-q} \\
&= d^{-q}\cdot\exp\Big\{\ln\Big(\prod_{j=1}^{d}(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau))^{\frac{1}{\tau}}\Big)\Big\} \\
&= d^{-q}\cdot\exp\Big\{\frac{1}{\tau}\sum_{j=1}^{d}\ln\big(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)\big)\Big\} \\
&\le d^{-q}\cdot\exp\Big\{\frac{1}{\tau}\sum_{j=1}^{d}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)\Big\} \\
&= d^{-q}\cdot\exp\Big\{\frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\zeta(\lceil\alpha_1\rceil\tau)}{\tau}\cdot\sum_{j=1}^{d}\gamma_j^{\tau}\Big\} \\
&\le d^{-q}\cdot\exp\Big\{\frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\zeta(\lceil\alpha_1\rceil\tau)}{\tau}\cdot\sum_{j=1}^{\infty}\gamma_j^{\tau}\Big\} \\
&< \infty
\end{aligned}
$$

for any $q \ge 0$ and $\tau > \frac{1}{\lceil\alpha_1\rceil}$. We further get

$$C_{\tau,q} = \sup_{d\in\mathbb{N}}\Big(\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau}\Big)^{\frac{1}{\tau}}d^{-q} < \infty$$

for any $q \ge 0$ and $\tau > \frac{1}{\lceil\alpha_1\rceil}$. It follows from Lemma 2.1 that APP is SPT or PT and the exponent $p^{\rm str} \le 2\tau$. Setting $\varepsilon \to 0$, we obtain

$$p^{\rm str} \le 2\tau \le 2\max\Big\{\frac{1}{\delta},\frac{1}{\lceil\alpha_1\rceil}\Big\}.$$

Hence the exponent of SPT is $p^{\rm str} = 2\max\{\frac{1}{\delta},\frac{1}{\lceil\alpha_1\rceil}\}$.

(2) Let $\tau > 0$. Due to

$$n\lambda_{d,n}^{\tau} \le \sum_{j=1}^{n}\lambda_{d,j}^{\tau} \le \sum_{j=1}^{\infty}\lambda_{d,j}^{\tau},$$

we have

$$\lambda_{d,n} \le \frac{(\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau})^{\frac{1}{\tau}}}{n^{\frac{1}{\tau}}}.$$

Noting that $\lambda_{d,n} \le \varepsilon^2$ holds for

$$n = \Big\lceil\sum_{j=1}^{\infty}\lambda_{d,j}^{\tau}\varepsilon^{-2\tau}\Big\rceil,$$

we get

$$\begin{aligned} n(\varepsilon, \mathrm{APP}_d) &= \min\{n | \lambda_{d,n+1} \le \varepsilon^2\} \\ &\le \lceil \sum_{j=1}^{\infty} \lambda_{d,j}^{\tau} \varepsilon^{-2\tau} \rceil \\ &\le 1 + \varepsilon^{-2\tau} \sum_{j=1}^{\infty} \lambda_{d,j}^{\tau} \\ &= 1 + \varepsilon^{-2\tau} \prod_{j=1}^{d} \Big(1 + \sum_{k=1}^{\infty} (R_{\alpha_j,\gamma_j}(k))^{\tau}\Big) \\ &\le 2\varepsilon^{-2\tau} \prod_{j=1}^{d} \Big(1 + \sum_{k=1}^{\infty} (R_{\alpha_j,\gamma_j}(k))^{\tau}\Big), \end{aligned} \tag{4.5}$$

where we used (4.2).

Set $R_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}} = r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}$. Assume that $\lim_{j\to\infty} \gamma_j < 1$. Then we have from (4.5) that

$$\begin{aligned} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\varepsilon^{-1}} &\le \frac{\ln \Big(2\varepsilon^{-2\tau} \prod_{j=1}^{d} \big(1 + \sum_{k=1}^{\infty} (R_{\alpha_j,\gamma_j}(k))^{\tau}\big)\Big)}{d+\varepsilon^{-1}} \\ &= \frac{\ln \Big(2\varepsilon^{-2\tau} \prod_{j=1}^{d} \big(1 + \sum_{k=1}^{\infty} (r_{\alpha_j,\gamma_j}(k))^{\tau}\big)\Big)}{d+\varepsilon^{-1}} \\ &= \frac{\ln \Big(2\varepsilon^{-2\tau} \prod_{j=1}^{d} \big(1 + \gamma_j^{\tau} \zeta(\lceil \alpha_j \rceil \tau)\big)\Big)}{d+\varepsilon^{-1}} \\ &\le \frac{\ln \Big(2\varepsilon^{-2\tau} \prod_{j=1}^{d} \big(1 + \gamma_j^{\tau} \zeta(\lceil \alpha_1 \rceil \tau)\big)\Big)}{d+\varepsilon^{-1}} \\ &\le \frac{\ln 2 + 2\tau \ln(\varepsilon^{-1})}{\varepsilon^{-1}} + \frac{\sum_{j=1}^{d} \ln \big(1 + \gamma_j^{\tau} \zeta(\lceil \alpha_1 \rceil \tau)\big)}{d} \\ &\le \frac{\ln 2 + 2\tau \ln(\varepsilon^{-1})}{\varepsilon^{-1}} + \frac{\sum_{j=1}^{d} \gamma_j^{\tau} \zeta(\lceil \alpha_1 \rceil \tau)}{d}. \end{aligned} \tag{4.6}$$

We will consider two cases:

- Case $\lim_{j\to\infty} \gamma_j = 0$: It means that for any $\delta > 0$ there exists a positive integer $J = J(\delta)$ such that

$$\gamma_j < \delta \quad \text{for all} \quad j \ge J.$$

Then we conclude from (4.6) that

$$\begin{aligned} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\varepsilon^{-1}} &\le \frac{\ln 2 + 2\tau \ln(\varepsilon^{-1})}{\varepsilon^{-1}} + \frac{\sum_{j=1}^{d} \gamma_j^{\tau} \zeta(\lceil \alpha_1 \rceil \tau)}{d} \\ &\le \frac{\ln 2 + 2\tau \ln(\varepsilon^{-1})}{\varepsilon^{-1}} + \frac{(J-1)\zeta(\lceil \alpha_1 \rceil \tau) + \sum_{j=J}^{\max(d,J)} \delta^{\tau} \zeta(\lceil \alpha_1 \rceil \tau)}{d}, \end{aligned}$$

which deduces that

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\varepsilon^{-1}} \le \delta^{\tau} \zeta(\lceil \alpha_1 \rceil \tau).$$

Setting $\delta \to 0$, we have $\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\varepsilon^{-1}} = 0$. This yields WT.

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• Case $\lim_{j\to\infty}\gamma_j \in (0,1)$: Then, for every $\lim_{j\to\infty}\gamma_j < \gamma_* < 1$ there exists a positive integer $J_0 = J(\gamma_*)$ such that

$$\gamma_j < \gamma_* \ \text{ for all } \ j \geq J_0.$$

We have from (4.6) that

$$\begin{aligned}
\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\varepsilon^{-1}} &\leq \frac{\ln 2 + 2\tau \ln(\varepsilon^{-1})}{\varepsilon^{-1}} + \frac{\sum_{j=1}^{d}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)}{d} \\
&\leq \frac{\ln 2 + 2\tau \ln(\varepsilon^{-1})}{\varepsilon^{-1}} + \frac{\sum_{j=1}^{J_0-1}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)}{d} \\
&+ \frac{\sum_{j=J_0}^{\max(J_0,d)}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)}{d} \\
&\leq \frac{\ln 2 + 2\tau \ln(\varepsilon^{-1})}{\varepsilon^{-1}} + \frac{(J_0-1)\zeta(\lceil\alpha_1\rceil\tau)}{d} \\
&+ \frac{\sum_{j=J_0}^{\max(J_0,d)}\gamma_*^{\tau}\zeta(\lceil\alpha_1\rceil\tau)}{d},
\end{aligned}$$

which means

$$\lim_{d+\varepsilon^{-1}\to\infty}\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\varepsilon^{-1}} \leq \gamma_*^{\tau}\zeta(\lceil\alpha_1\rceil\tau).$$

Noting that

$$\zeta(\alpha) = 1 + \sum_{k=2}^{\infty}\frac{1}{k^{\alpha}} \leq 1 + \int_1^{\infty}\frac{1}{x^{\alpha}}dx = 1 + \frac{1}{\alpha-1} \quad \text{for all } \ \alpha > 1,$$

and setting $\tau \to \infty$, we obtain

$$\lim_{d+\varepsilon^{-1}\to\infty}\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d+\varepsilon^{-1}} \leq \lim_{\tau\to\infty}\gamma_*^{\tau}\zeta(\lceil\alpha_1\rceil\tau) = 0.$$

This implies WT.

On the other hand, it suffices to show that WT yields $\lim_{j\to\infty}\gamma_j < 1$. Assume on the contrary that $\lim_{j\to\infty}\gamma_j = 1$. It yields that $\gamma_j \equiv 1$ for all $j \in \mathbb{N}$. It follows that

$$1 = r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) > \varepsilon^2$$

for all $\boldsymbol{k} \in \{0,1\}^d$. Then we have

$$n(\varepsilon, \mathrm{APP}_d) = |\{\boldsymbol{k} \in \mathbb{N}_0^d : r_{d,\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{k}) > \varepsilon^2\}| \geq 2^d.$$

Hence APP suffers from the curse of dimensionality. We cannot have WT.

(3) Let $\tau > 0$. Due to (4.5) and (3.8) we have

$$\begin{aligned}
n(\varepsilon, \mathrm{APP}_d) &\leq 2\varepsilon^{-2\tau}\prod_{j=1}^{d}\Big(1 + \sum_{k=1}^{\infty}(R_{\alpha_j,\gamma_j}(k))^{\tau}\Big) \\
&\leq 2\varepsilon^{-2\tau}\prod_{j=1}^{d}\big(1 + \lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)\big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^{t_1}+\varepsilon^{-t_2}} &\le \frac{\ln\left(2\varepsilon^{-2\tau}\prod_{j=1}^{d}\left(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)\right)\right)}{d^{t_1}+\varepsilon^{-t_2}} \\
&= \frac{\ln 2+2\tau\ln(\varepsilon^{-1})+\sum_{j=1}^{d}\ln\left(1+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)\right)}{d^{t_1}+\varepsilon^{-t_2}} \\
&\le \frac{\ln 2+2\tau\ln(\varepsilon^{-1})+\sum_{j=1}^{d}\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\gamma_j^{\tau}\zeta(\lceil\alpha_1\rceil\tau)}{d^{t_1}+\varepsilon^{-t_2}} \\
&\le \frac{\ln 2+2\tau\ln(\varepsilon^{-1})+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}\zeta(\lceil\alpha_1\rceil\tau)\sum_{j=1}^{d}\gamma_j^{\tau}}{d^{t_1}+\varepsilon^{-t_2}} \\
&\le \frac{\ln 2+2\tau\ln(\varepsilon^{-1})+\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}d\zeta(\lceil\alpha_1\rceil\tau)}{d^{t_1}+\varepsilon^{-t_2}} \\
&\le \frac{\ln 2+2\tau\ln(\varepsilon^{-1})}{\varepsilon^{-t_2}}+\frac{\lceil\alpha_1\rceil^{\lceil\alpha_1\rceil\tau}d\zeta(\lceil\alpha_1\rceil)}{d^{t_1}}.
\end{aligned}$$

We obtain for all $t_1 > 1$ and $t_2 > 0$,

$$\lim_{\varepsilon^{-1}+d\to\infty}\frac{\ln n(\varepsilon, \mathrm{APP}_d)}{d^{t_1}+\varepsilon^{-t_2}} = 0,$$

which means APP is (t_1, t_2)-WT for all $t_1 > 1$ and $t_2 > 0$.

□

In this paper we consider the SPT, PT, WT and (t_1, t_2)-WT for all $t_1 < 1$ and $t_2 > 0$ for worst case L_2-approximation in weighted Hilbert spaces $H_{R_{d,\alpha,\gamma}}$ with parameters $1 \ge \gamma_1 \ge \gamma_2 \ge \cdots \ge 0$ and $1 < \alpha_1 \le \alpha_2 \le \cdots$. We get the matching necessary and sufficient conditions on SPT or PT for $R_{d,\alpha,\gamma} \in \{\varphi_{d,\alpha,\gamma}, r_{d,\alpha,\gamma}, \omega_{d,\alpha,\gamma}\}$ and WT for $R_{d,\alpha,\gamma} = r_{d,\alpha,\gamma}$. In particular, it is (t_1, t_2)-WT for all $t_1 > 1$ and $t_2 > 0$.

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