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An Extended Version of the ${}_{r+1}R_{s,k}(B,C,z)$ Matrix Function

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Abstract: Recently, Shehata et al. [37] introduced the ${}_{r+1}R_s(B,C,z)$ matrix function and established some properties. The aim of this study established to devote and derive certain basic properties including analytic properties, recurrence matrix relations, differential properties, new integral representations, k-Beta transform, Laplace transform, fractional k-Fourier transform, fractional integral properties, the k-Riemann–Liouville and k-Weyl fractional integral and derivative operators an extended version of ${}_{r+1}R_{s,k}$ matrix function. We establish its relationships with other well known special matrix functions which have some particular cases in the context of three parametric Mittag-Leffler matrix function, k-Konhauser and k-Laguerre matrix polynomials. Finally, some special cases of the established formulas are also discussed.

Keywords: ${}_{r+1}R_{s,k}(B,C,z)$ matrix function; k-fractional integral operators; k-fractional derivative operators; Riemann-Liouville k-fractional integral; k-Gamma matrix function; k-Beta matrix function

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1. Introduction

Fractional calculus is the study of applications of derivatives and integrals of non-integer order. It is a generalized form of calculus, so it retains many properties of calculus. It is worth mentioning that, in recent times, theory of fractional calculus has developed quickly and played many important roles in science and engineering, serving as a powerful and very effective tool for various mathematical problems. It has been widely investigated in the last two decades. The hypergeometric function has a long history of mathematical and physical applications. They introduced integral representations of some k-confluent hypergeometric and k-hypergeometric functions. With the help of this new generalised pochhammer symbol. Joshi and Mittal [15], Mubeen and Habibullah [23,24], Rahman et al. [26], Sharma and Jain [33], Zhou et al. [39], introduced an integral representation of k-Gamma and kBeta functions and some generalized k-hypergeometric functions. Mubeen et al. [25] also introduced k-analogue of Kummer's first formula and solution of some integral equations involving confluent k-hypergeometric functions. These studies were extended by Ahmad et al. [1], Ali et al. [2], Diaz and Pariguan [6], Farid et al. [8], Gupta and Bhatt [10], Mittal and Joshi [19], Mittal et al. [20], Romero and Cerutti [28]. Jain et al. [16], Mubeen et al. [21], Rahman et al. [27] introduced and derived some identities of k-Gamma matrix function, k-Beta matrix function, k-hypergeometric matrix functions and k-fractional integrations.

For the past four decades or so, in both mathematics and science, certain special matrix functions are crucial. Jódar and Sastre [11], Jódar and Cortés[12–14] researched the matrix analogues of the gamma, beta, and Gauss hypergeometric functions, which provided the basis for the special matrix functions Bakhet et al.[3], Çekim et al. [4], Gezer and Kaanoglu [9].The extended work of ${}_{r+1}R_s(P,Q,z)$ matrix functions is examined in some detail in Sanjhira and Dave [30], Sanjhira and Dwivedi [31], Sanjhira et al. [32], Shehata [34–36], Shehata et al. [37], Varma et al. [38] for examples of several polynomials that have been introduced and investigated from a matrix perspective. The generalization of the ${}_{r+1}R_s(P,Q,z)$ function presented here was motivated by our investigations [31,37] of the properties of a class of polynomials which characterize is itself an interesting subject, the subsequent

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generalization of the ${}_{r+1}R_{s,k}(P,Q,z)$ matrix function would appear to be of mathematical interest on its own.

1.1. Preliminaries and some definitions

For this purpose, we will introduce the notations, properties, and definitions which we need in further sections. Throughout this paper, for a matrix $\mathbb{A}$ in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(\mathbb{A})$ denotes the set of all eigenvalues of $\mathbb{A}$. The two-norm of an $\mathbb{C}^{N\times N}$ matrix $\mathbb{A}$ will be denoted by $||\mathbb{A}||_2$ and it is defined by (see [11–14])

$$||\mathbb{A}||_2 = \sup_{\zeta\neq 0}\frac{||\mathbb{A}\zeta||_2}{||\zeta||_2},$$

where for a vector ζ in $\mathbb{C}^N$, $||\zeta||_2 = (\zeta^T\zeta)^{\frac{1}{2}}$ is the Euclidean norm of ζ, then the ζ^T vector is the Hermitian transpose of ζ (or, equivalently, the Hermitian transpose of the vector ζ that is viewed as a matrix) .

Let us denote the real numbers $\mathbb{M}(\mathbb{A})$ and $\mathbf{m}(\mathbb{A})$ as in the following

$$\mathbb{M}(\mathbb{A}) = \max\{Re(\zeta) : \zeta \in \sigma(\mathbb{A})\}; \quad \mathbf{m}(\mathbb{A}) = \min\{Re(\zeta) : \zeta \in \sigma(\mathbb{A})\}. \tag{1}$$

If $\mathbf{Y}(\zeta)$ and $\mathbf{\Psi}(\zeta)$ are holomorphic functions of the complex variable ζ, which are defined in an open set Ω of the complex plane, and $\mathbb{A},\mathbb{B}$ are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(\mathbb{A})\subset\Omega$ and $\sigma(\mathbb{B})\subset\Omega$, such that $\mathbb{A}\mathbb{B}=\mathbb{B}\mathbb{A}$, then the properties of the matrix functional calculus in [7], it follows that

$$\mathbf{Y}(\mathbb{A})\mathbf{\Psi}(\mathbb{B}) = \mathbf{\Psi}(\mathbb{B})\mathbf{Y}(\mathbb{A}).$$

Definition 1. *[21] For $k>0$, $\mathbb{A}$ and $\mathbb{B}$ are positive stable matrices in $\mathbb{C}^{N\times N}$, then the κ-Gamma and κ-Beta matrix functions are defined by (see [17,22])*

$$\Gamma_\kappa(\mathbb{A}) = \int_0^\infty t^{\mathbb{A}-I}e^{-\frac{t^\kappa}{\kappa}}dt = \kappa^{\frac{\mathbb{A}}{\kappa}-1}\Gamma(\frac{\mathbb{A}}{\kappa}) \tag{2}$$

and

$$\mathbb{B}_\kappa(\mathbb{A},\mathbb{B}) = \frac{1}{\kappa}\int_0^1 t^{\frac{\mathbb{A}}{\kappa}-I}(1-t)^{\frac{\mathbb{B}}{\kappa}-I}dt = \Gamma_\kappa(\mathbb{A})\Gamma_\kappa(\mathbb{B})\Gamma_\kappa^{-1}(\mathbb{A}+\mathbb{B}), \tag{3}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$.

Furthermore, if $\mathbb{A}$ is a matrix such that (see [21,27])

$$\mathbb{A}+k\ell I \quad \text{is an invertible matrix for all integers } \ell\geq 0, \forall k>0 \tag{4}$$

then the k-pochammer matrix symbol is defined as

$$(\mathbb{A})_{n,\kappa} = \mathbb{A}(\mathbb{A}+\kappa I)\ldots(\mathbb{A}+(n-1)\kappa I) = \Gamma_\kappa(\mathbb{A}+n\kappa I)\Gamma_\kappa^{-1}(\mathbb{A})\,; n\geq 1\,, (\mathbb{A})_{0,k} = I; (\mathbb{A})_{n,1} = (\mathbb{A})_n. \tag{5}$$

Also, they have provided some useful results

$$(\mathbb{A})_{n\ell,\kappa} = \ell^{n\ell}\left(\frac{\mathbb{A}}{\ell}\right)_{n,\kappa}\left(\frac{\mathbb{A}+kI}{\ell}\right)_{n,\kappa}\cdots\left(\frac{\mathbb{A}+k(\ell-1)I}{\ell}\right)_{n,\kappa} \tag{6}$$

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and

$$(1-k\zeta)^{-\frac{\mathbb{A}}{k}}=\sum_{n=0}^{\infty}\frac{(\mathbb{A})_{n,\kappa}}{n!}\zeta^{n}. \tag{7}$$

If $\Phi(\kappa,\ell)$ is matrix in $\mathbb{C}^{N\times N}$ for $\kappa\geq 0$, $\ell\geq 0$, then in an analogous way to the proof of Lemma **11** [12], it follows that

$$\begin{aligned}\sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{\infty}\Phi(\kappa,\ell)&=\sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{[\frac{1}{2}\kappa]}\Phi(\kappa-2\ell,\ell),\\ \sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{\infty}\Phi(\kappa,\ell)&=\sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{\kappa}\Phi(\kappa-\ell,\ell).\end{aligned} \tag{8}$$

Similarly to (8), we can write

$$\begin{aligned}\sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{[\frac{1}{2}\kappa]}\Phi(\kappa,\ell)&=\sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{\infty}\Phi(\kappa+2\ell,\ell),\\ \sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{\kappa}\Phi(\kappa,\ell)&=\sum_{\kappa=0}^{\infty}\sum_{\ell=0}^{\infty}\Phi(\kappa+\ell,\ell).\end{aligned} \tag{9}$$

Definition 2. *The hypergeometric matrix function ${}_2F_{1,\kappa}(\mathbb{A},\mathbb{B};C;\zeta)$ has been given in the form ([21,27])*

$${}_2F_{1,\kappa}(\mathbb{A},\mathbb{B};\mathbb{C};\zeta)={}_2F_1((\mathbb{A},\kappa),(\mathbb{B},\kappa);(C,\kappa);\zeta)=\sum_{\ell=0}^{\infty}\frac{(\mathbb{A})_{\ell,\kappa}(\mathbb{B})_{\ell,\kappa}[(C)_{\ell,\kappa}]^{-1}}{n!}\zeta^{\kappa}, \tag{10}$$

for matrices $\mathbb{A}$, $\mathbb{B}$ and C in $\mathbb{C}^{N\times N}$ such that $C+\ell I$ is an invertible matrix for all integers $\ell\geq 0$ and for $|\zeta|<1$. It has been seen by Jódar and Cortés [12] that the series is absolutely convergent for $|\zeta|=1$ when

$$\mathbf{m}(\mathbb{C})>\mathbb{M}(\mathbb{A})+\mathbb{M}(\mathbb{B}),$$

where $\mathbf{m}(\mathbb{A})$ and $\mathbb{M}(\mathbb{A})$ in (1) for any matrix $\mathbb{A}$ in $\mathbb{C}^{N\times N}$.

Definition 3. *For m and n are finite positive integers, the ${}_mR_n$ matrix function defined as (see [31,37])*

$$\begin{aligned}&{}_mR_n(A_1,A_2,\ldots,A_m;Q_1,Q_2,\ldots,Q_n;P,Q;\zeta)\\ &=\sum_{k=0}^{\infty}\frac{\zeta^k}{k!}(A_1)_k(A_2)_k\ldots(A_m)_k[(Q_1)_k]^{-1}[(Q_2)_k]^{-1}\ldots[(Q_n)_k]^{-1}\Gamma^{-1}(\ell P+Q)\\ &=\sum_{k=0}^{\infty}\frac{\zeta^k}{k!}\prod_{i=1}^{m}(A_i)_k\left[\prod_{j=1}^{n}(Q_i)_k\right]^{-1}\Gamma^{-1}(\ell P+Q),\end{aligned} \tag{11}$$

where A_i ; $1\leq i\leq m$ and Q_j; $1\leq j\leq n$ are matrices in $\mathbb{C}^{N\times N}$ such that

$$Q_j+kI \quad \text{are invertible matrices for all integers } k\geq 0. \tag{12}$$

Definition 4. *The k-Riemann Liouville fractional k-integral and derivative operators of order μ defined as follows (Mubeen and Habibullah see [23,24])*

$$I^{\mu}_{a,k}\mathbf{\Psi}(x)=\frac{1}{k\Gamma_k(\mu)}\int_a^x(x-t)^{\frac{\mu}{k}-1}\mathbf{\Psi}(t)dt,\mu\in R^{+} \tag{13}$$

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and

$$\left(\mathbb{D}^{\mu}_{a^+,k}\mathbf{\Psi}\right)(x) = \left(\frac{d}{dx}\right)^n \left(\mathbb{I}^{n-\mu}_{a^+,k}\mathbf{\Psi}\right)(x), \tag{14}$$

where $Re(\mu) > 0$ and k be any positive real number.

Definition 5. *For $\alpha \in C$, $Re(\alpha) > 0$ and $k \in R^+$, f belonging to $S(R)$, the Weyl fractional k-integral operator and k-Weyl fractional derivative are defined as (see [28,29])*

$$\left[\mathbb{W}^{\alpha}_{k}\mathbf{\Psi}\right](x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^{\infty}(t-x)^{\frac{\alpha}{k}-1}\mathbf{\Psi}(t)dt \tag{15}$$

and

$$\left[\mathbb{W}^{-\alpha}_{k}\mathbf{\Psi}\right](x) = -\frac{d}{dx}\left[\mathbb{W}^{1-\alpha}_{k}\mathbf{\Psi}\right](x). \tag{16}$$

Definition 6. *The $\mathfrak{B}[f(t);A,B]$ k-Beta transform, $\mathfrak{L}[f(t);\mathbf{s}]$ Laplace transform and $\mathfrak{F}[f(t)]$ Fractional k-Fourier transform of $f(t)$ are defined by*

$$\mathfrak{B}[\mathbf{\Psi}(t);A,B] = \frac{1}{k}\int_0^1 t^{\frac{A}{k}-I}(1-t)^{\frac{B}{k}-I}\mathbf{\Psi}(t)dt. \tag{17}$$

$$\mathfrak{L}[\mathbf{\Psi}(t);\mathbf{s}] = \int_0^{\infty} e^{-\mathbf{s}t}\mathbf{\Psi}(t)dt = F(\mathbf{s}), \mathbf{s} \in \mathcal{C}. \tag{18}$$

and

$$\mathfrak{F}[\mathbf{\Psi}(t)] = \int_{-\infty}^{0} e^{iw^{\frac{1}{\alpha}}z}\mathbf{\Psi}(z)dz. \tag{19}$$

The motive of the current study is to investigate the analytical and fractional integral and derivative properties of ${}_rR_{s,k}$ matrix function, as well as to emphasize the importance of their applications in diverse research areas. This function is an amalgamation of generalized Mittag–Leffler function and generalized hypergeometric function which plays an important role in the theory of mathematical analysis, fractional calculus and statistics and has significant applications in the field of free electron laser equations and fractional kinetic equations. For literature survey of fractional integral operators, researchers can refer to the papers of Farid et al. [8], Jain et al. [16], Kilbas et al. [18]. In the present sequel to the aforementioned and many other recent investigations. In Section 2, some properties, recurrence matrix relations, differential properties, new integral representations, k-Beta transform, Laplace Transform, Fractional k-Fourier transform, fractional integral properties, the k-Riemann–Liouville and k-Weyl fractional integral and derivative operators of the extended ${}_rR_{s,k}$ matrix functions are established and discussed. In section 3, we derive some properties of ${}_rR_s$ matrix functions. Finally, we give a definition of the novel generalized type ${}_rR_{s,k}$ matrix functions.

2. Main Results

For our aim section, we establish and discuss the properties convergence, recurrence matrix relations, k-integrals representations and differential properties, k-fractional operators, k-Beta, Laplace, Fractional k-Fourier transforms, fractional integral and derivative operators for the ${}_{r+1}R_{s,k}$ matrix function.

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Definition 7. *Let A, B, C, P_i, $1 \leq i \leq r$ and Q_j; $1 \leq j \leq s$ be matrices in $\mathbb{C}^{N\times N}$, $Re(B) > 0$, $Re(C) > 0$, $Re(P_i) > 0$, $Re(Q_j) > 0$, $k \in R^+$, then we define the ${}_{r+1}R_{s,k}(B, C, z)$ matrix function as*

$$\begin{aligned} &{}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; z) \\ &= \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!}(A)_{\ell,k} \prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1} \Gamma_k^{-1}(\ell B + C), \end{aligned} \tag{20}$$

for r and s are finite positive integers, such that

$$Q_j + k\ell I \quad \text{are invertible matrices for all integers } \ell \geq 0, \tag{21}$$

Now, we investigate the convergence of the following series, one gets

$$\begin{aligned} \frac{1}{R} &= \limsup_{\ell\to\infty} (\|U_\ell\|)^{\frac{1}{\ell}} = \limsup_{\ell\to\infty} \left(\left\| \frac{(A)_{\ell,k} \prod_{i=1}^{r}(P_i)_{\ell,k}[\prod_{j=1}^{s,k}(Q_j)_{\ell,k}]^{-1}\Gamma_k^{-1}(\ell B + C)}{\ell!} \right\| \right)^{\frac{1}{\ell}} \\ &= \limsup_{\ell\to\infty} \left\| \frac{\Gamma_k(A + k\ell I)\Gamma_k^{-1}(A) \prod_{i=1}^{r} \Gamma_k(P_i + k\ell I)\Gamma_k^{-1}(P_i) \prod_{j=1}^{s} \Gamma_k^{-1}(Q_j + k\ell I)\Gamma_k(Q_j)\Gamma_k^{-1}(\ell B + C)}{\ell!} \right\|^{\frac{1}{\ell}} \\ &= \limsup_{\ell\to\infty} \left\| \sqrt{2\pi} e^{-(A+k\ell I)}(A + k\ell I)^{A+k\ell I-\frac{1}{2}I} \left(\sqrt{2\pi} e^{-A}(A)^{A-\frac{1}{2}I} \right)^{-1} \right. \\ &\times \prod_{i=1}^{r} \sqrt{2\pi} e^{-(P_i+k\ell I)}(P_i + \ell I)^{P_i+k\ell I-\frac{1}{2}I} \left(\sqrt{2\pi} e^{-P_i}(P_i)^{P_i-\frac{1}{2}I} \right)^{-1} \\ &\times \prod_{j=1}^{s} \frac{1}{\sqrt{2\pi}} e^{(Q_j+k\ell I)}(Q_j + k\ell I)^{-Q_j-k\ell I+\frac{1}{2}I} \left(\frac{1}{\sqrt{2\pi}} e^{(Q_j)}(Q_j)^{-Q_j+\frac{1}{2}I} \right) \\ &\times \left. \frac{1}{\sqrt{2\pi}} e^{(\ell B+C)}(\ell B + C)^{-\ell B-C+\frac{1}{2}I} \frac{1}{\sqrt{2\pi} e^{-\ell-1}\ell^{\ell+\frac{1}{2}}} \right\|^{\frac{1}{\ell}} \\ &\approx \limsup_{\ell\to\infty} \left\| e^{-(A+k\ell I)}(A + k\ell I)^{A+k\ell I-\frac{1}{2}I} \prod_{i=1}^{r} e^{-(P_i+k\ell I)}(P_i + k\ell I)^{P_i+k\ell I-\frac{1}{2}I} \right. \\ &\times \left. \prod_{j=1}^{s} e^{(Q_j+k\ell I)}(Q_j + k\ell I)^{-Q_j-k\ell I+\frac{1}{2}I} e^{(\ell B+C)}(\ell B + C)^{-\ell B-C+\frac{1}{2}I} \frac{1}{e^{-\ell-1}\ell^{\ell+\frac{1}{2}}} \right\|^{\frac{1}{\ell}} \end{aligned}$$

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$$
\begin{aligned}
&\approx \limsup_{\ell\to\infty}\left\|(A+k\ell I)^{A+k\ell I-\frac{1}{2}I}\prod_{i=1}^{r}\prod_{j=1}^{s}e^{-(A+k\ell I)-(P_i+k\ell I)+Q_j+k\ell I+\ell B+C+\ell+1}(P_i+k\ell I)^{P_i+k\ell I-\frac{1}{2}I}\right.\\
&\left.\times(Q_j+k\ell I)^{-Q_j-k\ell I+\frac{1}{2}I}(\ell B+C)^{-\ell B-C+\frac{1}{2}I}\ell^{-\ell-\frac{1}{2}}\right\|^{\frac{1}{\ell}}\\
&\approx \limsup_{\ell\to\infty}\left\|(A+k\ell I)^{A+k\ell I-\frac{1}{2}I}\prod_{i=1}^{r}\prod_{j=1}^{s}e^{Q_j+C+\ell B-k\ell I+\ell I+I-A-P_i}(P_i+k\ell I)^{P_i+k\ell I-\frac{1}{2}I}\right.\\
&\left.\times(Q_j+k\ell I)^{-Q_j-k\ell I+\frac{1}{2}I}(\ell B+C)^{-\ell B-C+\frac{1}{2}I}\ell^{-\ell-\frac{1}{2}}\right\|^{\frac{1}{\ell}}\\
&\approx \limsup_{\ell\to\infty}\left\|(A+k\ell I)^{A+k\ell I-\frac{1}{2}I}\prod_{i=1}^{r}\prod_{j=1}^{s}e^{(B+(1-k)I)\ell}(P_i+k\ell I)^{P_i+k\ell I-\frac{1}{2}I}\right.\\
&\left.\times(Q_j+k\ell I)^{-Q_j-k\ell I+\frac{1}{2}I}(\ell B+C)^{-\ell B-C+\frac{1}{2}I}\ell^{-\ell-\frac{1}{2}}\right\|^{\frac{1}{\ell}}\\
&\approx \|e^{B+(1-k)I}\|\limsup_{\ell\to\infty}\left\|\prod_{i=1}^{r}\prod_{j=1}^{s}\frac{(A+k\ell I)(P_i+k\ell I)(Q_j+k\ell I)^{-1}(\ell B+C)^{-B}}{\ell}\right\|^{k}\\
&\times\left\|(A+k\ell I)^{A-\frac{1}{2}I}(P_i+k\ell I)^{P_i-\frac{1}{2}I}(Q_j+k\ell I)^{-Q_j+\frac{1}{2}I}(\ell B+C)^{-C+\frac{1}{2}I}\ell^{-\frac{1}{2}}\right\|^{\frac{1}{\ell}}.
\end{aligned}
$$

The above limit shows that the following:

1. If $r>s+1$, then the series in (20) diverges for $z\neq 0$.
2. If $r\leq s$, then the series in (20) converges for all finite z.
3. If $r=s+1$, then the series in (20) converges for all $|z|<\frac{1}{k}$ and diverges for all $|z|>\frac{1}{k}$.
4. If $r=s+1$, then the series in (20) is absolutely convergent on the circle $|z|=\frac{1}{k}$ when

$$\sum_{j=1}^{s}\mathbf{m}(Q_j)>\sum_{i=1}^{r}\mathbb{M}(P_i)+\mathbb{M}(A).$$

5. If $r=s+1$, then the series (20) is diverges for $|z|=\frac{1}{k}$ when

$$\sum_{j=0}^{s}\mathbf{m}(Q_j)\leq\sum_{i=0}^{r}\mathbb{M}(P_i)+\mathbb{M}(A)-k,$$

6. If $r=s+1$, then the series (20) is conditionally convergent for $|z|=\frac{1}{k}$ when

$$\sum_{i=0}^{r}\mathbb{M}(P_i)+\mathbb{M}(A)-k<\sum_{j=0}^{s}m(Q_j)\leq\sum_{i=0}^{r}\mathbb{M}(P_i)+\mathbb{M}(A).$$

where $\mathbb{M}(P_i); 1\leq i\leq r$ and $\mathbf{m}(Q_j); 1\leq j\leq s$ are defined in (1).

Theorem 1. *The following relations for ${}_{r+1}R_{s,k}$ hold true*

$$
\begin{aligned}
&(A-P_i)\,{}_{r+1}R_{s,k}=A\,{}_{r+1}R_{s,k}(A+kI)-P_i\,{}_{r+1}R_{s,k}(P_i+kI), i=1,2,3,\ldots,r,\\
&(P_v-P_i)\,{}_{r+1}R_{s,k}=P_v\,{}_{r+1}R_{s,k}(P_v+kI)-P_i\,{}_{r+1}R_{s,k}(P_i+kI)\,; i\neq v; i,v=1,2,3,\ldots,r,
\end{aligned}
\tag{22}
$$

$$(Q_v-Q_j)\,{}_{r+1}R_{s,k}=(Q_v-kI)\,{}_{r+1}R_{s,k}(Q_v-kI)-(Q_j-kI)\,{}_{r+1}R_{s,k}(Q_j-kI)\,; v\neq j; v,j=1,2,\ldots,s \tag{23}$$

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and

$$(A - Q_j + kI)\ {}_{r+1}R_{s,k} = A\ {}_{r+1}R_{s,k}(A + kI) - (Q_j - kI)\ {}_{r+1}R_{s,k}(Q_j - kI)\,; j = 1, 2, \ldots, s,$$
$$(P_i - Q_j + kI)\ {}_{r+1}R_{s,k} = P_i\ {}_{r+1}R_{s,k}(P_i + kI) - (Q_j - kI)\ {}_{r+1}R_{s,k}(Q_j - kI)\,; i = 1, 2, 3, \ldots, r,\ j = 1, 2, \ldots, s. \tag{24}$$

Proof. By using the following property

$$A(A + kI)_{\ell,k} = (A + k\ell I)(A)_{\ell,k},$$

we get the matrix contiguous function relation

$$\begin{aligned} A\ {}_{r+1}R_{s,k}(A + kI) &= A \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell\,!}(A + kI)_{\ell,k} \prod_{i=1}^{r}(P_i)_{\ell,k} \left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1} \Gamma_k^{-1}(\ell B + C) \\ &= \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!}(A + k\ell I)(A)_{\ell,k} \prod_{i=1}^{r}(P_i)_{\ell,k} \left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1} \Gamma_k^{-1}(\ell B + C) \\ &= \sum_{\ell=0}^{\infty}(A + k\ell I)\mathbf{\Psi}_{\ell,k}(z). \end{aligned} \tag{25}$$

where $\mathbf{\Psi}_{\ell,k}(z) = \frac{z^\ell}{\ell!}(A)_{\ell,k} \prod_{i=1}^{r}(P_i)_{\ell,k} \left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1} \Gamma_k^{-1}(\ell B + C)$

Similarly, we get

$$\begin{aligned} {}_{r+1}R_{s,k}(A - kI) &= (A - kI) \sum_{\ell=0}^{\infty} \Big(A + k(\ell - 1)I\Big)^{-1} \mathbf{\Psi}_{\ell,k}(z), \\ P_i\ {}_{r+1}R_{s,k}(P_i + kI) &= \sum_{\ell=0}^{\infty}(P_i + k\ell I)\mathbf{\Psi}_{\ell,k}(z), \\ {}_{r+1}R_{s,k}(P_i - kI) &= (P_i - kI) \sum_{\ell=0}^{\infty} \Big(P_i + k(\ell - 1)I\Big)^{-1} W_{\ell,k}(z)\mathbf{\Psi}_{\ell,k}(z), \\ {}_{r+1}R_{s,k}(Q_j + kI) &= Q_j \sum_{\ell=0}^{\infty} \Big(Q_j + k\ell I\Big)^{-1} \mathbf{\Psi}_{\ell,k}(z), \\ \Big(Q_j - kI\Big)\ {}_{r+1}R_{s,k}(Q_j - kI) &= \sum_{\ell=0}^{\infty}(Q_j + k(\ell - 1)I)\mathbf{\Psi}_{\ell,k}(z). \end{aligned} \tag{26}$$

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For all integers $n \geq 1$, we have

$$
\begin{aligned}
{}_{r+1}R_{s,k}(A+nkI) &= \sum_{\ell=0}^{\infty}\prod_{\mu=1}^{n}\Big(A+k(\mu-1)I\Big)^{-1}(A+k(\ell+\mu-1)I)\mathbf{\Psi}_{\ell,k}(z),\\
{}_{r+1}R_{s,k}(A-nkI) &= \sum_{\ell=0}^{\infty}\prod_{\mu=1}^{n}(A-k\mu I)\Big(A+k(\ell-\mu)I\Big)^{-1}\mathbf{\Psi}_{\ell,k}(z),\\
{}_{r+1}R_{s,k}(P_i+nkI) &= \sum_{\ell=0}^{\infty}\prod_{\mu=1}^{n}\Big(P_i+k(\mu-1)I\Big)^{-1}(P_i+k(\ell+\mu-1)I)\mathbf{\Psi}_{\ell,k}(z),\\
{}_{r+1}R_{s,k}(P_i-nkI) &= \sum_{\ell=0}^{\infty}\prod_{\mu=1}^{n}(P_i-k\mu I)\Big(P_i+k(\ell-\mu)I\Big)^{-1}\mathbf{\Psi}_{\ell,k}(z),\\
{}_{r+1}R_{s,k}(Q_j+nkI) &= \sum_{\ell=0}^{\infty}\prod_{\mu=1}^{n}(Q_j+k(\mu-1)I)\Big(Q_j+k(\ell+\mu-1)I\Big)^{-1}\mathbf{\Psi}_{\ell,k}(z),\\
{}_{r+1}R_{s,k}(Q_j-nkI) &= \sum_{\ell=0}^{\infty}\prod_{\mu=1}^{n}\Big(Q_j-k\mu I\Big)^{-1}(Q_j+k(\ell-\mu)I)\mathbf{\Psi}_{\ell,k}(z).
\end{aligned}
\tag{27}
$$

Using the differential operator $\theta = z\frac{d}{dz}$, we get

$$
\begin{aligned}
(k\theta I + A)\ {}_{r+1}R_{s,k} &= \sum_{\ell=0}^{\infty}(A+k\ell I)\mathbf{\Psi}_{\ell,k}(z),\\
(k\theta I + P_i)\ {}_{r+1}R_{s,k} &= \sum_{\ell=0}^{\infty}(P_i+k\ell I)\mathbf{\Psi}_{\ell,k}(z).
\end{aligned}
\tag{28}
$$

From (25) and (28), we have

$$
\begin{aligned}
(k\theta I + A)\ {}_{r+1}R_{s,k} &= A\ {}_{r+1}R_{s,k}(A+kI),\\
(k\theta I + P_i)\ {}_{r+1}R_{s,k} &= P_i\ {}_{r+1}R_{s,k}(P_i+kI)\,;\ i=1,2,\ldots,r.
\end{aligned}
\tag{29}
$$

Similarly, we get

$$
(k\theta I + Q_j - kI)\ {}_{r+1}R_{s,k} = (Q_j-kI)\ {}_{r+1}R_{s,k}(Q_j-kI)\,;\ j=1,2,\ldots,s. \tag{30}
$$

The elimination of $\theta\ {}_{r+1}R_{s,k}$ from (29) and (30), we obtain the recurrence matrix relations (22), (23) and (24) □

Theorem 2. *The following differential formulas hold true for* ${}_{r+1}R_{s,k}$

$$
{}_{r+1}R_{s,k} = C\ {}_{r+1}R_{s,k}(C+kI) + zB\frac{d}{dz}\ {}_{r+1}R_{s,k}(C+kI) \tag{31}
$$

and

$$
\begin{aligned}
D_z^{\mu}\ {}_{r+1}R_{s,k} =&(A)_{\mu,k}\Pi_{i=1}^{r}(P_i)_{\mu,k}\Big[\Pi_{j=1}^{s}(Q_j)_{\mu,k}\Big]^{-1}\ {}_{r+1}R_{s,k}(A+\mu kI, P_1+\mu kI,\ldots,P_r+\mu kI;\\
&Q_1+\mu kI,\ldots,Q_s+\mu kI; B, \mu B + C, z).
\end{aligned}
\tag{32}
$$

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Proof. The right hand side in (31) and using (20), we get

$$\begin{aligned}
&C\ {}_{r+1}R_{s,k}(C+kI) + zB\frac{d}{dz}\ {}_{r+1}R_{s,k}(C+kI) \\
&= C\ {}_{r+1}R_{s,k}(C+kI) + zB\left[\sum_{\ell=0}^{\infty}\frac{\ell z^{\ell-1}}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C+kI)\right] \\
&= C\ {}_{r+1}R_{s,k}(C+kI) + \sum_{\ell=0}^{\infty}\frac{z^{\ell}}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s,k}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C) \\
&- C\sum_{\ell=0}^{\infty}\frac{z^{\ell}}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C+kI)\right] = {}_{r+1}R_{s,k}.
\end{aligned}$$

Differentiating (20) with respect to z, we get

$$\begin{aligned}
D_z\ {}_{r+1}R_{s,k} =& A\Pi_{i=1}^{r}P_i\left(\Pi_{j=1}^{s}Q_j\right)^{-1}\ {}_{r+1}R_{s,k}(A+kI, P_1+kI, \\
&\ldots, P_r+kI; Q_1+kI, \ldots, Q_s+kI; B, B+C, z)\,;\ D_z = \frac{d}{dz}.
\end{aligned}$$

By repeating the above process that μ times, we get (32). □

Remark 1. 1. *If $k=1$, $A=B=C=I$ in (22)-(24), we get the results for the generalized hypergeometric matrix functions [34].*
2. *If $k=1$ in (31)-(32), we get the contiguous function relations for the ${}_{r+1}R_s$ matrix function [31,37].*

Theorem 3. *The ${}_{r+1}R_{s,k}$ matrix function has the following differential properties:*

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^{\mu}\left[z^{\frac{C}{k}-I}\ {}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; z^{\frac{B}{k}})\right] \\
&= \frac{1}{k^{\mu}}z^{\frac{C}{k}-(\mu+1)I}\ {}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C-\mu I; z^{\frac{B}{k}}),
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^{\mu}\left[z^{\frac{A}{k}+\mu I-I}\ {}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; z)\right] \\
&= \frac{1}{k^{\mu}}(A)_{\mu,k}z^{\frac{A}{k}-I}\ {}_{r+1}R_{s,k}(A+\mu kI, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; z),
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^{\mu}\left[z^{\frac{P_i}{k}+\mu I-I}\ {}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; z)\right] \\
&= \frac{1}{k^{\mu}}(P_i)_{\mu,k}z^{\frac{P_i}{k}-I}\ {}_{r+1}R_{s,k}(P_i+\mu kI), i=1,2,\ldots r
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^{\mu}\left[z^{\frac{Q_j}{k}}\ {}_{r+1}R_{s,k}(Q_j+kI; z)\right] \\
&= \frac{1}{k^{\mu}}\Gamma_k(Q_j)\Gamma_k^{-1}(Q_j-\mu kI)z^{\frac{Q_j}{k}-\mu I}\ {}_{r+1}R_{s,k}(Q_j-(\mu-1)kI; z), 1\le j\le s.
\end{aligned} \tag{36}$$

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Proof. To prove (33). Multiplying the equation (20) by $z^{\frac{C}{k}-I}$ and differentiating with respect to z, we get

$$\left(\frac{d}{dz}\right)\left[z^{\frac{C}{k}-I}\ {}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;z^{\frac{B}{k}})\right]$$
$$=\frac{1}{k}z^{\frac{C}{k}-2I}\ {}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C-I;z^{\frac{B}{k}}).$$

By repeating the above relation μ times, we get (33). On the same parallel lines, we get the results (34)-(36). So the detailed account of proof is omitted. □

Theorem 4. *The ${}_{r+1}R_{s,k}$ matrix functions satisfies the differential equation*

$$\theta\prod_{j=1}^{s}(k\theta\ I+Q_j-kI)\ {}_{r+1}R_{s,k}-z(k\theta I+A)\prod_{i=1}^{r}(k\theta\ I+P_i)\ {}_{r+1}R_{s,k}(B+C)=\mathbf{0},\tag{37}$$

where $\mathbf{0}$ is the null matrix in $\mathbb{C}^{N\times N}$.

Proof. Using the Euler differential operator $\theta=z\frac{d}{dz}$, we get

$$\theta\prod_{j=1}^{s}(k\theta\ I+Q_j-kI)\ {}_{r+1}R_{s,k}=\sum_{\ell=1}^{\infty}\frac{\ell\, z^{\ell}}{\ell!}\prod_{j=1}^{s}(k\ell I+Q_j-kI)(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left(\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)$$
$$=\sum_{\ell=1}^{\infty}\frac{z^{\ell}}{(\ell-1)!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left(\prod_{j=1}^{s}(Q_j)_{\ell-1,k}\right)^{-1}\Gamma_k^{-1}(\ell B+C).$$

Replace ℓ by $\ell+1$, we have

$$\theta\prod_{j=1}^{s}(k\theta\ I+Q_j-kI)\ {}_{r+1}R_{s,k}=\sum_{\ell=0}^{\infty}\frac{z^{\ell+1}}{\ell!}(A)_{\ell+1,k}\prod_{i=1}^{r}(P_i)_{\ell+1,k}\left(\prod_{j=1}^{s}(Q_j)_{\ell,k}\right)^{-1}\Gamma_k^{-1}(\ell B+B+C)$$
$$=z(k\theta I+A)\prod_{i=1}^{r}(k\theta\ I+P_i)\ {}_{r+1}R_{s,k}(B+C).$$

□

Theorem 5. *The following integral representations for ${}_{r+1}R_{s,k}$ matrix function hold true:*

$$\begin{aligned}&{}_{r+1}R_{s,k}\left(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,z\right)\\&=\frac{1}{k}\Gamma_k\left(Q_j\right)\Gamma_k^{-1}\left(A\right)\Gamma_k^{-1}\left(Q_j-A\right)\int_0^1\xi^{\frac{1}{k}A-I}\left(1-\xi\right)^{\frac{1}{k}(Q_j-A)-I}\\&\times\ {}_{r}R_{s-1,k}\left(\begin{matrix}P_1,\ldots,P_{i-1},P_i,P_{i+1}\ldots,P_r;\\Q_1,\ldots,Q_{j-1},Q_{j+1}\ldots,Q_s\end{matrix};B,C,z\xi\right)d\xi,\end{aligned}\tag{38}$$

$$\begin{aligned}&{}_{r+1}R_{s,k}\left(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,z\right)\\&=\frac{1}{k}\Gamma_k\left(Q_j\right)\Gamma_k^{-1}\left(P_i\right)\Gamma_k^{-1}\left(Q_j-P_i\right)\int_0^1\xi^{\frac{1}{k}P_i-I}\left(1-\xi\right)^{\frac{1}{k}(Q_j-P_i)-I}\\&\times\ {}_{r}R_{s-1,k}\left(\begin{matrix}A,P_1,\ldots,P_{i-1},P_{i+1}\ldots,P_r;\\Q_1,\ldots,Q_{j-1},Q_{j+1}\ldots,Q_s\end{matrix};B,C,z\xi\right)d\xi,\end{aligned}\tag{39}$$

$$
\begin{aligned}
&{}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;P,Q,z)=\Gamma_k^{-1}(A)\\
&\int_0^\infty \xi^{A-I}e^{-\frac{\xi^k}{k}}\,{}_rR_{s,k}(P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,z\xi^k)d\xi
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
&{}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;P,Q,z)=\Gamma_k^{-1}(P_i)\\
&\int_0^\infty \xi^{P_i-I}e^{-\frac{\xi^k}{k}}\,{}_rR_{s,k}(A,P_1,\ldots,P_{i-1},P_{i+1}\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,z\xi^k)d\xi.
\end{aligned}
\tag{41}
$$

Proof. By using (3) and (5), we get

$$
\begin{aligned}
&(A)_{\ell,k}\Big[(Q_j)_{\ell,k}\Big]^{-1}=\Gamma_k(Q_j)\Gamma_k(A+k\ell)\Gamma_k^{-1}(A)\Gamma_k^{-1}(Q_j+k\ell)\\
&=\Gamma_k(Q_j)\Gamma_k^{-1}(A)\Gamma_k^{-1}(Q_j-A)\mathbb{B}_k(A+k\ell I,Q_j-A)\\
&=\frac{1}{k}\Gamma_k(Q_j)\Gamma_k^{-1}(A)\Gamma_k^{-1}(Q_j-A)\int_0^1 \xi^{\frac{1}{k}A+(\ell-1)I}\,(1-\xi)^{\frac{1}{k}(Q_j-A)-I}\,d\xi.
\end{aligned}
\tag{42}
$$

Using the above equation (42) and (20), we obtain (38). In a similar way, we obtain to the desired results (39)-(41). □

Theorem 6. *The following integrals representations hold true:*

$$
\begin{aligned}
&\int_0^x \xi^{\frac{Q_j}{k}-I}(x-\xi)^{\frac{E}{k}-I}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,z\xi)d\xi\\
&=k\mathbb{B}_k(Q_j,E)x^{\frac{Q_j+E}{k}-I}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_j+E;B,C;zx),1\le j\le s
\end{aligned}
\tag{43}
$$

and

$$
\begin{aligned}
&\int_a^x (x-\xi)^{\frac{E}{k}-I}(\xi-a)^{\frac{Q_j}{k}-I}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_j;B,C,a-\xi)d\xi\\
&=k\mathbb{B}_k(E,Q_j)(x-a)^{\frac{Q_j+E}{k}-I}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_j+E;B,C;a-x),1\le j\le s.
\end{aligned}
\tag{44}
$$

Proof. Taking left hand side of (43), we get

$$
\begin{aligned}
&\int_0^x \xi^{\frac{Q_j}{k}-I}(x-\xi)^{\frac{E}{k}-I}\sum_{\ell=0}^\infty \frac{(z\xi)^\ell}{\ell!}(A)_{\ell,k}\prod_{i=1}^r(P_i)_{\ell,k}\left[\prod_{j=2}^s(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)d\xi\\
&=\sum_{\ell=0}^\infty\int_0^x \xi^{\frac{Q_j}{k}+(\ell-1)}(x-\xi)^{\frac{E}{k}-I}\frac{z^\ell}{\ell!}(A)_{\ell,k}\prod_{i=1}^r(P_i)_{\ell,k}\left[\prod_{j=2}^s(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)d\xi.
\end{aligned}
$$

Changing order of summation and integration with put $\xi=xu$ and $d\xi=xdu$, we get

$$
\begin{aligned}
&\sum_{\ell=0}^\infty x^{\frac{Q_j+E}{k}+(\ell-1)I}\int_0^1 u^{\frac{Q_j}{k}+\ell-I}(1-u)^{\frac{E}{k}-I}\frac{z^\ell}{\ell!}(A)_{\ell,k}\prod_{i=1}^r(P_i)_{\ell,k}\left[\prod_{j=2}^s(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)du\\
&=\sum_{\ell=0}^\infty x^{\frac{Q_j+E}{k}+(\ell-1)I}\frac{z^\ell}{\ell!}(A)_{\ell,k}\prod_{i=1}^r(P_i)_{\ell,k}\left[\prod_{j=2}^s(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)k\mathbb{B}_k(Q_j+\ell kI,E)\\
&=k\sum_{\ell=0}^\infty x^{\frac{Q_j+E}{k}+(\ell-1)I}\frac{z^\ell}{\ell!}(A)_{\ell,k}\prod_{i=1}^r(P_i)_{\ell,k}\left[\prod_{j=2}^s(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)\Gamma_k(Q_j+\ell kI)\Gamma_k(E)\Gamma_k^{-1}(Q_j+E+\ell kI)
\end{aligned}
$$

Which is desired result (43). Letting $t=\frac{\xi-a}{x-a},\xi-a=(x-a)t,x-\xi=(x-a)(1-t)$ and using the k-beta matrix function (3) in the left hand side of (44), we obtain (44). □

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Theorem 7. *Let μ be a positive integer, then the ${}_{r+1}R_{s,k}$ matrix function satisfies the following Euler-Type integral representation*

$$\begin{aligned}
&{}_{r+\mu}R_{s+\mu}\left(A, P_1, P_2, \ldots, P_r, \Delta(E;\mu); Q_1, Q_2, \ldots, Q_s, \triangle(E+M;\mu); B, C, cz^{\frac{\mu}{k}}\right)\\
&= \frac{1}{k} z^{I-\frac{E+M}{k}} \Gamma_k^{-1}(E)\, \Gamma_k(E+M)\, \Gamma_k^{-1}(M) \int_0^z \xi^{\frac{E}{k}-I} (z-\xi)^{\frac{M}{k}-I}\\
&\times\ {}_{r+1}R_{s,k}\left(\begin{array}{c} A, P_1, P_2, \ldots, P_r; \\ Q_1, Q_2, \ldots, Q_s \end{array}; B, C, c\xi^{\frac{\mu}{k}}\right) d\xi,
\end{aligned} \tag{45}$$

where $\Delta(E,\mu)$ is given the array of a parameters

$$\Delta(E,\mu) = \frac{1}{\mu}E, \frac{1}{\mu}(E+kI), \frac{1}{\mu}(E+2kI), \ldots, \frac{1}{\mu}(E+k(\mu-1)I).$$

and

$$\begin{aligned}
&{}_{r+\mu+\imath}R_{s+\mu+\imath}\left(A, P_1, P_2, \ldots, P_r, \Delta(E;\mu), \Delta(M;\imath); Q_1, Q_2, \ldots, Q_s, \triangle(E+M;\mu+\imath); B, C, \frac{\alpha\mu^{\mu}\imath^{\imath}}{(\mu+\imath)^{\mu+\imath}}\right)\\
&= \Gamma_k^{-1}(E)\, \Gamma_k(E+M)\, \Gamma_k^{-1}(M) \int_0^1 \xi^{\frac{E}{k}-I} (1-\xi)^{\frac{M}{k}-I}\\
&\times\ {}_{r+1}R_{s,k}\left(\begin{array}{c} A, P_1, P_2, \ldots, P_r; \\ Q_1, Q_2, \ldots, Q_s \end{array}; B, C, \alpha\xi^{\frac{\mu}{k}}(1-t)^{\frac{\imath}{k}}\right) d\xi.
\end{aligned} \tag{46}$$

Proof. If we apply the substitution $\chi = zu$, $\chi = z\xi$, $\chi = zd\xi$ and use the k-Beta matrix function, we obtain our desired results (45)-(46). □

Theorem 8. *The following results hold true:*

$$\begin{aligned}
&\mathfrak{B}\left[{}_{r+1}R_{s,k}(A+E, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C, zt); A, E\right]\\
&= \frac{1}{k}\int_0^1 t^{\frac{A}{k}-I}(1-t)^{\frac{E}{k}-I}\ {}_{r+1}R_{s,k}(A+E, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C, zt)dt\\
&= \Gamma_k(A)\Gamma_k(E)\Gamma_k^{-1}(A+E)\ {}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; z)
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
&\mathfrak{B}\left[{}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_{i-1}, P_i+E, P_{i+1}, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C, zt); P_i, E\right]\\
&= \frac{1}{k}\int_0^1 t^{\frac{P_i}{k}-I}(1-t)^{\frac{E}{k}-I}\ {}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_{i-1}, P_i+E, P_{i+1}, , \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C, zt)dt\\
&= \Gamma_k(P_i)\Gamma_k(E)\Gamma_k^{-1}(P_i+E)\ {}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; z),
\end{aligned} \tag{48}$$

Proof. Using the k-Beta transform (17), we obtain (47)-(48). □

Theorem 9. *The Laplace transform for ${}_{r+1}R_{s,k}$ matrix function are given by*

$$\begin{aligned}
&\mathfrak{L}\left[t^{\frac{C}{k}-I}\ {}_{r+1}R_{s,k}(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C, zt^{\frac{B}{k}}); \mathbf{s}\right]\\
&= k(\mathbf{s}k)^{-\frac{C}{k}}\ {}_{r+1}F_{s,k}(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; z(k\mathbf{s})^{-\frac{B}{k}}),
\end{aligned} \tag{49}$$

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$$\begin{aligned}&\mathfrak{L}\left[t^{\frac{E}{k}-I}\ {}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,zt);\mathbf{s}\right]\\&=k\Gamma_k(E)(\mathbf{s}k)^{-\frac{E}{k}}\ {}_{r+2}R_{s,k}(A,E,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;\frac{z}{\mathbf{s}k}).\end{aligned}\tag{50}$$

$$\begin{aligned}&\mathfrak{L}\left[\ {}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;zx);s\right]\\&\frac{1}{s}\Gamma_k(k)\ {}_{r+2}R_{s,k}\left(A,kI,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;\frac{z}{ks}\right),\end{aligned}\tag{51}$$

$$\begin{aligned}&\mathfrak{L}\left[x^{\frac{A}{k}-I}\ {}_{r}R_{s,k}(P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;zx)\right]\\&=\frac{1}{s}(sk)^{I-\frac{A}{k}}\Gamma_k(A)\ {}_{r+1}R_{s,k}\left(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;\frac{z}{ks}\right)\end{aligned}\tag{52}$$

and

$$\begin{aligned}&\mathfrak{L}\left[x^{\frac{P_i}{k}-I}\ {}_{r}R_{s,k}(A,P_1,P_2,\ldots,P_{i-1},P_{i+1},\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;zx)\right]\\&=\frac{1}{s}(sx)^{-\frac{P_i}{k}}\Gamma_k(P_i)\ {}_{r+1}R_{s,k}\left(A,P_1,P_2,\ldots,P_{i-1},P_i,P_{i+1},\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;\frac{z}{sk}\right),1\leq i\leq r.\end{aligned}\tag{53}$$

Proof. On applying the Laplace transform (18), this yields the right hand side of (49)-(53), we obtain desired results (49)-(53). □

Theorem 10. *The Fractional k-Fourier transform for ${}_{r+1}R_{s,k}$ matrix function is given by*

$$\begin{aligned}&\mathfrak{F}\left[\ {}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,z)\right]\\&=\frac{\Gamma_k(k)}{iw^{\frac{1}{\alpha}}}\ {}_{r+2}R_{s,k}\left(kI,A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,\frac{i}{kw^{\frac{1}{\alpha}}}\right).\end{aligned}\tag{54}$$

Proof. Applying the Fractional k-Fourier transform of ${}_{r+1}R_{s,k}$ for $z<0$, we get

$$\begin{aligned}&\mathfrak{F}\left[\ {}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,z)\right]\\&=\int_{-\infty}^{0}e^{iw^{\frac{1}{\alpha}}z}\ {}_{r+1}R_{s,k}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C,z)dz\\&=\sum_{\ell=0}^{\infty}\frac{z^{\ell}}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)\int_{-\infty}^{0}e^{iw^{\frac{1}{\alpha}}z}z^{\ell}dz.\end{aligned}$$

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Putting $\chi = -iw^{\frac{1}{\alpha}}z$, $d\chi = -iw^{\frac{1}{\alpha}}dz$ and changing the order of integration and summation, we have

$$
\begin{aligned}
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)\int_0^1 e^{-\chi}\left(\frac{\chi}{-iw^{\frac{1}{\alpha}}}\right)^{\ell}\frac{1}{-iw^{\frac{1}{\alpha}}}d\chi\\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)(-1)^{\ell}i^{-(\ell+1)}w^{-\frac{\ell+1}{\alpha}}\Gamma(\ell+1)\\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)i^{\ell-1}w^{-\frac{\ell+1}{\alpha}}k^{-\ell}\Gamma_k(k)(k)_{\ell,k}\\
&= \frac{\Gamma_k(k)}{iw^{\frac{1}{\alpha}}}\sum_{\ell=0}^{\infty}\frac{1}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)\left(\frac{i}{kw^{\frac{1}{\alpha}}}\right)^{\ell}(k)_{\ell,k}.
\end{aligned}
$$

□

Theorem 11. *Let $\mu > 0$, $0 < \mu \le 1$ and $\mathbb{I}_k^{\mu}$ be the operators of Riemann-Liouville fractional integral and $\mathbb{D}_k^{\beta}$ fractional derivative then there hold the relations:*

$$
\begin{aligned}
&\mathbb{I}_k^{\mu}\left[t^{\frac{E}{k}}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\dots,P_r;Q_1,Q_2,\dots,Q_s;B,C;t)\right](z)\\
&= \Gamma_k(E+kI)\Gamma_k^{-1}(E+(\mu+k)I)z^{\frac{(\mu+k)I+E}{k}}\\
&\times\, {}_{r+2}R_{s+1,k}(A,E+kI,P_1,P_2,\dots,P_r;E+(\alpha+k)I,Q_1,Q_2,\dots,Q_s;B,C;z),
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
&\mathfrak{D}_k^{\mu}z^{\frac{E}{k}}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\dots,P_r;Q_1,Q_2,\dots,Q_s;B,C,z)\\
&= \frac{1}{k}\Gamma_k(E+kI)\Gamma_k^{-1}(E+(1-\mu)I)z^{\frac{E+(1-\mu)I}{k}-I}\\
&\times\, {}_{r+2}R_{s+1,k}(A,E+kI,P_1,P_2,\dots,P_r;E+(1-\mu)I;B,C,z),Q_1,Q_2,\dots,Q_s;B,C,z),
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
&\mathbb{I}_{a^+,k}^{\mu}\left[(t-a)^{\frac{E}{k}-I}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\dots,P_r;Q_1,Q_2,\dots,Q_s;B,C;\nu(t-a))\right](z)\\
&= \Gamma_k(E)\Gamma^{-1}(E+\mu I)(z-a)^{\frac{E+\mu I}{k}-I}\\
&\times\, {}_{r+2}R_{s+1,k}(E,A,P_1,P_2,\dots,P_r;E+\mu I,Q_1,Q_2,\dots,Q_s;B,C;\nu(z-a))
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
&\mathbb{D}_{a^+}^{\mu}\left[(t-a)^{\frac{E}{k}-I}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\dots,P_r;Q_1,Q_2,\dots,Q_s;B,C;\nu(t-a))\right](z)\\
&= \Gamma_k(E)\Gamma_k^{-1}(E-\mu I)(z-a)^{\frac{E-\mu I}{k}-I}\\
&\times\, {}_{r+2}R_{s+1,k}(E,A,P_1,P_2,\dots,P_r;E-\mu I,Q_1,Q_2,\dots,Q_s;B,C;\nu(z-a)).
\end{aligned}
\tag{58}
$$

Proof. To prove assertion (55). Using equations (20) and (13) this yields the right hand side of (55), we get

$$
\begin{aligned}
&\left[\mathbb{I}_k^{\mu}t^{\frac{E}{k}}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\dots,P_r;Q_1,Q_2,\dots,Q_{s,k};B,C;t)\right](x)\\
&= \frac{1}{k\Gamma_k(\mu)}\int_0^z(z-u)^{\frac{\mu}{k}-1}\,{}_{r+1}R_{s,k}(A,P_1,P_2,\dots,P_r;Q_1,Q_2,\dots,Q_s;B,C;t)t^{\frac{E}{k}}dt\\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)\frac{1}{k\Gamma_k(\mu)}\int_0^z(z-t)^{\frac{\mu}{k}-1}t^{\frac{E}{k}+\ell}dt
\end{aligned}
$$

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Putting $t = z\chi$, $dt = zd\chi$ and changing the order of summation and integration, we obtain (55). Similarly applying Riemann-Liouville fractional integral and derivative operators, one find the required results (56)-(58). □

Theorem 12. *For the* ${}_{r+1}R_{s,k}$ *matrix function, we have*

$$\mathbb{W}_k^{\beta}\left[(u+a)^{-\frac{E}{k}}\ {}_{r+1}R_{s,k}\left(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; \frac{1}{u+a}\right)\right](z)$$
$$= (z+a)^{\frac{\beta I-E}{k}}\Gamma_k^{-1}(E)\Gamma_k(E-\beta I)\ {}_{r+2}R_{s+1,k}\left(A, E-\beta I, P_1, P_2, \ldots, P_r; E, Q_1, Q_2, \ldots, Q_s; B, C; \frac{1}{z+a}\right), \tag{59}$$

$$\begin{aligned}&\mathbb{W}_k^{\beta}\left[(u+a)^{-\frac{C}{k}}\ {}_{r+1}R_{s,k}\left(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C-\beta I; \nu(u+a)^{-\frac{B}{k}}\right)\right]\\ &= (z+a)^{\frac{\beta I-C}{k}}\ {}_{r+1}R_{s,k}\left(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; \nu(z+a)^{-\frac{B}{k}}\right),\end{aligned} \tag{60}$$

$$\mathbb{W}_k^{-\beta}\left[\ {}_{r+1}R_{s,k}\left(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_{s,k}; B, C; \frac{1}{u+a}\right)\right](z)$$
$$= \frac{1}{k}(z+a)^{\frac{(1-\beta)I-E}{k}-I}\Gamma_k^{-1}(E)\Gamma_k(E+\beta I)\ {}_{r+2}R_{s+1,k}\left(A, E+\beta I, P_1, P_2, \ldots, P_r; E, Q_1, Q_2, \ldots, Q_s; B, C; \frac{1}{z+a}\right) \tag{61}$$

and

$$\begin{aligned}&\mathbb{W}_k^{-\beta}\left[(u+a)^{-\frac{C}{k}}\ {}_{r+1}R_{s,k}\left(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C+\beta I; \nu(u+a)^{-\frac{B}{k}}\right)\right]\\ &= (z+a)^{\frac{(1-\beta)I-C}{k}-I}\ {}_{r+1}R_{s,k}\left(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; \nu(z+a)^{-\frac{B}{k}}\right).\end{aligned} \tag{62}$$

Proof. Applying the k-Weyl fractional integral operator (15), one gets

$$\begin{aligned}&\mathbb{W}_k^{\beta}\left[(u+a)^{-\frac{E}{k}}\ {}_{r+1}R_{s,k}\left(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_{s,k}; B, C; \frac{1}{u+a}\right)\right](z)\\ &= \frac{1}{k\Gamma_k(\beta)}\int_z^{\infty}(u-z)^{\frac{\beta}{k}-1}(u+a)^{-\frac{E}{k}}\ {}_{r+1}R_{s,k}\left(A, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; \frac{1}{u+a}\right)du\\ &= \sum_{\ell=0}^{\infty}\frac{z^{\ell}}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)\frac{1}{k\Gamma_k(\beta)}\int_z^{\infty}(u-z)^{\frac{\beta}{k}-1}(u+a)^{-\ell I-\frac{E}{k}}du.\end{aligned}$$

Making the change of variables $t = \frac{u-z}{u+a}$, $u = \frac{z+at}{1-t}$, $u = z \Rightarrow t = 0$, $u = \infty \Rightarrow t = 1$ $du = \frac{z+a}{(1-t)^2}dt$, $u + a = \frac{z+a}{1-t}$, $u - z = \frac{t(z+a)}{1-t}$, we get

$$\begin{aligned}&(z+a)^{\frac{\beta I-E}{k}}\Gamma_k(-\beta)\sum_{\ell=0}^{\infty}\frac{z^{\ell}}{\ell!}(A)_{\ell,k}\prod_{i=1}^{r}(P_i)_{\ell,k}\left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1}\Gamma_k^{-1}(\ell B+C)(z+a)^{-\ell}\Gamma_k^{-1}(E)\\ &\times\Gamma_k(E-\beta I)(E-\beta I)_{\ell,k}[(E)_{\ell,k}]^{-1} = (z+a)^{\frac{\beta I-E}{k}}\Gamma_k^{-1}(E)\Gamma_k(E-\beta I)\\ &\times\ {}_{r+2}R_{s+1,k}\left(A, E-\beta I, P_1, P_2, \ldots, P_r; E, Q_1, Q_2, \ldots, Q_{s,k}; B, C; \frac{1}{z+a}\right).\end{aligned}$$

Again applying the k-Weyl fractional integral and derivative operators, we get the results (60)-(62). □

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3. Some special cases and applications

In this section, we develop a integral of the ${}_{r+1}R_{s,k}$ matrix function involving relation with some special cases related to integral representations of ${}_{r+1}R_{s,k}$ matrix functions have also been explained below.

Theorem 13. *For $|z| < 1$, $Re(B) > Re(A) > 0$, the ${}_{r+1}R_{r,k}$ matrix function satisfies the following Euler-type integral representation:*

$$ {}_{r+1}R_{r,k}(A,\Delta(P,r);\Delta(Q,r);B,C;z) = \Gamma_k(Q)\Gamma_k^{-1}(P)\Gamma_k^{-1}(Q-P)\int_0^1 t^{\frac{P}{k}-I}(1-t)^{\frac{Q-P}{k}-I}\mathbf{E}_{A,B,C}(zt^{\frac{r}{k}})dt \tag{63}$$

where $\mathbf{E}_{A,B,C}(z)$ is three parametric Mittag–Leffler matrix function [31,32].

Proof. For convenience, let ${}_{r+1}R_{r,k}$ be left-hand side of (63), then

$$ \begin{aligned} {}_{r+1}R_{r,k}(A,\Delta(P,r);\Delta(Q,r);B,C;z) &= \sum_{\ell=0}^{\infty}\frac{z^\ell}{\ell!}(A)_{\ell,k}(\frac{1}{r}P)_{\ell,k}(\frac{1}{r}(P+I))_{\ell,k}\dots\frac{1}{r}(P+(r-1)I)_{\ell,k} \\ \times[(\frac{1}{r}Q)_{\ell,k}]^{-1}[(\frac{1}{r}(Q+I))_{\ell,k}]^{-1}\dots[\frac{1}{r}(Q+(r-1)I)_{\ell,k}]^{-1}\Gamma_k^{-1}(\ell B+C). \end{aligned} \tag{64}$$

Taking the following properties [34]

$$ (P)_{\ell r,k} = r^{\ell r}\prod_{i=1}^{r}\left(\frac{P+(i-1)I}{r}\right)_{\ell,k}, \ell = 0,1,2,\dots, \tag{65}$$

and

$$ (P)_{r\ell,k}[(Q)_{r\ell,k}]^{-1} = \Gamma_k(Q)\Gamma_k^{-1}(P)\Gamma_k^{-1}(Q-P)\mathbf{B}_k(P+rk\ell,Q-P). \tag{66}$$

where r is a positive integer. Using (65) and (66) into (64), we arrive at

$$ \begin{aligned} &{}_{r+1}R_{r,k}(A,\Delta(P,r);\Delta(Q,r);B,C;z) \\ &= \sum_{\ell=0}^{\infty}\frac{z^\ell}{\ell!}(A)_{\ell,k}(\frac{1}{r}P)_{\ell,k}(\frac{1}{r}(P+I))_{\ell,k}(\frac{1}{r}(P+(r-1)I))_{\ell,k} \\ &[(\frac{1}{r}Q)_{\ell,k}]^{-1}[(\frac{1}{r}(Q+I))_{\ell,k}]^{-1}[(\frac{1}{r}(Q+(r-1)I))_{\ell,k}]^{-1}\Gamma_k^{-1}(\ell B+C) \\ &= \sum_{\ell=0}^{\infty}\frac{z^\ell}{\ell!}(A)_{\ell,k}(P)_{r\ell}[(Q)_{r\ell}]^{-1}\Gamma_k^{-1}(\ell B+C) \\ &=\Gamma_k(Q)\Gamma_k^{-1}(A)\Gamma_k^{-1}(Q-P)\sum_{\ell=0}^{\infty}\frac{z^\ell}{\ell!}(A)_{\ell,k}\mathbf{B}_k(P+r\ell,Q-P)\Gamma_k^{-1}(\ell B+C) \\ &=\Gamma_k(Q)\Gamma_k^{-1}(A)\Gamma_k^{-1}(Q-P)\sum_{\ell=0}^{\infty}\frac{z^\ell}{\ell!}(A)_{\ell,k}\Gamma_k^{-1}(\ell B+C)\int_0^1 t^{\frac{P+r\ell I}{k}-I}(1-t)^{\frac{Q-P}{k}-I}dt \\ &=\Gamma_k(Q)\Gamma_k^{-1}(P)\Gamma_k^{-1}(Q-P)\int_0^1 t^{\frac{P}{k}-I}(1-t)^{\frac{Q-P}{k}-I}\sum_{\ell=0}^{\infty}\frac{(zt^{\frac{r}{k}})^\ell}{\ell!}(A)_{\ell,k}\Gamma_k^{-1}(\ell B+C)dt \\ &=\Gamma_k(Q)\Gamma_k^{-1}(P)\Gamma_k^{-1}(Q-P)\int_0^1 t^{\frac{P}{k}-I}(1-t)^{\frac{Q-P}{k}-I}\mathbf{E}_{A,B,C}(zt^{\frac{r}{k}})dt. \end{aligned}$$

□

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Theorem 14. *For $n,k \in N$, the ${}_{r+1}R_{r,k}$ matrix function satisfy the following Euler type integral representation:*

$$\begin{aligned}&{}_{r+1}R_{r,k}(-nI,\Delta(P,r);\Delta(Q,r);mI,C;z) = \Gamma_k(Q)\Gamma_k^{-1}(P)\Gamma_k^{-1}(Q-P)\\&\times \Gamma_k(n+1)\Gamma_k^{-1}(nkI+C)\int_0^1 t^{\frac{P}{k}-I}(1-t)^{\frac{Q}{k}-I}\mathbf{Z}_{n,k}^{C-I}(zt^{\frac{r}{k}};m)dt\end{aligned} \tag{67}$$

where $\mathbf{Z}_{n,k}^{C-I}(z;m)$ is the Konhauser matrix polynomials [30,32,38] of degree n in z^k.

Proof. By performing $A=-nI$ and $B=mI$ then (63) reduces to

$$\begin{aligned}&{}_{r+1}R_{r,k}(-nI,\Delta(P,r);\Delta(Q,r);mI,C;z)\\&=\Gamma_k(Q)\Gamma_k^{-1}(P)\Gamma_k^{-1}(Q-P)\int_0^1 t^{\frac{P}{k}-I}(1-t)^{\frac{Q-P}{k}-I}E_{mI,C;-nI}(zt^r)dt\end{aligned}$$

Using the result defined in [32], this leads to right-hand side of (67). □

Theorem 15. *For $n \in N$, there reduces to the following integral representation*

$$\begin{aligned}&{}_{r+1}R_{r,k}(-nI,\Delta(P,r);\Delta(Q,r);1I,C;z) = \Gamma_k(Q)\Gamma_k^{-1}(P)\Gamma_k^{-1}(Q-P)\\&\times \Gamma_k(n+1)\Gamma_k^{-1}(Q+nI)\int_0^1 t^{\frac{P}{k}-I}(1-t)^{\frac{Q-P}{k}-I}\mathbf{L}_{n,k}^{C-I}(zt^{\frac{r}{k}})dt,\end{aligned} \tag{68}$$

where $\mathbf{L}_{n,k}^{C-I}(z)$ is the Laguerre matrix polynomials [11].

3.1. Special cases

Case 1. On setting $k=1$, $P_i=I$, and $Q_j=I$, (20) becomes

$${}_1R_0(A;-;B,C;z)=\sum_{\ell=0}^{\infty}\frac{z^\ell}{\ell!}(A)_\ell\Gamma^{-1}(\ell B+C)=E_{B,C}^{A}(z), \tag{69}$$

where $E_{B,C}^{A}(z)$ is the three parameter Mittag–Leffler matrix function given by Sanjhira et al. [32]

Case 2. On setting $k=1$, (20) reduces to

$$\begin{aligned}&{}_{r+1}R_{s,1}(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;z)\\&={}_rK_s(A,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;z),\end{aligned} \tag{70}$$

where ${}_rK_s$ is the matrix K-function given by Sharma and Jain [33].

Case 3. By taking $k=1$ and $A=I$, (20) becomes

$$\begin{aligned}&{}_{r+1}R_{s,1}(I,P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;z)\\&={}_rM_s(P_1,P_2,\ldots,P_r;Q_1,Q_2,\ldots,Q_s;B,C;z),\end{aligned} \tag{71}$$

where ${}_rM_s$ is the generalized matrix M-series defined by Sharma and Jain [33].

Case 4. On setting $k=1$, $A=I$ and $Q_1=I$, (20) reduces to

$$\begin{aligned}&{}_{r+1}R_{s,1}(I,P_1,P_2,\ldots,P_r;I,Q_2,\ldots,Q_s;B,C;z)\\&={}_rR_{s-1}(P_1,P_2,\ldots,P_r;Q_2,\ldots,Q_s;B,C;z),\end{aligned} \tag{72}$$

where ${}_rR_{s-1}$ matrix function defined by Sanjhira and Dwivedi [31] and Shehata et al. [37].

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Case 5. On taking $k = 1$, $A = I$ and $B = C = I$, (20) becomes

$$\begin{aligned} &{}_{r+1}R_{s,1}(I, P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; I, I; z) \\ &= {}_rF_s(P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; z), \end{aligned} \tag{73}$$

where ${}_rF_s(z)$ is the generalized hypergeometric matrix function by Shehata [34].

4. Conclusion

It should be observed in conclusion, we are motivated to obtain the our interest recurrence matrix relation, differential properties, new integral representations, Fractional k-Beta, Laplace and k-Fourier transforms, the k-Riemann–Liouville and k-Weyl fractional integral and derivative operators of this ${}_{r+1}R_{s,k}$ matrix function, which is an important crucial in theory of classical analysis, integral transforms, mathematical analysis, operational techniques, mathematical physics, fractional calculus, applied mathematics and statistics. Hence, for our purposes, the results appear in this study are seemed novel to the literature. From this view, we gives some special cases, such as hypergeometric and Mittag-Leffler matrix functions, k-Konhauser and k-Laguerre matrix polynomials for the ${}_{r+1}R_{s,k}$ matrix function using transform method with its application to the Mittag–Leffler matrix function and Euler-type integral representations including many special cases and so on. Lastly, we conclude this paper by hoping that we will be able also extend these generalized type k-fractional derivatives and their consequences for by analytical continuation. We have been asked to publish in the present form in the hope that others, more qualified than ourselves in this field, may find the results suggestive, even indicative, of the usefulness of a more systematic study.

Definition 8. *Our definitions of extended ${}_{r+1}R_{s,k}$ matrix function can further results be generalized Pochhammer symbol $(A, \mu)_{\ell,k}$ to the following form:*

$$\begin{aligned} &{}_{r+1}R_{s,k}((A, \mu), P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; z) \\ &= \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!}(A, \mu)_{\ell,k} \prod_{i=1}^{r}(P_i)_{\ell,k} \left[\prod_{j=1}^{s}(Q_j)_{\ell,k}\right]^{-1} \Gamma_k^{-1}(\ell B + C), \end{aligned} \tag{74}$$

where

$$(A, \mu)_{\ell,k} = \Gamma_k^{-1}(A) \int_0^\infty t^{\frac{A+\ell}{k}-1} e^{-t-\frac{\mu}{t}} dt; Re(A) > 0, \mu > 0 \tag{75}$$

and the extension of extended ${}_{r+1}R_{s,k}$ matrix function will be defined as follows

$$\begin{aligned} &{}_{r+1}R_{s,k}((A, \mu), P_1, P_2, \ldots, P_r; Q_1, Q_2, \ldots, Q_s; B, C; \tau; z) \\ &= \prod_{i=1}^{r}\prod_{j=1}^{s} \Gamma_k^{-1}(P_i)\Gamma_k(Q_j) \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!}(A, \mu)_{\ell,k}\Gamma_k(P_i + k\ell\tau)\Gamma_k^{-1}(Q_j + k\ell\tau)\Gamma_k^{-1}(\ell B + C). \end{aligned} \tag{76}$$

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