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Landau Problem in Dynamical Noncommutative Space

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Abstract: This paper aims to investigate the Landau problem within a two-dimensional dynamical noncommutative (DNC) space. We address the deformed Landau problem using time-independent perturbation theory, where the eigenenergies are successfully determined. Notably, the energy shift depends on the DNC parameter τ. Using the accuracy of energy measurement, we put an upper bound on the parameter τ. Moreover, we study magnetoconductivity by employing the Kubo formula. This approach has allowed us to test the effects of noncommutative and DNC spaces on the behavior of magnetoconductivity. We show that dynamical noncommutativity of space has no effects on the x-component of the conductivity.

Keywords: dynamical noncommutative space; deformed Landau system; time-independent perturbation theory; magnetoresistance; Kubo formula

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I. Introduction

Two-dimensional (2D) systems in quantum mechanics (QM) are of great interest and offer a unique base for exploring the intriguing behavior of particles in confined geometries at the atomic and subatomic scales. In these systems, particles are confined to move in two dimensions. These systems exhibit distinct quantum mechanical phenomena that significantly differ from their three-dimensional counterparts. In 2D systems, the motion of particles is restricted to specific paths, leading to quantization of energy levels and discrete spectra. This quantization arises due to the confinement of particles in two dimensions, causing the wave functions to exhibit characteristic patterns and behaviors. The discrete energy levels give rise to specific electronic, optical, and magnetic properties. The 2D systems have been extensively studied and have led to significant discoveries and advancements in various areas of physics. Here are a few key examples: (**i**) Graphene [1–3], which is a single layer of carbon atoms arranged in a 2D honeycomb lattice, has attracted enormous attention in recent years. Its remarkable electronic properties arise due to the unique behavior of electrons in a honeycomb lattice structure. Graphene exhibits phenomena such as massless Dirac fermions and high electron mobility. It has potential applications in electronics, optoelectronics, energy storage, and other areas. (**ii**) Quantum dots [4–6], which are artificially engineered nanoscale structures that confine electrons or other particles in two dimensions. By carefully controlling the size and shape of the quantum dot, quantum mechanical effects become dominant, leading to discrete energy levels and interesting electronic and optical properties. Quantum dots have applications in quantum computing, nanotechnology, biological imaging and optoelectronics. (**iii**) Quantum hall effect [7–9], which is a fascinating phenomenon that occurs in 2D electron systems subjected to a strong magnetic field (MF). It manifests as quantized conductance, where the electrical resistance becomes quantized in integer multiples of a fundamental value. This effect has paved the way for the discovery of topological insulators and the study of topological states of matter. (**iv**) Thin films [10–12], which are 2D layers of materials that are only a few atomic or molecular layers thick. In these 2D films, quantum confinement effects significantly influence the electronic, magnetic, and optical properties of the material such as electrical conductivity, magnetism, and optical absorption. Thin film structures are widely used in electronics, solar cells, sensors and surface coatings, and their behavior can be tailored by controlling the film thickness and composition. These and other notable examples such as 2D electron gas [13], Weyl semimetals [14] and other materials, e.g. [15–18] demonstrate the rich and diverse nature of 2D systems in QM and highlight their importance in various technological applications and fundamental research. The 2D systems have been instrumental in advancing our understanding of fundamental physics and have paved the way for technological advancements.

In QM, the terms "*Landau system*" and "*Landau problem*" are often used interchangeably, but can refer to slightly different aspects. However, the Landau System is generally referring to a charged particle (usually an electron) moving in a 2D plane under the effect of a uniform MF perpendicular to the plane. This system is named after Lev Landau. On the other hand, Landau Problem is specifically referring to the quantum mechanical description of a charged particle confined to a 2D plane and subjected to a perpendicular MF. In this scenario, the charged particle experiences a Lorentz force due to the MF, causing it to move in circular orbits around the field lines. The Landau problem is often used to study the effects of a magnetic field on the conduct of electrons in a solid, particularly in the context of condensed matter physics. It has important applications in understanding phenomena like the quantum Hall effect, which exhibits remarkable conduct related to the quantization of electrical conductance. Overall, the Landau system refers to the general physical setup of a charged particle moving in a MF in two dimensions, while the Landau problem refers to the specific quantum mechanical treatment of this system.

On the other side, in the recent years the study of quantum systems in noncommutative (NC) spaces has been an issue of great interest. Certainly, delving into NC geometry holds significant importance in gaining insights into phenomena occurring at small scales and exerts a profound influence across various domains of contemporary physics, including cosmology, gravitation, high energy and astrophysics. To explore further the subject of NC spaces, refer to, for instance, the Refs. [19–30]. Various forms of NC spaces have been explored, see Refs. [31–37] for an overview. However, one type is particularly important to us, in which the noncommutativity of space is position dependent, called dynamical NC (DNC) space.

In this work, we seek to examine the influences of a DNC space on the Landau problem. In the same context, we have investigated the Landau problem in NC framework [38] and obtained the energy spectrum and the wave function of the deformed system. Besides, in the classical limit, we have derived the NC semi-classical partition function for one and N-particles systems. Then, the thermodynamic properties such as the Helmholtz free energy, mean energy, specific heat and entropy in NC and commutative settings are determined. Likewise, we have obtained the three-dimensional Pauli equation in the presence of an electromagnetic field in a NC phase-space [39]. Besides the corresponding deformed continuity equation, where the cases of a constant and non-constant Mfs are considered. It is shown that the non-constant MF lifts the order of the noncommutativity parameter in both the Pauli equation and the corresponding continuity equation. By using a classical treatment, also we have derived the semi-classical NC partition function of the deformed system of the one and N-particles systems, then, obtaining the corresponding Helmholtz free energy followed by the magnetization and the magnetic susceptibility of electrons in both commutative and NC settings. In addition, Dossa et al [40] have studied NC Landau problem in the presence of a minimal length, where through the Nikiforov-Uvarov method, they obtained the energy eigenvalues and the corresponding wave functions, which were expressed in terms of hypergeometric functions. Also, Aslam Halder and Sunandan Gangopadhyay [41] have considered the thermodynamics of the Landau system in a NC phase-space. The partition function of the NC system is computed and used to found the magnetization and the susceptibility. Besides, they investigated the de Hass–van Alphen effect in the NC setting.

This paper is outlined as follows. In Section II, the DNC space is briefly reviewed. In Section III, the DNC Landau problem is investigated, where we study the 2D Pauli equation in the DNC space in Subsection A. Then, in Sub-sections B and C, we obtain the solutions of the 2D Pauli equation in DNC space. Moreover, in Sub-section D we study the magnetoconductivity and examine the effects of DNC space on it. Based on the energy shift and the accuracy of energy measurement, an upper bound on the DNC parameter is found in in Sub-section E . We present our conclusion in Section IV.

II. Basic overview of dynamical noncommutative space

We first shortly review the essential formulas of the DNC space algebra we use in this work. As known at the tiny scale (string scale), the position coordinates do not commute with each other, thus the canonical variables satisfy the following deformed Heisenberg commutation relation

$$\left[x_{\mu}^{nc}, x_{\nu}^{nc}\right] = i\Theta_{\mu\nu}, \tag{1}$$

with $\Theta_{\mu\nu}$ is an anti-symmetric tensor. In simplest way, the deformation parameter is considered a real constant. But, in general, $\Theta_{\mu\nu}$can be a function of coordinates. Fring et al [35] made a generalization of NC space to a position-dependent space by introducing a set of new variables X, Y, P_x,P_y and convert the constant Θ into a function of coordinates as $\theta(X,Y) = \Theta\left(1+\tau Y^2\right)$. As another example of $\theta(X,Y)$, we mention that Gomes et al [42] chose in their study $\theta(X,Y) = \Theta/\left[1+\Theta\alpha\left(1+Y^2\right)\right]$. However, a deformation of this NC parameter form, will almost inevitably lead to non-Hermitian coordinates. It was pointed out [43,44] that these types of structures are related directly to non-Hermitian Hamiltonian systems. Later, it is explained how this problem was solved.

In the new type of two-dimensional NC space, which is known as the DNC space or $\tau-$space, the commutation relations are [35]

$$\begin{array}{llll} [X,Y] = & i\Theta\left(1+\tau Y^2\right), & \left[Y,P_y\right] = & i\hbar\left(1+\tau Y^2\right), \\ [X,P_x] = & i\hbar\left(1+\tau Y^2\right), & [Y,P_x] = & 0, \\ \left[X,P_y\right] = & 2i\tau Y\left(\Theta P_y + \hbar X\right), & \left[P_x,P_y\right] = & 0. \end{array} \tag{2}$$

It is interesting to note that $\sqrt{\Theta}$ and $\sqrt{\tau}$ have dimensions of L and L^{-1}, respectively. In the limit $\tau \to 0$, we recover the following non-dynamical NC commutation relations

$$\begin{array}{llll} [x^{nc},y^{nc}] = & i\Theta, & \left[y^{nc},p_y^{nc}\right] = & i\hbar, \\ [x^{nc},p_x^{nc}] = & i\hbar, & [y^{nc},p_x^{nc}] = & 0, \\ \left[x^{nc},p_y^{nc}\right] = & 0, & \left[p_x^{nc},p_y^{nc}\right] = & 0. \end{array} \tag{3}$$

The coordinate X and the momentum P_y are not Hermitian, which make the Hamiltonian that includes these variables non-Hermitian. We represent algebra (2) in terms of the standard Hermitian NC variables operators $x^{nc}, y^{nc}, p_x^{nc}, p_y^{nc}$ as follows

$$\begin{array}{ll} X = \left(1+\tau\left(y^{nc}\right)^2\right)x^{nc}, & Y = y^{nc}, \\ P_y = \left(1+\tau\left(y^{nc}\right)^2\right)p_y^{nc}, & P_x = p_x^{nc}. \end{array} \tag{4}$$

From the representation (4), some of the operators involved are no longer Hermitian. However, to convert the non-Hermitian variables into Hermitian ones, we use a similarity transformation as a Dyson map $\eta O \eta^{-1} = o = O^{\dagger}$ with $\eta = (1+\tau Y^2)^{-\frac{1}{2}}$, as stated in [35]. Therefore, we express the new Hermitian variables x, y, p_x and p_y in terms of NC variables as follows

$$\begin{array}{llll} x = \eta X \eta^{-1} & = & (1+\tau Y^2)^{-\frac{1}{2}} X (1+\tau Y^2)^{\frac{1}{2}} & \\ & = & (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} x^{nc} (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} & \\ y = \eta Y \eta^{-1} & = & (1+\tau\left(y^{nc}\right)^2)^{-\frac{1}{2}} y^{nc} (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} & = y^{nc} \\ p_x = \eta P_x \eta^{-1} & = & (1+\tau\left(y^{nc}\right)^2)^{-\frac{1}{2}} p_x^{nc} (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} & = p_x^{nc} \\ p_y = \eta P_y \eta^{-1} & = & (1+\tau\left(y^{nc}\right)^2)^{-\frac{1}{2}} P_y (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} & \\ & = & (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}} p_y^{nc} (1+\tau\left(y^{nc}\right)^2)^{\frac{1}{2}}. & \end{array} \tag{5}$$

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These new Hermitian DNC variables satisfy the following commutation relations

$$\begin{array}{llll} [x,y]= & i\Theta\left(1+\tau y^2\right), & [y,p_y]= & i\hbar\left(1+\tau y^2\right), \\ [x,p_x]= & i\hbar\left(1+\tau y^2\right), & [y,p_x]= & 0, \\ [x,p_y]= & 2i\tau y\left(\Theta p_y+\hbar x\right), & [p_x,p_y]= & 0. \end{array} \tag{6}$$

Consequently, using Bopp-shift transformation, one can represent the NC variables in terms of the standard commutative variables [3]

$$\begin{array}{ll} x^{nc}=x^s-\frac{\Theta}{2\hbar}p_y^s, & p_x^{nc}=p_x^s, \\ y^{nc}=y^s+\frac{\Theta}{2\hbar}p_x^s, & p_y^{nc}=p_y^s, \end{array} \tag{7}$$

where the index s refers to the standard commutative space. The interesting point is that in the DNC space there is a minimum length for X in a simultaneous X, Y measurement [35]:

$$\triangle X_{\min}=\Theta\sqrt{\tau}\sqrt{1+\tau\langle Y\rangle_\rho^2}, \tag{8}$$

as well, in a simultaneous Y, P_y measurement we find a minimal momentum as

$$\triangle\left(P_y\right)_{\min}=\hbar\sqrt{\tau}\sqrt{1+\tau\langle Y\rangle_\rho^2}. \tag{9}$$

The motivation and interesting physical consequence for position-dependent noncommutativity is that objects in two-dimensional spaces are string-like [35]. However, investigating Landau problem in DNC space gives rise to some phenomenological consequences that may be very important and useful.

III. Dynamical Noncommutative Landau Problem

A. Extension to dynamical noncommutative space

The time-independent Pauli equation can be given by [38]

$$\left\{\frac{1}{2m_e}\left(\vec{p}^{\,s}-e\vec{A}^{\,s}\right)^2+e\phi+\mu_B\vec{\sigma}\cdot\vec{B}\right\}\psi_{n,l}\left(\vec{r}^{\,s}\right)=E_{n,l}\psi_{n,l}\left(\vec{r}^{\,s}\right), \tag{10}$$

where $\psi_{n,l}$ is the wave function, with n, l denote the radial and angular quantum numbers. $\vec{p}^{\,s}=-i\hbar\vec{\nabla}$ is the momentum operator, m_e, e are the mass and charge of the electron. $\vec{B}$ is the applied MF vector, with $\vec{A}^{\,s}$ and ϕ^s are the the vector and the Coulomb potentials. $\mu_B=\frac{|e|\hbar}{2m_e}$ is the Bohr magneton (given in SI units), and $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ are the Pauli matrices. In the Landau system, $\vec{B}$ is oriented along the (Oz) axis and hence, we use the following symmetric gauge

$$\vec{A}^{\,s}=\frac{B}{2}\left(-y^s,x^s,0\right),\ A_0^s=e\phi^s=0, \tag{11}$$

where the electron is unbound, i.e. $\phi^s=0$. Now, the 2D time-independent Pauli equation is

$$\left\{\frac{(p_x^s)^2+\left(p_y^s\right)^2}{2m_e}-\frac{eB}{2m_e}\left(p_y^sx^s-p_x^sy^s\right)+\frac{e^2B^2}{8m_e}\left\{(x^s)^2+(y^s)^2\right\}+\sigma_z\mu_BB\right\}\psi_{n,l}=E_{n,l}\psi_{n,l}. \tag{12}$$

Therefore, the Hamiltonian from equation (12) can be given as follows

$$\mathscr{H}_P\left(x_i^s,p_i^s\right)=\frac{(p_x^s)^2+\left(p_y^s\right)^2}{2m_e}-\frac{eB}{2m_e}\left(p_y^sx^s-p_x^sy^s\right)+\frac{e^2B^2}{8m_e}\left\{(x^s)^2+(y^s)^2\right\}+\sigma_z\mu_BB. \tag{13}$$

The above Pauli Hamiltonian in DNC space turns to

$$\mathscr{H}_P\left(x_i,p_i\right)=\frac{p_x^2+p_y^2}{2m_e}-\frac{eB}{2m_e}\left(p_y^s x^s-p_x y\right)+\frac{e^2B^2}{8m_e}\left(x^2+y^2\right)+\sigma_z\mu_B B. \tag{14}$$

Now, using equation (5), we express the Pauli Hamiltonian (14 in terms of NC variables as follows

$$\begin{aligned}\mathscr{H}_P\left(x_i^{nc},p_i^{nc}\right)=&\tfrac{1}{2m_e}\left\{\left(p_x^{nc}\right)^2+\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac12}p_y^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)p_y^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac12}\right\}\\&-\tfrac{eB}{2m_e}\left\{-p_x^{nc}y^{nc}+\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac12}p_y^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)x^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac12}\right\}\\&+\tfrac{e^2B^2}{8m_e}\left\{\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac12}x^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)x^{nc}\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac12}+\left(y^{nc}\right)^2\right\}+\sigma_z\mu_B B.\end{aligned} \tag{15}$$

As long as τ is very small, the parentheses can be expanded to the first-order as follows

$$\left(1+\tau\left(y^{nc}\right)^2\right)^{\frac12}=1+\frac12\tau\left(y^{nc}\right)^2+0\left(\tau^2\right), \tag{16}$$

thereafter, equation (15) turns to

$$\begin{aligned}\mathscr{H}_P\left(x_i^{nc},p_i^{nc}\right)=&\tfrac{1}{2m_e}\left\{\left(p_x^{nc}\right)^2+\left(p_y^{nc}\right)^2+\tau p_y^{nc}\left(y^{nc}\right)^2p_y^{nc}+\tfrac{\tau}{2}\left(y^{nc}\right)^2\left(p_y^{nc}\right)^2+\tfrac{\tau}{2}\left(p_y^{nc}\right)^2\left(y^{nc}\right)^2\right\}\\&-\tfrac{eB}{2m_e}\left\{-p_x^{nc}y^{nc}+p_y^{nc}x^{nc}+\tfrac12\tau p_y^{nc}x^{nc}\left(y^{nc}\right)^2+\tfrac12\tau\left(y^{nc}\right)^2p_y^{nc}x^{nc}+\tau p_y^{nc}\left(y^{nc}\right)^2x^{nc}\right\}+\sigma_z\mu_B B\\&+\tfrac{e^2B^2}{8m_e}\left\{\left(x^{nc}\right)^2+\left(y^{nc}\right)^2+\tau x^{nc}\left(y^{nc}\right)^2x^{nc}+\tfrac{\tau}{2}\left(y^{nc}\right)^2\left(x^{nc}\right)^2+\tfrac{\tau}{2}\left(x^{nc}\right)^2\left(y^{nc}\right)^2\right\}+0\left(\tau^2\right),\end{aligned} \tag{17}$$

then with the help of the Bopp-shift (7), the above Hamiltonian can be stated in terms of the standard commutative variables as follows

$$\begin{aligned}\mathscr{H}_P\left(x_i^s,p_i^s\right)=&\tfrac{1}{2m_e}\left\{\left(p_x^s\right)^2+\left(p_y^s\right)^2+\tau p_y^s\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2p_y^s\right.\\&+\tfrac{\tau}{2}\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2\left(p_y^s\right)^2+\tfrac{\tau}{2}\left(p_y^s\right)^2\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2\Big\}-\tfrac{eB}{2m_e}\Big\{p_y^s\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)\\&-p_x^s\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)+\tfrac{\tau}{2}p_y^s\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2+\tfrac{\tau}{2}\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2p_y^s\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)\\&+\tau p_y^s\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)\Big\}+\tfrac{e^2B^2}{8m_e}\Big\{\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)^2+\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2\\&+\tfrac{\tau}{2}\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)^2+\tfrac{\tau}{2}\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)^2\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2\\&\left.+\tau\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)\left(y^s+\tfrac{\Theta}{2\hbar}p_x^s\right)^2\left(x^s-\tfrac{\Theta}{2\hbar}p_y^s\right)\right\}+\sigma_z\mu_B B.\end{aligned} \tag{18}$$

Now, to the first order in Θ and τ, equation (18) becomes

$$\begin{aligned}\mathscr{H}_P\left(x_i^s,p_i^s\right)=&\tfrac{1}{2m_e}\left\{\left(p_x^s\right)^2+\left(p_y^s\right)^2\right\}-\tfrac{eB}{2m_e}L_z+\tfrac{e^2B^2}{8m_e}\left\{\left(x^s\right)^2+\left(y^s\right)^2\right\}+\sigma_z\mu_B B\\&+\tfrac{eB}{2m_e}\tfrac{\Theta}{2\hbar}\left\{\left(p_x^s\right)^2+\left(p_y^s\right)^2-\tfrac{eB}{2}L_z\right\}+\tfrac{\tau}{2m_e}\Big\{\tfrac{e^2B^2}{2}\left(x^s\right)^2\left(y^s\right)^2+p_y^s\left(y^s\right)^2p_y^s\\&+\tfrac{\left(y^s\right)^2\left(p_y^s\right)^2}{2}+\tfrac{\left(p_y^s\right)^2\left(y^s\right)^2}{2}-eB\left(p_y^s\left(y^s\right)^2x^s+\tfrac{p_y^sx^s\left(y^s\right)^2}{2}+\tfrac{\left(y^s\right)^2p_y^sx^s}{2}\right)\Big\},\end{aligned} \tag{19}$$

taking into account that terms containing $\Theta\tau$ also are neglected. We continue to simplify the Hamiltonian (19) to get

$$\mathscr{H}_P=H_0+H_\Theta+H_\tau, \tag{20}$$

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with

$$H_0 = \frac{(p_x^s)^2 + \left(p_y^s\right)^2}{2m_e} - \frac{eB}{2m_e} L_z + \frac{e^2B^2}{8m_e} \left\{(x^s)^2 + (y^s)^2\right\} + \mu_B \sigma_z B, \tag{21}$$

$$H_\Theta = \frac{eB}{2m_e}\frac{\Theta}{2\hbar} \left\{(p_x^s)^2 + \left(p_y^s\right)^2 - \frac{eB}{2} L_z\right\}, \tag{22}$$

$$\begin{aligned} H_\tau = \frac{\tau}{2m_e} \Bigg\{ & \frac{e^2B^2}{2} (x^s)^2 (y^s)^2 + \frac{(y^s)^2 (p_y^s)^2}{2} + \frac{(p_y^s)^2 (y^s)^2}{2} \\ & + p_y^s (y^s)^2 p_y^s - eB \left(p_y^s (y^s)^2 x^s + \frac{p_y^s x^s (y^s)^2}{2} + \frac{(y^s)^2 p_y^s x^s}{2} \right) \Bigg\}, \end{aligned} \tag{23}$$

where

$$L_z = \left(\overrightarrow{r}^s \times \overrightarrow{p}^s\right)_z = p_y^s x^s - p_x^s y^s. \tag{24}$$

Knowing that H_τ and H_Θ are the perturbation Hamiltonians in which they reflect the effects of dynamical and non-dynamical NC spaces on the Pauli Hamiltonian. In the next step, we will solve the following energy equation

$$\mathscr{H}_P \left|\psi_{n,l}\right\rangle = (H_0 + H_\Theta + H_\tau) \left|\psi_{n,l}\right\rangle = E_{n,l}^{DNC} \left|\psi_{n,l}\right\rangle, \tag{25}$$

with

$$\left|\psi_{n,j}\right\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix}. \tag{26}$$

We are now investigating the effect of perturbed Hamiltonians on the Landau problem. Given that the DNC and NC parameters τ and Θ are both non-zero and very small, we can employ perturbation theory to determine the spectrum of the systems under consideration.

B. Unperturbed system

We consider the following NC system

$$(H_0 + H_\Theta) |\psi\rangle = E_{n,l}^{NC} |\psi\rangle. \tag{27}$$

Thereafter, the corresponding NC Pauli equation is given by

$$\left\{ \frac{1 + \frac{\Theta}{2l_B^2}}{2m_e} \left\{(p_x^s)^2 + \left(p_y^s\right)^2\right\} - \frac{eB\left(1 + \frac{\Theta}{4l_B^2}\right)}{2m_e} L_z + \frac{e^2B^2}{8m_e} \left\{(x^s)^2 + (y^s)^2\right\} + \sigma_z \mu_B B \right\} \left|\psi_{n,l}\right\rangle = E_{n,l}^{NC} |\psi\rangle, \tag{28}$$

with $l_B = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length. Now, we have

$$\tilde{m} = \frac{m_e}{1 + \frac{\Theta}{2l_B^2}}, \quad \tilde{\omega} = \frac{eB}{2m_e}\left(1 + \frac{\Theta}{4l_B^2}\right), \tag{29}$$

and

$$\frac{1}{2}\tilde{m}\tilde{\omega}^2 = \frac{e^2B^2}{8m_e} \frac{\left(1 + \frac{\Theta}{4l_B^2}\right)^2}{1 + \frac{\Theta}{2l_B^2}}, \tag{30}$$

but, as long as Θ is very small, therefore we use the series expansion $\left(1 + \frac{\Theta}{4l_B^2}\right)^2 = 1 + \frac{\Theta}{2l_B^2}$, then equation (30) becomes

$$\frac{1}{2}\tilde{m}\tilde{\omega}^2 = \frac{e^2B^2}{8m_e}. \tag{31}$$

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Using the above two-equations (29), (31), the Pauli equation reads

$$\left\{\frac{(p_x^s)^2+\left(p_y^s\right)^2}{2\tilde{m}}+\frac{\tilde{m}\tilde{\omega}^2}{2}\left\{(x^s)^2+(y^s)^2\right\}-\tilde{\omega}L_z+\sigma_z\mu_B B\right\}\left|\psi_{n,j}\right\rangle=E_{n,j}^{NC}\left|\psi_{n,j}\right\rangle. \tag{32}$$

We suppose that $\tilde{\omega}$ is a deformed cyclotron frequency, and in $\Theta\rightarrow 0$ limit, $\tilde{\omega}$ is reduced to $\frac{\omega_c}{2}=\frac{eB}{2m_e}$.

Our system looks like a 2D harmonic oscillator (HO) with an additional interaction terms, namely $\sigma_z\mu_B B-\tilde{\omega}L_z$. This system corresponds to the Landau level problem, it corresponds to the motion of a charged particle in the xy plane and subjected to a uniform MF and in interaction with its orbital and spin angular momentum. However, the equation (32) can be written as

$$\left\{\mathscr{H}_{\mathrm{HO}}^{NC}-\tilde{\omega}L_z+\mu_B\sigma_z B\right\}\left|\psi_{n,l}\right\rangle=E_{n,j}^{NC}\left|\psi_{n,l}\right\rangle. \tag{33}$$

This system will be merely solved by introducing the following deformed creation and annihilation operators of the HO

$$\begin{aligned}\tilde{a}_d&=\frac{1}{2}\sqrt{\frac{m_e\tilde{\omega}}{\hbar}}(x^s-iy^s)+\frac{i}{2}\sqrt{\frac{1}{\hbar\tilde{\omega}m_e}}\left(p_x^s-ip_y^s\right),\\ \tilde{a}_g&=\frac{1}{2}\sqrt{\frac{m_e\tilde{\omega}}{\hbar}}(x^s+iy^s)+\frac{i}{2}\sqrt{\frac{1}{\hbar\tilde{\omega}m_e}}\left(p_x^s+ip_y^s\right),\end{aligned} \tag{34}$$

and

$$\begin{aligned}\tilde{a}_d^\dagger&=\frac{1}{2}\sqrt{\frac{m_e\tilde{\omega}}{\hbar}}(x^s+iy^s)-\frac{i}{2}\sqrt{\frac{1}{\hbar\tilde{\omega}m_e}}\left(p_x+ip_y\right),\\ \tilde{a}_g^\dagger&=\frac{1}{2}\sqrt{\frac{m_e\tilde{\omega}}{\hbar}}(x^s-iy^s)-\frac{i}{2}\sqrt{\frac{1}{\hbar\tilde{\omega}m_e}}\left(p_x^s-ip_y^s\right),\end{aligned} \tag{35}$$

which satisfy the following commutation relations

$$\left[\tilde{a}_i,\tilde{a}_i^\dagger\right]=1,\quad [\tilde{a}_i,a_i]=\left[\tilde{a}_i^\dagger,\tilde{a}_i^\dagger\right]=0,\quad i=d,g. \tag{36}$$

In terms of the ladder operators (34, 35), the Hamiltonian terms in equation (33) can be re-written as

$$L_z=\hbar\left(\tilde{a}_d^\dagger\tilde{a}_d-\tilde{a}_g^\dagger\tilde{a}_g\right), \tag{37}$$

$$\mathscr{H}_{\mathrm{HO}}^{NC}=\hbar\tilde{\omega}\left(\tilde{a}_d^\dagger\tilde{a}_d+\tilde{a}_g^\dagger\tilde{a}_g+1\right)-\hbar\tilde{\omega}\left(\tilde{a}_d^\dagger\tilde{a}_d-\tilde{a}_g^\dagger\tilde{a}_g\right)=\hbar\tilde{\omega}\left(2\tilde{a}_g^\dagger\tilde{a}_g+1\right). \tag{38}$$

The eigenstates of our Hamiltonian terms are labeled by the numbers n and l of excitation quanta of the oscillators $\tilde{a}_g$ and $\tilde{a}_d$, respectively.

$$\tilde{a}_d^\dagger\tilde{a}_d\left|n,l\right\rangle=l\left|n,l\right\rangle \text{ and } \tilde{a}_g^\dagger\tilde{a}_g\left|n,l\right\rangle=n\left|n,l\right\rangle, \tag{39}$$

where both n and j can take on any positive integer value. Therefore, our NC Pauli system (33) becomes

$$\left\{\hbar\tilde{\omega}\left(3\tilde{a}_g^\dagger\tilde{a}_g-\tilde{a}_d^\dagger\tilde{a}_d+1\right)+\sigma_z\mu_B B\right\}\left|n,l\right\rangle=E_{n,l}^{NC}\left|n,l\right\rangle, \tag{40}$$

with ± 1 are the eigenvalue of σ_z, therefore, our system energy spectrum is

$$E_{n,l}^{NC}=\hbar\tilde{\omega}\left(3n-l+1\right)\pm\mu_B B. \tag{41}$$

The effect of the NC space is reduced in the parameter $\tilde{\omega}$, then for positive eigenvalue of σ_z, one can write

$$E_{n,l}^{NC}=\left(1+\frac{\Theta}{(2l_B)^2}\right)\frac{eB}{2m_e}\left(3n-l+1\right)+\mu_B B. \tag{42}$$

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Note that the NC Pauli system has been thoroughly examined in Ref. [38]. Now, as we approach the limit $\Theta \to 0$, we can observe the behavior of the unperturbed energy spectrum

$$E_{n,l} = \frac{eB}{2m_e}(3n - l + 1) + \mu_B B,$$

where the NC energy spectrum becomes commutative one, i.e. commutative Landau system [38,45].

C. Perturbed system

As long as the DNC parameterτ is very small compared to the energy scales of the system, thus we treat the DNC effect as a perturbation. We use the time-independent perturbation theory to investigate the system in τ-space. Note that in the commutative space, $\Theta = 0$ and $\tilde{\omega} \to \omega = \frac{eB}{2m_e}$, and the equations (34), (35) give rise to creation and annihilation operators of the HO [46], i.e. a_d, $a_d^\dagger$ and a_g, $a_g^\dagger$. Now, we consider the first term in equation (23)

$$\mathscr{T}_1 = \frac{\tau}{2m_e}\frac{e^2B^2}{2}(x^s)^2(y^s)^2, \tag{43}$$

then using the ladder operators of the HO, one has

$$\begin{cases} x^s = \frac{1}{2\beta}\left(a_d + a_d^\dagger + a_g + a_g^\dagger\right), \\ y^s = \frac{i}{2\beta}\left(a_d - a_d^\dagger - a_g + a_g^\dagger\right), \end{cases} \qquad \begin{cases} p_x^s = \frac{i\hbar\beta}{2}\left(-a_d + a_d^\dagger - a_g + a_g^\dagger\right), \\ p_y^s = \frac{\hbar\beta}{2}\left(a_d + a_d^\dagger - a_g - a_g^\dagger\right), \end{cases} \tag{44}$$

with

$$\beta = \sqrt{\frac{m_e\omega_c}{2\hbar}} = \frac{1}{l_B\sqrt{2}} \text{ and } \omega_c = 2\omega = \frac{eB}{m_e}. \tag{45}$$

Then by introducing

$$n = n_d + n_g \text{ and } l = n_d - n_g. \tag{46}$$

The eigenkets of the system are given as follows

$$\left| n_d = \frac{n+l}{2}, n_g = \frac{n-l}{2} \right\rangle. \tag{47}$$

So

$$\begin{aligned} (x^s)^2 &= \tfrac{1}{4\beta^2}\left(a_d + a_d^\dagger + a_g + a_g^\dagger\right)\left(a_d + a_d^\dagger + a_g + a_g^\dagger\right) \\ &= \tfrac{1}{4\beta^2}\Big\{a_d^2 + a_d a_d^\dagger + a_d a_g + a_d a_g^\dagger + a_d^\dagger a_d + \left(a_d^\dagger\right)^2 + a_d^\dagger a_g \\ &+ a_d^\dagger a_g^\dagger + a_g a_d + a_g a_d^\dagger + a_g^2 + a_g a_g^\dagger + a_g^\dagger a_d + a_g^\dagger a_d^\dagger + a_g^\dagger a_g + \left(a_g^\dagger\right)^2\Big\}, \end{aligned} \tag{48}$$

and

$$\begin{aligned} (y^s)^2 &= \tfrac{-1}{4\beta^2}\left(a_d - a_d^\dagger - a_g + a_g^\dagger\right)\left(a_d - a_d^\dagger - a_g + a_g^\dagger\right) \\ &= \tfrac{-1}{4\beta^2}\Big\{a_d^2 - a_d a_d^\dagger - a_d a_g + a_d a_g^\dagger - a_d^\dagger a_d + \left(a_d^\dagger\right)^2 + a_d^\dagger a_g \\ &- a_d^\dagger a_g^\dagger - a_g a_d + a_g a_d^\dagger + a_g^2 - a_g a_g^\dagger + a_g^\dagger a_d - a_g^\dagger a_d^\dagger - a_g^\dagger a_g + \left(a_g^\dagger\right)^2\Big\}. \end{aligned} \tag{49}$$

It is obvious that $(x^s)^2(y^s)^2$contains 256 terms. However, in what follows we calculate the correction due to this term on the energy of the ground state of the system ($n = 0$, $l = 0$) i.e., $\langle 0,0|\,\mathscr{T}_1\,|0,0\rangle = \langle 0,0|\,\tau\frac{e^2B^2}{4m_e}(x^s)^2(y^s)^2\,|0,0\rangle$. After detailed calculations, we find out that only the contributions of the following terms are non-zero

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$$\begin{cases} \langle 0,0|\, a_d^2 \left(a_d^\dagger\right)^2 |0,0\rangle = 2, \\ \langle 0,0|\, -a_d a_d^\dagger a_d a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_d a_g a_g^\dagger a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_d a_d^\dagger a_g a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_d a_g a_d^\dagger a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_g a_d a_g^\dagger a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_g a_g^\dagger a_d a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_g a_d a_d^\dagger a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_g a_g^\dagger a_g a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, a_g^2 \left(a_g^\dagger\right)^2 |0,0\rangle = 2. \end{cases} \tag{50}$$

Therefore, we have

$$\langle 0,0|\, \mathscr{T}_1 |0,0\rangle = \frac{e^2B^2}{16m_e\beta^4}\tau = \frac{1}{4}\tau\frac{\hbar^2}{m_e}. \tag{51}$$

Now, by the same method, one can calculate the contributions of the other terms on the energy of the ground state of the system. So let us consider the term $\mathscr{T}_2 = \frac{-\tau eB}{2m_e} p_y^s \left(y^s\right)^2 x^s$, again there are 256 terms, however, the non-zero contributions are as follows

$$\begin{cases} \langle 0,0|\, -a_d^2 \left(a_d^\dagger\right)^2 |0,0\rangle = -2, \\ \langle 0,0|\, a_d a_d^\dagger a_d a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, a_d a_g a_g^\dagger a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, a_d a_d^\dagger a_g a_g^\dagger |0,0\rangle = 1, \\ \langle 0,0|\, a_d a_g a_d^\dagger a_g^\dagger |0,0\rangle = 1, \\ \langle 0,0|\, -a_g a_d a_g^\dagger a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_g a_g^\dagger a_d a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, a_g a_d a_d^\dagger a_g^\dagger |0,0\rangle = 1, \\ \langle 0,0|\, a_g a_g^\dagger a_g a_g^\dagger |0,0\rangle = 1, \\ \langle 0,0|\, a_g^2 \left(a_g^\dagger\right)^2 |0,0\rangle = 2. \end{cases} \tag{52}$$

So, we have

$$\langle 0,0|\, \mathscr{T}_2 |0,0\rangle = \langle 0,0|\, \frac{-eB\tau}{2m_e} p_y^s \left(y^s\right)^2 x^s |0,0\rangle = 0. \tag{53}$$

As long as the operators x^s and $\left(y^s\right)^2$ are commutative, thus the contribution of $\mathscr{T}_3 = \frac{-eB\tau}{2m_e}\frac{p_y^s x^s \left(y^s\right)^2}{2}$ on the energy of the ground state also vanishes. Now, we consider the term $\mathscr{T}_4 = \frac{-eB\tau}{2m_e}\frac{\left(y^s\right)^2 p_y^s x^s}{2}$, so the non-vanishing contributions are as follows

$$\begin{cases} \langle 0,0|\, a_d^2 \left(a_d^\dagger\right)^2 |0,0\rangle = 2, \\ \langle 0,0|\, -a_d a_d^\dagger a_d a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, a_d a_g a_g^\dagger a_d^\dagger |0,0\rangle = 1, \\ \langle 0,0|\, a_d a_d^\dagger a_g a_g^\dagger |0,0\rangle = 1, \\ \langle 0,0|\, -a_d a_g a_d^\dagger a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, a_g a_d a_g^\dagger a_d^\dagger |0,0\rangle = 1, \\ \langle 0,0|\, -a_g a_g^\dagger a_d a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, -a_g a_d a_d^\dagger a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0|\, a_g a_g^\dagger a_g a_g^\dagger |0,0\rangle = 1, \\ \langle 0,0|\, a_g^2 \left(a_g^\dagger\right)^2 |0,0\rangle = -2. \end{cases} \tag{54}$$

Also, the contribution of $\mathscr{T}_4$ vanishes. Then, we consider $\mathscr{T}_5 = \frac{\tau}{2m_e} p_y^s (y^s)^2 p_y^s$, so the non-vanishing contributions are

$$\begin{cases} \langle 0,0| - a_d^2 \left(a_d^\dagger\right)^2 |0,0\rangle = -2, \\ \langle 0,0| - a_d a_d^\dagger a_d a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0| - a_d a_g a_g^\dagger a_d^\dagger |0,0\rangle = -1, \\ \langle 0,0| - a_d a_d^\dagger a_g a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0| - a_d a_g a_d^\dagger a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0| a_g a_d a_g^\dagger a_d^\dagger |0,0\rangle = 1, \\ \langle 0,0| a_g a_g^\dagger a_d a_d^\dagger |0,0\rangle = 1, \\ \langle 0,0| - a_g a_d a_d^\dagger a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0| - a_g a_g^\dagger a_g a_g^\dagger |0,0\rangle = -1, \\ \langle 0,0| - a_g^2 \left(a_g^\dagger\right)^2 |0,0\rangle = -2, \end{cases} \tag{55}$$

however, the contribution of $\mathscr{T}_5$ on the energy shift of the ground state of the system is

$$\langle 0,0| \mathscr{T}_5 |0,0\rangle = \frac{1}{4}\tau\frac{\hbar^2}{m_e}. \tag{56}$$

Now with the same manner, the non-vanishing contributions of $\mathscr{T}_6 = \frac{\tau}{2m_e}\frac{(y^s)^2 \left(p_y^s\right)^2}{2}$ leads to

$$\langle 0,0| \mathscr{T}_6 |0,0\rangle = -\frac{1}{16}\tau\frac{\hbar^2}{m_e}, \tag{57}$$

and finally, the non-vanishing contributions of $\mathscr{T}_7 = \frac{\tau}{2m_e}\frac{\left(p_y^s\right)^2 (y^s)^2}{2}$ also leads to obtaining

$$\langle 0,0| \mathscr{T}_7 |0,0\rangle = -\frac{1}{16}\tau\frac{\hbar^2}{m_e}. \tag{58}$$

Using the above different terms (51, 56, 57 and 58), the first-order correction to the energy eigenvalues, i.e. the energy shift for the ground state, i.e. is given by

$$\triangle E = E_0^{(1)} = \frac{3}{8}\tau\frac{\hbar^2}{m_e}. \tag{59}$$

Now, the expression for the energy eigenvalues of the DNC Landau problem, encompassing first-order correction, is given as

$$E_{n,l}^{DNC} = \left\{\left(1 + \frac{\Theta}{(2l_B)^2}\right)\frac{(3n - l + 1)}{2m_e}e + \mu_B\right\}B + \frac{3}{8}\tau\frac{\hbar^2}{m_e}. \tag{60}$$

After acquiring the energy eigenvalues for the DNC Landau problem, we can delve into a range of calculations to thoroughly explore the system's characteristics. These calculations extend to scrutinizing the Landau level degeneracy, elucidating the energy gap, performing a detailed wave function analysis, and investigating various quantum transport properties. The latter includes crucial quantities like Hall conductivity, Hall resistivity, and magnetoconductivity. These analyses contribute to a comprehensive understanding of the system's behavior and pave the way for further insights into its quantum properties. Furthermore, it is crucial to examine the effects of the DNC space on these properties and compare them with the results obtained from the ordinary Landau problem. Through these comparative analyses, we can gain valuable insights into the specific characteristics of the DNC

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Landau problem and discern its implications across diverse physical contexts. These calculations have the potential to reveal novel phenomena that emerge from the interplay between DNC space, quantum mechanics, and external influences. This exploratory approach significantly contributes to a more profound comprehension of the system, shedding light on its conduct under unique and intriguing conditions. Next, our focus shifts towards investigating the magnetoconductivity property of the DNC Landau problem.

D. Magnetoconductivity

The magnetoconductivity $\triangle\sigma$ is a physical quantity that quantifies how electrical conductivity of a material changes in the presence of an external MF. It is measure of how the variation in a material's electrical conductivity due to the effect of the MF. By using the eigenvalues (60) of the deformed Landau problem, one can calculate the magnetoconductivity, which is typically expressed in terms of the change in conductivity as a function of the MF. It can be calculated using the following formula [47]:

$$\triangle\sigma = \sigma\left(B\right) - \sigma\left(0\right), \tag{61}$$

with $\sigma(B)$ and $\sigma(0)$ correspond to the presence and absence of MF, respectively. However, the conductivity can be related to the eigenenergy through a concept called the '*Kubo formula*' [48] (for an overview, see [49,50]). This formula relates the conductivity to the current-current correlation function, which can be calculated using the eigenstates and eigenvalues of the system. The Kubo formula for the conductivity in a general system is given by [51]

$$\sigma\left(B\right) = \frac{e}{\Omega^2}\sum_{n,m}\frac{f\left(E_n\right) - f\left(E_m\right)}{E_n - E_m}\left|\langle n|\,\overrightarrow{\mathscr{J}}\,|m\rangle\right|^2, \tag{62}$$

where Ω is the volume of the system, $f(E_n)$ and $f(E_m)$are the Fermi-Dirac distribution functions corresponding to the energy eigenvalues E_n and E_m, $|n\rangle$ and $|m\rangle$ are the eigenstates. The term $\frac{f(E_n)-f(E_m)}{E_n-E_m}$ captures the difference in occupation probabilities between the states and the energy difference and $\overrightarrow{\mathscr{J}}$ is the current density operator. The modulus squared of the matrix element $\left|\left\langle n \mid \overrightarrow{\mathscr{J}} \mid m\right\rangle\right|^2$ is taken to obtain the contribution to the conductivity. The formula (62) is derived from the linear response theory and is valid in the regime of non-interacting systems. It assumes that the system is in thermal equilibrium with a temperature and that the Fermi-Dirac distribution function $f(E)$ characterizes the occupation probabilities of the energy levels. It is important to note that the Kubo formula is a general framework, and its practical application depends on the system under consideration and the availability of suitable models or approximations.

Now, for the electron with mass $\mathrm{m_e}$ and charge e,

$$\overrightarrow{\mathscr{J}} = e\overrightarrow{v} = \frac{e}{m_e}\overrightarrow{p}, \tag{63}$$

and in the case of non-relativistic QM, the velocity operator is given by the following commutator

$$\overrightarrow{v} = \frac{1}{i\hbar}\left[\overrightarrow{r}^s, \mathscr{H}_P\right]. \tag{64}$$

But in the x-direction, one has

$$\mathscr{J}_x = ev_x = \frac{e}{i\hbar}\left[x^s, \mathscr{H}_P\right], \tag{65}$$

then, we have

$$\sigma_x\left(B\right) = \frac{e}{\Omega^2}\sum_{n,m}\frac{f\left(E_n\right) - f\left(E_m\right)}{E_n - E_m}\left|\langle n|\,\mathscr{J}_x\,|m\rangle\right|^2. \tag{66}$$

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The y-component, $\mathscr{J}_y$ can be treated similarly. The x-component of the magnetoconductivity can be considered as follows

$$\triangle\sigma_x = \sigma_x(B) - \sigma_x(0), \tag{67}$$

note that the x-component of the magnetoconductivity specifically quantifies the change in conductivity in the x-direction due to the applied MF. Now, substituting the expression for the Hamiltonian $\mathscr{H}_P$ and evaluating the derivatives, we obtain the simplified expression for v_x

$$v_x = \frac{1}{m_e}\left(1 + \frac{\Theta}{2l_B^2}\right)p_x^s + \frac{eB}{2m_e}\left(1 + \frac{\Theta}{4l_B^2}\right)y^s, \tag{68}$$

thereafter, the current density operator expression is

$$\mathscr{J}_x = \mathscr{A}p_x^s + \mathscr{B}y^s. \tag{69}$$

with

$$\mathscr{A} = \frac{e}{m_e}\left(1 + \frac{\Theta}{2l_B^2}\right), \text{and } \mathscr{B} = \frac{e^2B}{2m_e}\left(1 + \frac{\Theta}{4l_B^2}\right). \tag{70}$$

Then, using equation (44), we calculate $\langle n|\,\mathscr{A}p_x^s\,|m\rangle$

$$\begin{aligned}\langle n|\,\mathscr{A}p_x^s\,|m\rangle &= \left(\tfrac{i\hbar\beta}{2}\right)\mathscr{A}\Big(-\langle n|\,a_d\,|m\rangle + \langle n|\,a_d^\dagger\,|m\rangle - \langle n|\,a_g\,|m\rangle + \langle n|\,a_g^\dagger\,|m\rangle\Big)\\ &= 2\left(\tfrac{i\hbar\beta}{2}\right)\mathscr{A}\left\{\sqrt{m+1}\delta_{n,m+1} - \sqrt{m}\delta_{n,m-1}\right\},\end{aligned} \tag{71}$$

and for $(\langle n|\,\mathscr{J}_x\,|m\rangle)^* = \langle m|\,\mathscr{J}_x^\dagger\,|n\rangle$, we have

$$\begin{aligned}\langle m|\,\mathscr{A}p_x^s\,|n\rangle &= \left(\tfrac{i\hbar\beta}{2}\right)^*\mathscr{A}\Big(-\langle m|\,a_d^\dagger\,|n\rangle + \langle m|\,a_d\,|n\rangle - \langle m|\,a_g^\dagger\,|n\rangle + \langle m|\,a_g\,|n\rangle\Big)\\ &= 2\left(\tfrac{i\hbar\beta}{2}\right)^*\mathscr{A}\left\{\sqrt{n}\delta_{m,n-1} - \sqrt{n+1}\delta_{m,n+1}\right\}.\end{aligned} \tag{72}$$

For $\mathscr{B}y^s$, we have

$$\begin{aligned}\langle n|\,\mathscr{B}y^s\,|m\rangle &= \left(\tfrac{i}{2\beta}\right)\mathscr{B}\Big(\langle n|\,a_d\,|m\rangle - \langle n|\,a_d^\dagger\,|m\rangle - \langle n|\,a_g\,|m\rangle + \langle n|\,a_g^\dagger\,|m\rangle\Big)\\ &= 2\left(\tfrac{i}{2\beta}\right)\mathscr{A}\left\{\sqrt{m}\delta_{n,m-1} - \sqrt{m+1}\delta_{n,m+1} - \sqrt{m}\delta_{n,m-1} + \sqrt{m+1}\delta_{n,m+1}\right\} = 0,\end{aligned} \tag{73}$$

so the contribution of y^s is zero. Now, for $|\langle n|\,\mathscr{J}_x\,|m\rangle|^2$ we have

$$\begin{aligned}|\langle n|\,\mathscr{J}_x\,|m\rangle|^2 &= |\langle n|\,\mathscr{A}p_x^s\,|m\rangle|^2 = \left|\tfrac{i\hbar\beta}{2}\right|^2|\mathscr{A}|^2\left|\langle n|-a_d + a_d^\dagger - a_g + a_g^\dagger\,|m\rangle\right|^2\\ &= 4\left|\tfrac{i\hbar\beta}{2}\right|^2|\mathscr{A}|^2\Big\{\sqrt{n}\sqrt{(m+1)}\delta_{n,m+1}\delta_{m,n-1} - \sqrt{(n+1)}\sqrt{(m+1)}\delta_{n,m+1}\delta_{m,n+1}\\ &\quad -\sqrt{n}\sqrt{m}\delta_{n,m-1}\delta_{m,n-1} + \sqrt{m}\sqrt{(n+1)}\delta_{n,m-1}\delta_{m,n+1}\Big\}.\end{aligned} \tag{74}$$

Note that $\delta_{n,m+1}$ and $\delta_{m,n-1}$ are the same, but for the second term of the above equation, and from $\delta_{n,m+1}$, we have n = m + 1, and from $\delta_{m,n+1}$, we have n = m − 1, thus it is impossible to have both these relations at the same time, so that the contribution of this term is zero. With the same method one can show that the contribtuion of the third term is also zero, and for the fourth term, from $\delta_{n,m-1}$, $\delta_{m,n+1}$, we have n = m − 1, which means $\delta_{n,m-1}$, $\delta_{m,n+1}$ are the same. Therefore, equation (74) becomes

$$|\langle n|\,\mathscr{J}_x\,|m\rangle|^2 = 4\left|\frac{i\hbar\beta}{2}\right|^2|\mathscr{A}|^2\left\{\sqrt{n(m+1)}\delta_{n,m+1} + \sqrt{m(n+1)}\delta_{n,m-1}\right\}. \tag{75}$$

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Then equation (66) is

$$\begin{aligned}\sigma_x(B) &= \frac{4e\left|\frac{i\hbar\beta}{2}\right|^2|\mathscr{A}|^2}{\Omega^2}\sum_{n,m}\frac{f(E_n)-f(E_m)}{E_n-E_m}\left\{\sqrt{n(m+1)}\delta_{n,m+1}+\sqrt{m(n+1)}\delta_{n,m-1}\right\} \\ &= \frac{4e\left|\frac{i\hbar\beta}{2}\right|^2|\mathscr{A}|^2}{\Omega^2}\left\{(m+1)\frac{f(E_{m+1})-f(E_m)}{E_{m+1}-E_m}+m\frac{f(E_{m-1})-f(E_m)}{E_{m-1}-E_m}\right\}.\end{aligned} \tag{76}$$

Now, we assume that if $m \leq M$ then $f(E_m) = 1$; and if $m > M$ then $f(E_m) = 0$, where M stands for the Fermi level, i.e., $E_M = E_{\text{Fermi}}$, so equation (76) becomes

$$\sigma_x(B) = \frac{4e\left|\frac{i\hbar\beta}{2}\right|^2|\mathscr{A}|^2}{\Omega^2}\left\{(M+1)\frac{-1}{E_{M+1}-E_M}+M\frac{1-1}{E_{M-1}-E_M}\right\}, \tag{77}$$

with $E_{M+1} - E_M = \hbar\omega$, then

$$\sigma_x(B) = \frac{-4e\left|\frac{i\hbar\beta}{2}\right|^2|\mathscr{A}|^2}{\hbar\omega\Omega^2}(M+1), \tag{78}$$

by using equations (45, 70), we simplify equation (78) to have

$$\begin{aligned}\triangle\sigma_x = \sigma_x(B) &= \frac{-\frac{\hbar^2}{2l_B^2}\frac{e^3}{m_e^2}\left(1+\frac{\Theta}{2l_B^2}\right)^2}{\hbar\omega\Omega^2}(M+1) \\ &= \frac{-2e}{\hbar\omega}\left(\frac{\mu_B}{\Omega l_B}\right)^2\left\{1+\frac{\Theta}{l_B^2}\right\}(M+1)+\mathcal{O}(\Theta^2)\end{aligned}. \tag{79}$$

M refers to Fermi level. For instance, for low temperatures, only the ground state $M = 0$ and first excited states are occupied, $M = 1$.

Therefore, in summary, as noted, the magnetoconductivity of the DNC Landau problem in the low-temperature regime is independent of the DNC parameter but is associated with the NC parameter.

E. Upper bound on dynamical parameter

In this section we impose an upper bound on the DNC parameters τ. So, using equation (59) and the accuracy of energy measurement 10^{-12} eV [52], one can find an upper bound on τ

$$\frac{3}{8}\tau\frac{\hbar^2}{m_e} \leq 10^{-12}\ \text{eV}, \tag{80}$$

after some calculations we have

$$\sqrt{\tau} \leq 5.916 \times 10^3\ \text{m}^{-1}. \tag{81}$$

Using the relation 1 $\text{Fermi}^{-1} \approx 200$ MeV, one can find

$$\sqrt{\tau} \leq 1.1832 \times 10^{-9}\ \text{MeV} = 1.1832 \times 10^{-3}\ \text{eV}, \tag{82}$$

which is much better than our previous upper bound [3].

IV. Conclusion

In this study, we explored the effects of both NC and DNC spaces on the Landau problem, a system governed by a 2D Pauli equation using time-independent perturbation theory. We obtained the energy spectrum, which depends on the parameters Θ and τ. Moreover, we delved into the study of magnetoconductivity using the Kubo formula, providing useful insights into magnetoconductivity behavior. It is noteworthy that in the field of condensed matter physics, various techniques, such as the Boltzmann transport equation and Landauer-Büttiker formalism, are employed to calculate conductivity in conducting materials. However, our findings revealed that DNC space had no effect on the x-component of magnetoconductivity, while NC space exhibits some influences. Furthermore, using the accuracies of energy measurement, we established an upper bound on the DNC parameter, i.e., $\sqrt{\tau} \leq 1.1832 \times 10^{-3}$ eV. Note that exploring fundamental phenomena within DNC spaces is intriguing, where the motivation for studying these spaces lies in the fact that the objects in DNC spaces are naturally of string type. In the limits of $\tau \to 0$ and $\Theta \to 0$, the deformed Landau problem reduces to that of ordinary QM, confirming that our results are compatible and reductive. Obviously, our results can be considered important and helpful tool for investigating more related studies, including thermodynamical properties and other scenarios of deformation parameters.

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