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Numerical Study of the Impacts of Stochastic Forcing on the Vortex in Fluid Flow

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Abstract: This paper provides the numerical investigation of two-dimensional stochastic Navier-Stokes equation driven by additive white noises. The vorticity-stream function method is presented to calculate the incompressible fluid flow. In order to solve the formulation of steam function equation, we propose a Crank-Nicolson Fourier pseudo-spectral method. Numerical results obviously show that the presence of stochastic forcing can have an impact upon the shapes of vortex in fluid flow. To a certain extent, the phenomenon of eddy shedding and the formation of some new vortex in a random medium can be observed, which make it possible for us to conduct optimal control experiments in the development of vortex structures.

Keywords: stochastic navier-stokes equations; vorticity-stream function; fourier pseudo-spectral method; fluid flow

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1. Introduction

Stochastic Navier-Stokes equation (SNSE) has aroused wide concern mainly because of their potential applications in fluid mechanics, which can be quite useful for modelling the random flow motions of Newtonian incompressible fluids and explaining the mechanisms of turbulence phenomenon (see [6,10,11,13] and references therein). Unlike deterministic Navier-Stokes equation, solutions of SNSE are random fields with some statistical characteristics, complex dynamics and interesting properties ([8,10]). Since the seminal work of Bensoussan and Temam [7], the theoretical analysis of SNSE has been intensively investigated in the literatures (see above referred works). However, rather little has been done in numerical technique for SNSE.

In this study, we focus on the numerical investigation of time-dependent incompressible SNSE with the additive spatial white noise as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{Re}\Delta\mathbf{u} + \nabla p = \dot{\mathbf{W}}(\mathbf{x}),\ \mathbf{x}\in\Omega,\ 0<t\leq T, \tag{1.1}$$

with the incompressibility constraint:

$$\text{div}\ \mathbf{u} = 0,\ \mathbf{x}\in\Omega,\ 0<t\leq T, \tag{1.2}$$

subject to the homogeneous Dirichlet boundary conditions:

$$\mathbf{u}(\mathbf{x},t) = 0,\ \mathbf{x}\in\partial\Omega,\ 0<t\leq T, \tag{1.3}$$

and the problem is supplemented by the initial condition:

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ \mathbf{x}\in\Omega, \tag{1.4}$$

in which $\Omega\subset\mathbb{R}^2$ is a bounded domain with regular boundary $\partial\Omega$; $\mathbf{u} = \mathbf{u}(\mathbf{x},t) = (u(\mathbf{x},t), v(\mathbf{x},t))$ is the Eulerian velocity field; Re is the Reynolds number; $p = p(\mathbf{x},t)$ is the associated pressure

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field; $\mathbf{W} = (W_1, W_2)$ stands for the stochastic external forcing, where W_1 and W_2 are independent components, considered as the space-dependent white noise.

Currently, Wiener Chaos expansion for SNSE is considered in [9], this algorithm can only work on relatively short-time intervals. The implicit and semi-discrete Euler method and finite element discretizations are studied in [2,4], the strong convergence is proved in the mean-square sense, but the numerical simulations are not presented in their works. In [5], the author proposes the semi-group and cubature techniques for SNSE, a weak convergence is proved for the method proposed above. Breit et al. [3] studied convergence rates for a finite element based on space-time approximation in terms of convergence in probability. Wu et al. [12] established numerical ergodicity of two dimensional stochastic Navier-Stokes equations. However, as we know, the numerical investigation of impacts of noises on the vortex in fluid flow has not been addressed yet, which is indeed a fascinating problem. This paper aims to develop the vorticity-stream function combined with the Fourier pseudo-spectral method for solving the SNSE. Typically, making use of the stochastic solution to observe the effects of random forcing on the vortex in fluid flow.

The paper is organized as follows: In Section 2, we describe the vorticity-stream function method to deal with the SNSE and propose the Fourier pseudo-spectral method to solve the stream-function equation. The results of numerical experiments on the two test models are offered in Section 3. The conclusions are provided in the final section.

2. Numerical algorithms

In this section, the vorticity-stream function method [7] combined with the Fourier pseudo-spectral method are proposed for the numerical investigation of two-dimensional time-dependent incompressible SNSE. To begin with, we give the derivation of the vorticity-stream function formulation for the incompressible flows. The basic idea of this method is to replace the vorticity in the vorticity-stream function formulation by the Poisson equation for the stream-function. Secondly, a Crank-Nicolson Fourier pseudo-spectral method is used to solve the derived vorticity-stream function equation.

2.1. The vorticity-stream function method

The incompressibility constraint (1.2) entails the existence of a suitable stream-function $\psi(x, y, t)$ such that, for any fixed $t \geq 0$,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{curl}\ \psi = (-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}). \tag{2.1}$$

We introduce the evolution of the vorticity ω of $\mathbf{u}$ as

$$\omega = \mathrm{curl}\ \mathbf{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{2.2}$$

which, substituted into the curl condition (2.1), yields the Poisson equation for the stream function

$$\omega = \mathrm{curl}\ \mathbf{curl}\ \psi = \Delta \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}.$$

To eliminate the pressure, we take the curl to both sides of (1.1):

$$\frac{\partial \omega}{\partial t} - \frac{1}{Re}\Delta \omega + \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} = \mathrm{curl}\ \dot{\mathbf{W}}\,. \tag{2.3}$$

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Therefore, based on (2.1)-(2.3), the momentum equation (1.1) can be reformulated as the vorticity transport equation:

$$\frac{\partial \omega}{\partial t} - \frac{1}{Re}\Delta\omega + \nabla^{\perp}\psi \cdot \nabla\omega = \widetilde{\mathbf{W}}, \tag{2.4}$$

where

$$\widetilde{\mathbf{W}} = \frac{\partial^2 W_2}{\partial x \partial y} - \frac{\partial^2 W_1}{\partial y \partial x}, \ \mathbf{u} = \nabla^{\perp}\psi = (-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}).$$

Here, the space-dependent white noise W_l $(l = 1, 2)$ can be numerically approximated using the Fourier modes as

$$\frac{\partial^2 W_1}{\partial x \partial y} = \sum_{k=1}^{\infty} \sigma_k \eta_{1k}\phi_k(x, y), \ \frac{\partial^2 W_2}{\partial y \partial x} = \sum_{k=1}^{\infty} \sigma_k \eta_{2k}\phi_k(x, y)$$

with the mutual independence random variables $\eta_{1k}, \eta_{2k} \sim \mathcal{N}(0, 1)$, σ_k is a decay coefficient and ϕ_k is Fourier modes truncated with $\phi_k(x, y) = \sqrt{2}\sin(k\pi x)\sin(k\pi y)$.

Noticing that the velocity field $\mathbf{u}$ is divergence-free due to (2.1). Furthermore, the Dirichlet boundary condition (1.3) now read

$$\nabla\psi(x, y, t) = 0, \text{ for } (x, y) \in \partial\Omega, t \geq 0. \tag{2.5}$$

We can simply rewrite the boundary condition (2.5) as

$$\psi(x, y, t) = \frac{\partial \psi}{\partial n}(x, y, t) = 0, \ (x, y) \in \partial\Omega, t \geq 0, \tag{2.6}$$

where $\frac{\partial \psi}{\partial n}$ denotes the outward normal derivative.

Finally, the initial condition (1.4) can be rewritten in terms of ψ as

$$\psi(x, y, 0) = \psi_0(x, y), \ (x, y) \in \Omega. \tag{2.7}$$

2.2. Crank-Nicolson Fourier pseudo-spectral method

Denote by $t_n = n\Delta t, n = 0, 1, \cdots N$ the time mesh point with a uniform time step $\Delta t > 0$, which satisfy $0 \leq t_n \leq T$ and $\Delta t = T/N$. Let ω^n and ψ^n denote as the approximation of ω and ψ at time t^n. Now, we approximate the Eq.(2.4) by the second-order Crank-Nicolson scheme in time

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} + \frac{1}{2}(\nabla^{\perp}\psi \cdot \nabla\omega - \frac{1}{Re}\Delta\omega)^{n+1} = -\frac{1}{2}(\nabla^{\perp}\psi \cdot \nabla\omega - \frac{1}{Re}\Delta\omega)^n + \widetilde{\mathbf{W}}. \tag{2.8}$$

Using the operator notation, the above equations look like the following

$$[1 + \frac{\Delta t}{2}(\nabla^{\perp}\psi^{n+1} \cdot \nabla - \frac{1}{Re}\Delta)]\omega^{n+1} = [1 - \frac{\Delta t}{2}(\nabla^{\perp}\psi^n \cdot \nabla - \frac{1}{Re}\Delta)]\omega^n + \Delta t\widetilde{\mathbf{W}}. \tag{2.9}$$

For the spatial discretization, we use a classical Fourier pseudo-spectral method on a periodic domain Ω. Firstly, the vorticity field is represented as Fourier series

$$\omega(\mathbf{x}, t) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \widehat{\omega}(\mathbf{k}, t)e^{i\mathbf{k}\cdot\mathbf{x}},$$

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where $i^2 = -1$ and the Fourier transform of ω is defined as

$$\widehat{\omega}(\mathbf{k}, t) = \frac{1}{4\pi^2} \int_{\Omega} \omega(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d\Omega,$$

with $\mathbf{k} = (k_x, k_y)$ and $\mathbf{x} = (x, y) \in [0, L_x] \times [0, L_y]$. The uniform Fourier discretization is adapted in space and is truncated at $k_x = -N_x/2$ and $k_x = N_x/2 + 1$, $k_y = -N_y/2$ and $k_y = N_y/2 + 1$, where N_x and N_y are the number of grids in x and y directions, respectively.

Using the Fourier transform of both sides of the Eq.(2.9), we obtain

$$(1 + \frac{\Delta t}{2} \cdot (i\mathbf{k}) \cdot \nabla^{\perp}\psi^{n+1} + \frac{|\mathbf{k}|^2}{Re})\widehat{\omega}^{n+1} = (1 - \frac{\Delta t}{2} \cdot (i\mathbf{k}) \cdot \nabla^{\perp}\psi^{n} - \frac{|\mathbf{k}|^2}{Re})\widehat{\omega}^{n} + \Delta t\widetilde{\mathbf{W}}, \tag{2.10}$$

in which the term $\nabla^{\perp}\psi^m$ ($m = n + 1$ or n) also can be transformed by

$$\nabla^{\perp}\psi^m = \sum_{\mathbf{k}\in\mathbb{Z}^2} (i\mathbf{k}^{\perp})\widehat{\psi}^m(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.11}$$

where $\mathbf{k}^{\perp} = (-k_y, k_x)$. Therefore, the approximation of velocity $\mathbf{u}^n$ can be computed from the vorticity be means of Biot-Savart's law in Fourier formulation

$$\mathbf{u}^n(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^2} \frac{i\mathbf{k}^{\perp}}{|\mathbf{k}|^2} \widehat{\omega}^n(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.12}$$

It should be noted that the Fast Fourier transform (FFT) is used for the transformation between physical and Fourier space.

3. Numerical experiments

In this section, we apply the Fourier pseudo-spectral schemes to solve the two-dimensional SNSE. To help us better understand the effects of the random forcing on the vortex in fluid dynamics, we can first solve the deterministic Navier-Stokes equation (without stochastic forcing), and then calculate the stochastic moments from (2.10) and (2.12) by numerical integrations. Under this framework, the stochastic solution and deterministic solution of the corresponding unforced problem can be compared in our numerical experiments.

3.1. Numerical example 1

In this numerical experiment, the computational domain is taken as $\Omega = [0, 2\pi] \times [0, 2\pi]$. We consider the following initial condition for vorticity $\omega = v_x - u_y$ as

$$\begin{aligned}\omega(x, y, 0) = &\exp((x-\pi)^2 + 5(y - 3\pi/4)^2) - 0.1 \times \operatorname{sech}((x - 3\pi/4)^2 + 5(y - 3\pi/4)^2) \\ &- 0.2 \times \exp((x - 5\pi/4)^2 + 5(y - 5\pi/4)^2),\end{aligned}$$

where the initial velocity field components $u = -\frac{1}{2}\int_0^y \omega(x, s, 0)ds$ and $v = \frac{1}{2}\int_0^x \omega(s, y, 0)ds$.

We choose the time step $\Delta t = 0.001$ and the number of grids $N_x = N_y = 128$. Before solving the flow problem for the stochastic case, we present some random vorticity field with white noise in Figure 1, where the random noise is generated by Monte-Carlo sampling. We give the numerical contour plots for the vorticity of deterministic and stochastic problems at time $t = 0.2, 1.0, 2.0, 3.0$ for the Reynolds number $Re = 100$ in Figure 2. Clearly, the presence of stochastic noise can change the structures of vortex in fluid flow. Furthermore, when the time $t = 3.0$, we can observe that the eddies are shedding from the double vortexes and some new vortexes are formatting in the random medium.

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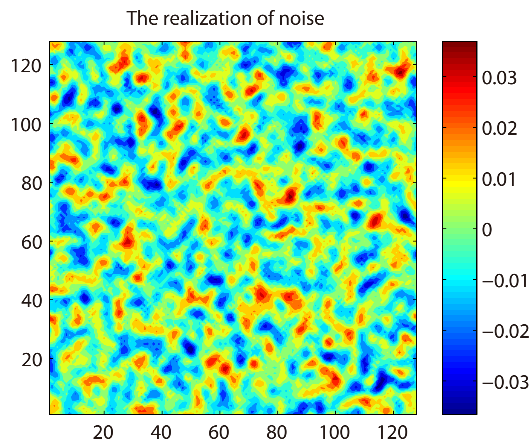

Figure 1. The realization of random vorticity field with white noise.

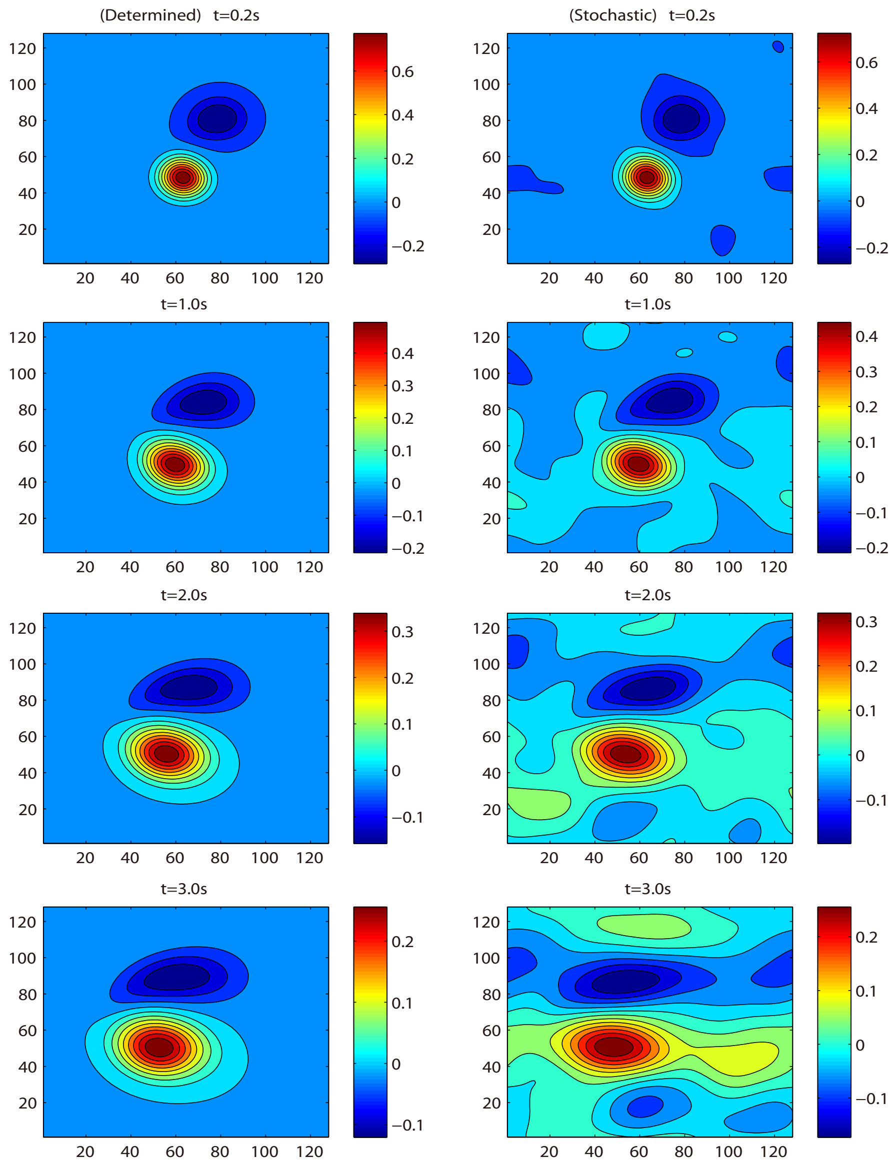

Figure 2. Numerical results for the velocity field of solutions to SNSE (right) and its corresponding deterministic problem (left).

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3.2. Numerical example 2

We consider the solutions of the vorticity ω subject to the following initial condition:

$$\omega(x,y,0) = \exp((x-\pi)^2 + 5(y-3\pi/4)^2) + \exp((x-\pi)^2 + 5(y-5\pi/4)^2)$$
$$- 0.5 \times \exp((x-5\pi/4)^2 + 2.5(y-5\pi/4)^2), (x,y) \in \Omega,$$

where the domain $\Omega = [0, 2\pi] \times [0, 2\pi]$.

The Reynolds number is chosen as $Re = 10$. We also set the time step $\Delta t = 0.001$ and the number of grids $N_x = N_y = 128$. The stochastic and deterministic Navier-Stokes equation are solved at time $t = 0.2, 1.0, 2.0, 3.0$, respectively. In Figure 3, it is clearly exhibited that the patterns of vortexes are changed with the evolution of random forcing. Moreover, at time $t = 3.0$, we can see that the presence of stochastic forcing can accelerate the eddy separation from the warm vortex. The generation of turbulence flow with a random noise forcing can be commonly observed in the fluid flow.

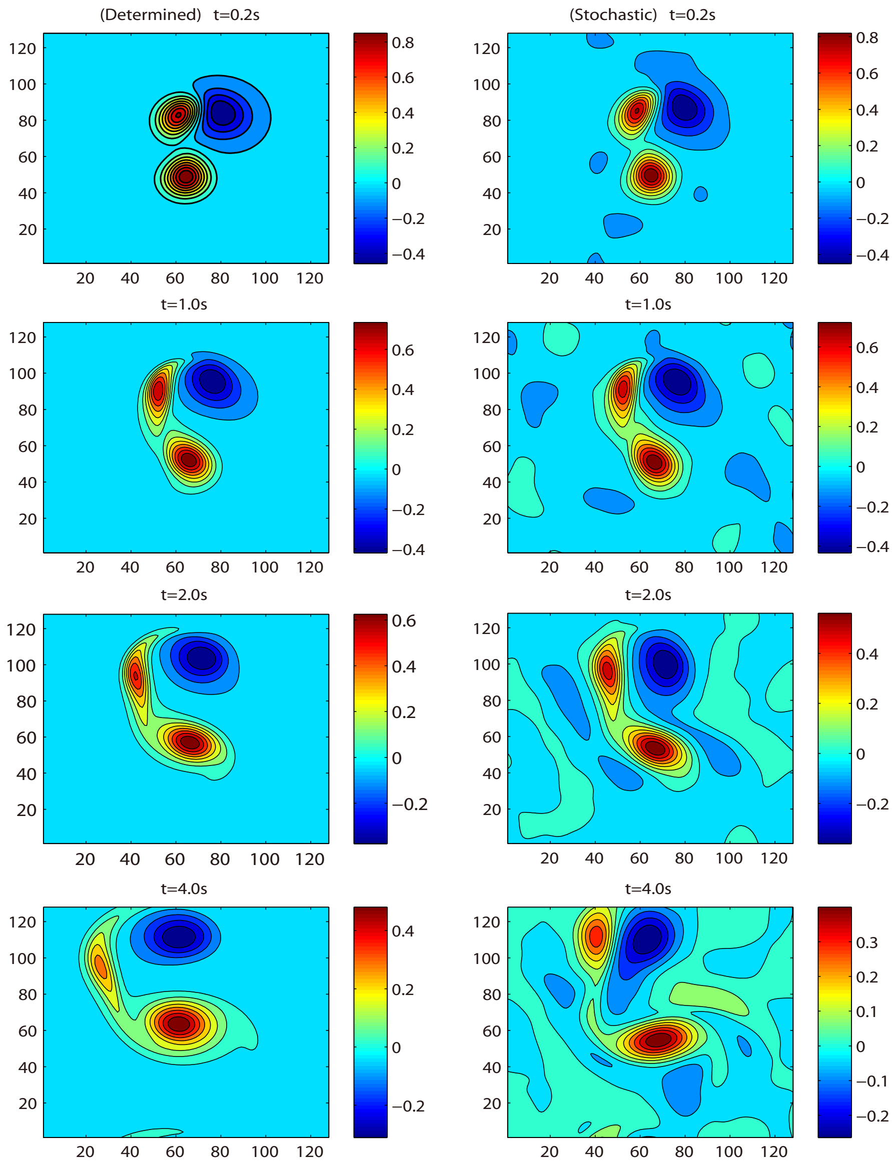

Figure 3. Numerical results for the velocity field of solutions to SNSE (right) and its corresponding deterministic problem (left).

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4. Conclusions

In this paper, we propose the vorticity-stream function method combined with the Fourier pseudo-spectral method to simulate the two-dimensional incompressible SNSE. Numerical results demonstrate that the proposed method can be quite useful for modelling the random flow motions of incompressible fluids. Furthermore, the proposed method in this paper can be better used to study the effects of the random forcing on the vortex in fluid dynamics. However, there is a number of unresolved issues. In particular, further work on this subject is required on:

1. Numerical analysis. This study is concerned with the numerical simulations of the impacts of random forcing on the vorticity in fluid dynamics. It appears that the error analysis and convergence order of the proposed method is beyond the scope of this paper.

2. Random forcing. This paper is limited to the case of the additive random forcing dependent of spatial variables. The sources of randomness might not describe the actual physical behavior in fluid flow. The SNSE driven by multiplicative space-time white noise is being addressed in our ongoing research.

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References

1. A. Bensoussan, R. Temam, Equations stochastiques du type Navier-Stokes, J. Funct. Anal. 13 (2) (1973) 195-222.
2. Z. Brzeźniak, E. Carelli, A. Prohl, Finite element based discretizations of the incompressible Navier-Stokes equations with multiplicative random forcing, IMA J. Numer. Anal. 34 (2014) 502-549.
3. D. Breit, A. Dodgson, Convergence rates for the numerical approximation of the 2D stochastic Navier-Stokes equations, Numer. Math. 147 (2021) 553-578.
4. E. Carelli, A. Prohl, Rates of convergence for discretizations of the stochastic incompressible Navier-Stokes equations, SIAM J. Numer. Anal. 50 (2012) 2467-2496.
5. P. Dörsek, Semigroup splitting and cubature approximations for the stochastic Navier-Stokes equations, SIAM J. Numer. Anal. 50 (2012) 729-746.
6. F. Flandoli, D. Gatarek, Martingale and stationary solutions for the stochastic Navier-Stokes equation, Probab. Theory Related Fields 102 (1995) 367-391.
7. M.M. Gupta, J.C. Kalita, A new paradigm for solving Navier-Stokes equations: streamfunction-velocity formulation, J. Comput. Phys. 207 (2005) 52-68.
8. M. Hairer, J.C. Mattingly, Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, Ann. Math. 164 (2006) 993-1032.
9. T.Y. Hou, W. Luo, B. Rozovskii, H.-M. Zhou, Wiener Chaos expansions and numerical solutions of randomly forced equations of fluid mechanics, J. Comput. Phys. 216 (2006) 687-706.
10. J.C. Mattingly, E. Pardoux, Malliavin calculus and the randomly forced Navier-Stokes equation, Comm. Pure Appl. Math. 59 (2006) 1742-1790.
11. R. Mikulevicius, B. Rozovskii, Stochastic Navier-Stokes equations for turbulent flows, SIAM J. Math. Anal. 35 (2004) 1250-1310.
12. S. Wu, X. Wang, J.H. Huang, Numerical ergodicity of two dimensional stochastic Navier-Stokes equations with Gaussian noise, Comput. Math. Appl., 141 (2023) 15-32.
13. G. Zou, G. Lv, J. Wu, Stochastic Navier-Stokes equations with Caputo derivative driven by fractional noises, J. Math. Anal. Appl. 461 (1) (2018) 595-609.