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Article

Phase Diffusion Mitigation in the Truncated Mach-Zehnder Interferometer

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Abstract: The presence of phase diffusion can lead to the loss of the advantages of quantum measurements. Squeezing is considered to be an effective method to reduce the detrimental effect of phase diffusion on the measurement. The Mach-Zehnder (MZ) interferometer has been exploited as a generic tool for precise phase measurement. Describing the reduction in quantum advantage caused by phase diffusion in an MZ interferometer that can be mitigated by squeezing is not easy to handle analytically, because the input state will change from a pure state to a mixed state after experiencing the diffusion noise in the MZ interferometer. We introduce a truncated MZ interferometer that can achieve the same potential phase sensitivity as the conventional MZ interferometer. This scheme can theoretically explain how phase diffusion reduces phase estimation and why squeezing counteracts in the presence of phase diffusion. Using the Gaussian property of the input state and the characteristic of Gaussian operation in the squeezing, the two orthogonal field quantities of the quantum state are squeezed and anti-squeezed to different degrees, and the analytic results are obtained. The truncated MZ interferometer is simpler to build and operate than the conventional MZ interferometer, and also mitigates the phase diffusion noise by the squeezing operation, thus making it useful for applications in precision metrology.

Keywords: phase diffusion; truncated Mach-Zehnder interferometer; squeezing operation; noise mitigation

1. Introduction

Quantum metrology is fundamental in physics and invaluable for a wide range of applications, attracting many researchers' attention [1–3]. Sometimes, the parameters cannot be measured directly in the quantum light field, so the phase of the light field is often chosen as an ideal parameter for measurement [4–6] and the scheme of phase estimation is theoretically proposed [7–9], mainly optical interferometry and homodyne measurements [10,11]. Interferometers can provide the most precise measurements. Recently, physicists with the advanced Laser Interferometer Gravitational-Wave Observatory (LIGO) observed the gravitational waves. The Mach-Zehnder (MZ) interferometer and its variants have been used as a generic model to realize precise measurement of phase.

However, the light field will inevitably interact with the environment in the dynamic process and be disturbed by noise, such as photon loss and the most damaging phase diffusion noise, which have been studied by many researchers [12–24]. For interferometers, the photon losses is a typical decoherence process which should be taken into account. The general frame for estimating the ultimate precision limit in the presence of photon loss has been analyzed [12–15], where this decoherence process can be described by a set of Kraus operators, and the corresponding lower bounds in quantum

metrology is given by the quantum Cramér-Rao bound (QCRB) usage of quantum Fisher information (QFI) [25]. It establishes the best precision that can be attained with a given quantum probe [26–37].

The role of phase-diffusive noise in phase estimation has been investigated for qubit [38–40], condensate systems [41,42], and a continuous-variable system [22]. This noise is considered the most detrimental because it destroys the off-diagonal elements of the density matrix. If this input quantum state is not affected by phase diffusion, its phase shift will not change, but this quantum state is utterly useless for phase estimation [22]. Escher et al. [43] presented a variational approach to show an analytical bound based on purification techniques, which has an explicit dependence on the mathematical description of the noise. The weak measurements and joint estimation were used to deal with the phase and phase diffusion simultaneously for quantum metrology [44,45].

The optimizations of phase estimation with photon losses have been discussed [46]. According to the QCRB, the first beam splitter should be unbalanced to improve the precision when photon losses are unbalanced, regardless of the detection method. Unbalanced losses are unavoidable for multipass interferometry, such as, in the LISA proposal [47,48], where the signal interference arm suffering significant propagation loss is designed to return and interfere with the lossless local reference arm. The optimization of splitting ratio is helpful for applications with significant unbalanced loss, which is realized in experiment [49,50]. On the other hand, how to counteract the adverse effects caused by phase diffusion was also studied. In recent years, the scheme of using an optical parametric oscillator (OPO) to counteract phase diffusion has attracted attention and has been implemented experimentally [51,52]. Inspired by this, injecting squeezing directly after encoding is also considered an ideal solution. Squeezing has been theoretically explained as an effective means of reducing phase diffusion, consistent with existing experimental phenomena [53], thus providing a basis for subsequent loss research and new ideas. However, due to the emergence of phase noise, we will not be able to obtain the upper bound of the QFI that can be analyzed [54], which makes the result not look so clear. Therefore, it is worth studying whether the phase diffusion can be reduced in the MZ interferometer and its working principle can be explained analytically.

In this paper, we introduce a truncated MZ interferometer, similar to the truncated SU(1,1) interferometer [55], which not only achieves the same potential phase sensitivity as the traditional MZ interferometer, but also simplifies the theoretical treatment of phase diffusion noise. In this truncated MZ interferometer, combined with the relevant characteristics of the Gaussian state [56], the Gaussian state is injected, and then a model is theoretically constructed to counteract the phase noise, and the phase uncertainty is selected as an appropriate parameter for the measurement results. Mixed states caused by phase noise are challenging to theoretically handle in conventional interferometers. However, a truncated MZ interferometer can solve this problem.

2. Truncated MZ Interferometer

2.1. Model

In the traditional MZ interferometer, the light field is divided into two beams after passing through the first beam splitter and entering the interferometer's two arms. After experiencing different physical processes, the light field is combined in the second beam splitter. Then, it is output to the detector to measure the quantity we want to measure. However, the feasibility of the above scheme has certain limitations. When there is phase diffusion noise in one or both arms of the interferometer, the light field will become a mixed state before combining, making it difficult to obtain a suitable and convenient expression method that is used to describe the output state of the light field after passing through the interferometer. According to Ref. [55], we introduced a truncated MZ interferometer as an ideal measurement tool, in which the upper arm is the detection arm and the lower arm is the conjugate arm, as shown in Figure 1. After the first beam splitter splits the light field, then suffers phase shift, it finally enters the detectors behind the two arms of the interferometer respectively, avoiding the process of combining beams.

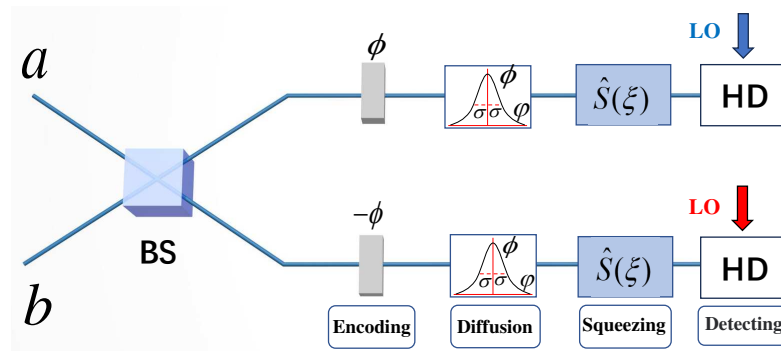


Figure 1. Schematic of the truncated Mach-Zehnder interferometer. a and b denote two light modes in the interferometer. The phase diffusion corresponds to the application of a random, Gaussian-distributed phase shift φ with variance σ^2 and the mean equal to the estimated parameter ϕ . The squeezing operation $\hat{S}(\xi)$ is introduced after the phase noise. BS: beam splitter; ϕ : phase shift; HD: homodyne detection; LO: local oscillator.

For homodyne detection, we define the quadrature operators of the probe and conjugate arms, which are given by

$$\hat{J}_p = \frac{1}{\sqrt{2}}(e^{-i\phi_p}\hat{a}_p + e^{i\phi_p}\hat{a}_p^\dagger), \quad (1)$$

$$\hat{J}_c = \frac{1}{\sqrt{2}}(e^{-i\phi_c}\hat{a}_c + e^{i\phi_c}\hat{a}_c^\dagger), \quad (2)$$

where ϕ_p and ϕ_c are the local oscillator phases at the homodyne detectors. The joint quadrature operator is

$$\hat{J} = \hat{J}_p + \hat{J}_c. \quad (3)$$

We estimate phase by two measurements performed on the state after phase encoding. In particular, when the two results of \hat{J} we measure in Equation (3) by changing ϕ_p and ϕ_c are orthogonal, we call the two results \hat{X} and \hat{Y} , then the phase to be estimated is [51,52]

$$\phi = \arctan \frac{\langle \hat{Y} \rangle}{\langle \hat{X} \rangle}. \quad (4)$$

Then, according to the law of variance propagation, we can get

$$\Delta^2\phi = \frac{\langle \hat{Y} \rangle^2 \text{Var}(\hat{X}) + \langle \hat{X} \rangle^2 \text{Var}(\hat{Y})}{[\langle \hat{Y} \rangle^2 + \langle \hat{X} \rangle^2]^2}, \quad (5)$$

where $\text{Var}(\hat{O}) = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$ ($\hat{O} = \hat{X}, \hat{Y}$). The phase uncertainty is a significant physical quantity. The smaller it is, the more accurate our measurement is.

2.2. Phase Estimation

In the truncated MZ Interferometer, we input the coherent state $|\sqrt{2}\alpha\rangle$ into one arm and the vacuum state $|0\rangle$ into the other arm. After passing through the beam splitter, there is a coherent state $|\alpha\rangle$ and $|\alpha e^{\frac{\pi}{2}i}\rangle$. Then, the phase encoding information of ϕ and $-\phi$ are programmed into the probe arm and the conjugate arm, as shown in Figure 2.

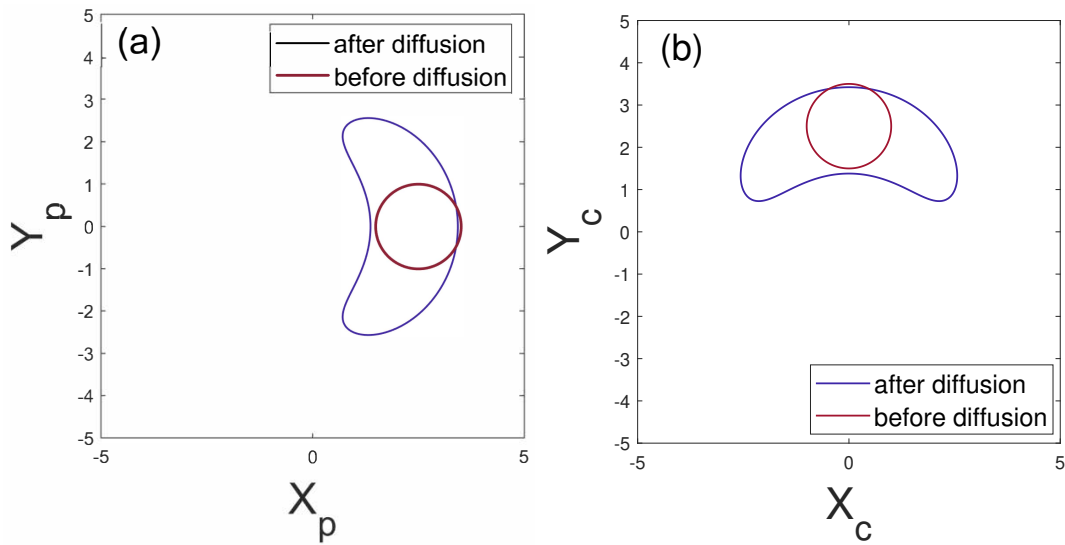


Figure 2. Phase-space representation of a coherent state $|\alpha\rangle$ (red) and its phase-diffused counterpart (blue). (a) Probe arm; (b) Conjugate arm. Parameters: $\alpha = 2.5$, $\sigma = 1.2$.

In order to obtain a set of suitable orthogonal field quantities, we choose $\phi_p = 0$, $\phi_c = \frac{\pi}{2}$ for the \hat{X} , and $\phi_p = \frac{\pi}{2}$, $\phi_c = 0$ for the \hat{Y} . Now, let us look at the results of the measurement. When $\phi_p = 0$ and $\phi_c = \frac{\pi}{2}$, we obtain

$$\langle \hat{X}_p \rangle = \sqrt{2\alpha} \cos(\phi_p + \phi) = \sqrt{2\alpha} \cos(\phi), \quad (6)$$

$$\langle \hat{X}_c \rangle = \sqrt{2\alpha} \sin(\phi_c - \phi) = \sqrt{2\alpha} \cos(\phi), \quad (7)$$

and

$$\text{Var}(\hat{X}_p) = \text{Var}(\hat{X}_c) = \frac{1}{2}, \quad (8)$$

then

$$\langle \hat{X} \rangle = \langle \hat{X}_p \rangle + \langle \hat{X}_c \rangle = 2\sqrt{2\alpha} \cos(\phi), \quad (9)$$

$$\text{Var}(\hat{X}) = \text{Var}(\hat{X}_p) + \text{Var}(\hat{X}_c) = 1. \quad (10)$$

Similarly, when $\phi_p = \frac{\pi}{2}$ and $\phi_c = 0$, we have

$$\langle \hat{Y}_p \rangle = \sqrt{2\alpha} \cos(\phi_p + \phi) = -\sqrt{2\alpha} \sin(\phi), \quad (11)$$

$$\langle \hat{Y}_c \rangle = \sqrt{2\alpha} \sin(\phi_c - \phi) = -\sqrt{2\alpha} \sin(\phi), \quad (12)$$

and

$$\text{Var}(\hat{Y}_p) = \text{Var}(\hat{Y}_c) = \frac{1}{2}, \quad (13)$$

then

$$\langle \hat{Y} \rangle = \langle \hat{Y}_p \rangle + \langle \hat{Y}_c \rangle = -2\sqrt{2\alpha} \sin(\phi), \quad (14)$$

$$\text{Var}(\hat{Y}) = \text{Var}(\hat{Y}_p) + \text{Var}(\hat{Y}_c) = 1. \quad (15)$$

Using Equation (5), we can therefore obtain the phase uncertainty in the lossless case

$$\Delta^2\phi = \frac{1}{8\alpha^2}. \quad (16)$$

This is the standard quantum limit (SQL) of truncated MZ Interferometer for coherent state $|\sqrt{2\alpha}\rangle$ input.

2.3. Phase Estimation with Phase Diffusion

Decoherence is the result of the interaction between the quantum system and the environment during its evolution. Three typical decoherence modes are phase diffusion, photon loss, and detection losses. The photon losses in the interferometer are modeled by adding fictitious beam splitters, which have been studied [13].

Here, this scheme mainly studies phase diffusion, which represents the phase noise of collective dephasing. Assuming that the phase diffusion corresponds to applying a random, Gaussian-distributed phase shift φ with variance σ^2 and the mean equal to the estimated parameter ϕ . The evolution of the initial state ρ_{in} is expressed as

$$\rho_t = \int_{-\infty}^{+\infty} d\varphi p_\varphi(\varphi) \hat{U}_\varphi \rho_{in} \hat{U}_\varphi^\dagger,$$

where $p_\varphi(\varphi) = e^{-\frac{(\varphi-\phi)^2}{2\sigma^2}} / \sqrt{2\pi\sigma^2}$, $\hat{U}_\varphi = e^{-i\varphi\hat{n}}$, and the symbol σ represents the amplitude of phase diffusion. Considering the initial input state $|\psi\rangle_{in} = \sum_{n=0}^N c_n |n\rangle$, then the evolved state may be written as

$$\begin{aligned} \rho_t &= \int_{-\infty}^{+\infty} d\varphi p_\varphi(\varphi) \hat{U}_\varphi \rho_{in} \hat{U}_\varphi^\dagger \\ &= \sum_{n,m} e^{-i\varphi(n-m)} e^{-\frac{1}{2}\sigma^2(n-m)^2} \rho_{n,m} |n\rangle\langle m|, \end{aligned} \quad (17)$$

where $\rho_{n,m} = c_n c_m^*$. The density matrix of the output mixed state is obtained when the phase noise is Gaussian noise. From the form of the above matrix, we can see that when phase diffusion exists, the matrix elements on the diagonal of the matrix are not affected by noise, indicating energy conservation, but the off-diagonal items of the matrix are suppressed exponentially, indicating that the phase information carried by the state is destroyed.

We assume that the amplitude of phase diffusion in the probe arm and conjugate arm are σ_1 and σ_2 , respectively. When noise appears, the light field in the two arms will change from a pure state to a mixed state. When $\phi_p = 0$ and $\phi_c = \frac{\pi}{2}$, we get

$$\langle \hat{X} \rangle_D = \langle \hat{X}_p \rangle_D + \langle \hat{X}_c \rangle_D = \sqrt{2}\alpha(e^{-\frac{1}{2}\sigma_1^2} + e^{-\frac{1}{2}\sigma_2^2}) \cos(\phi), \quad (18)$$

and

$$\begin{aligned} [Var(\hat{X})]_D &= [Var(\hat{X}_p)]_D + [Var(\hat{X}_c)]_D \\ &= 1 + \alpha^2[2 + (e^{-2\sigma_1^2} + e^{-2\sigma_2^2}) \cos(2\phi) \\ &\quad - 2(e^{-\sigma_1^2} + e^{-\sigma_2^2}) \cos^2(\phi)]. \end{aligned} \quad (19)$$

Similarly, when $\phi_p = \frac{\pi}{2}$ and $\phi_c = 0$, we get

$$\langle \hat{Y} \rangle_D = \langle \hat{Y}_p \rangle_D + \langle \hat{Y}_c \rangle_D = -\sqrt{2}\alpha(e^{-\frac{1}{2}\sigma_1^2} + e^{-\frac{1}{2}\sigma_2^2}) \sin(\phi), \quad (20)$$

and

$$\begin{aligned} [Var(\hat{Y})]_D &= [Var(\hat{Y}_p)]_D + [Var(\hat{Y}_c)]_D \\ &= 1 + \alpha^2[2 - (e^{-2\sigma_1^2} + e^{-2\sigma_2^2}) \cos(2\phi) \\ &\quad - 2(e^{-\sigma_1^2} + e^{-\sigma_2^2}) \sin^2(\phi)]. \end{aligned} \quad (21)$$

In the presence of phase diffusion in the truncated MZ interferometer, we get the phase uncertainty

$$\begin{aligned}
 (\Delta^2\phi)_D &= \frac{(\frac{1}{2}e^{-\frac{1}{2}\sigma_1^2} + \frac{1}{2}e^{-\frac{1}{2}\sigma_2^2})^{-2}}{8\alpha^2} + [2(e^{-\frac{1}{2}\sigma_1^2} + e^{-\frac{1}{2}\sigma_2^2})^{-2} \\
 &- (e^{-2\sigma_1^2} + e^{-2\sigma_2^2})(e^{-\frac{1}{2}\sigma_1^2} + e^{-\frac{1}{2}\sigma_2^2})^{-2} \cos^2(2\phi) \\
 &- (e^{-\frac{1}{2}\sigma_1^2} + e^{-\frac{1}{2}\sigma_2^2})^{-2}(e^{-\sigma_1^2} + e^{-\sigma_2^2}) \sin^2(2\phi)]/2.
 \end{aligned}
 \tag{22}$$

Define a physical quantity $\Lambda_{\sigma_1, \sigma_2}^j = e^{-j\sigma_1^2} + e^{-j\sigma_2^2}$ ($j = 1/2, 1, 2$) related to the loss of the two arms, then the above Equation (22) can be simplified to

$$\begin{aligned}
 (\Delta^2\phi)_D &= \frac{(\frac{1}{2}\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2}}{8\alpha^2} + \frac{1}{2}[-\Lambda_{\sigma_1, \sigma_2}^2(\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2} \cos^2(2\phi) \\
 &+ 2(\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2} - \Lambda_{\sigma_1, \sigma_2}^1(\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2} \sin^2(2\phi)].
 \end{aligned}
 \tag{23}$$

From the above calculation results, we can see that when phase diffusion noise appears, the orthogonal field quantities of the quantum states in the two arms will be expanded to varying degrees, thus making the phase uncertainty even greater. Moreover, due to the beam-splitting effect of the MZ interferometer, there is an initial phase difference between the two arms, so the quantum states in the detection arm and the conjugate arm will change in different directions in the phase space, as shown in Fig. 2. At the optimal point $\phi = 0$, the phase uncertainty is given by

$$\begin{aligned}
 (\Delta^2\phi)_D &= \frac{1}{8\alpha^2}(\frac{1}{2}\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2} \\
 &+ \frac{2(\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2} - \Lambda_{\sigma_1, \sigma_2}^2(\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2}}{2}.
 \end{aligned}
 \tag{24}$$

When $\sigma_1 = \sigma_2 = \sigma$, the above Equation (24) becomes

$$(\Delta^2\phi)_D = \frac{1}{8\alpha^2}e^{\sigma^2} + \frac{1}{2}\sinh(\sigma^2).
 \tag{25}$$

Comparing Equation (16) with Equation (25), we can see that when there is no loss in the interferometer, the square of the phase uncertainty is inversely proportional to the average number of photons, reaching the SQL $\Delta^2\phi = \frac{1}{8\alpha^2}$. In the presence of phase diffusion, the phase uncertainty will increase by a factor, and an additional noise term. Next, we use a method that injects squeezing in opposite directions into each arm to counteract this detrimental effect.

3. Phase Noise Optimization

In order to reduce the impact of this noise as much as possible, as shown in Figure 1, after the phase diffusion noise a squeezed operation is applied, which is given by

$$\hat{S}(\zeta) = \exp\left[\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})\right],
 \tag{26}$$

where $\zeta = re^{i\eta}$ is the squeezed parameter, r is the squeezed amplitude, and η is the squeezed angle. When we take $\eta = \pi$, it means that we adopt the anti-squeezed scheme, and the above equation is $\hat{S}(r) = \exp[\frac{1}{2}(r\hat{a}^2 - r\hat{a}^{\dagger 2})]$.

In any arm of the truncated interferometer, when losses are not present, we can obtain the measurement results for any detection reference phase

$$\begin{aligned}\langle \hat{J}_r \rangle &= \sqrt{2}\alpha [\cosh r \cos(\phi_{in} + \phi_r) + \sinh r \cos(\phi_{in} - \phi_r)], \\ \text{Var}(\hat{J}_r) &= \frac{1}{2} + \cos 2\phi_r \sinh r \cosh r + \sinh^2 r.\end{aligned}\quad (27)$$

Especially, when $\phi_r = 0$ or $\phi_r = \pi/2$, we have

$$\langle \hat{J}_r \rangle_{\phi_r=0} = \sqrt{2}\alpha e^r \cos \phi_{in}, \quad [\text{Var}(\hat{J}_r)]_{\phi_r=0} = \frac{1}{2}e^{2r}, \quad (28)$$

or

$$\langle \hat{J}_r \rangle_{\phi_r=\frac{\pi}{2}} = -\sqrt{2}\alpha e^{-r} \sin \phi_{in}, \quad [\text{Var}(\hat{J}_r)]_{\phi_r=\frac{\pi}{2}} = \frac{1}{2}e^{-2r}. \quad (29)$$

The coherent state is a Gaussian state commonly used in measurement schemes. After defining a set of orthogonal field operators \hat{X} and \hat{Y} , we will be able to obtain two crucial physical quantities of the Gaussian state: one-dimensional moment vector R

$$\begin{aligned}R &= \begin{pmatrix} \langle \hat{X} \rangle \\ \langle \hat{Y} \rangle \end{pmatrix} = \sqrt{2} \begin{pmatrix} \alpha \cos(-\phi_{in}) \\ \alpha \sin(-\phi_{in}) \end{pmatrix} \\ &= \sqrt{2} \begin{pmatrix} \tilde{\alpha}_q \cos(-\phi_{out}) \\ \tilde{\alpha}_p \sin(-\phi_{out}) \end{pmatrix},\end{aligned}\quad (30)$$

and covariance matrix CM

$$CM = \begin{pmatrix} (\text{Var}\hat{X}) & 0 \\ 0 & (\text{Var}\hat{Y}) \end{pmatrix}. \quad (31)$$

The squeezed operation is also a Gaussian operation, affecting the evolution of the one-dimensional moment vector R and covariance matrix CM of the Gaussian state in the dynamic process. Since the phase noise model we established here is a Gaussian noise model, the effect of squeezing on the evolution of Gaussian states in phase space will not be destroyed by noise.

The phase diagrams of the two arms are different due to the existence of an initial phase difference. We add squeezed operation in the opposite direction in the two arms. The probe arm injects reverse squeezing, and the conjugate arm injects positive squeezing. Therefore, in the presence of phase diffusion, by adding squeezed operation, we can get

$$\begin{aligned}\langle \hat{X} \rangle_S &= \sqrt{2}\alpha e^r (e^{-\frac{1}{2}\sigma_1^2} + e^{-\frac{1}{2}\sigma_2^2}) \cos(\phi), \\ [\text{Var}(\hat{X})]_S &= e^{2r} + (\alpha e^r)^2 [2 + (e^{-2\sigma_1^2} + e^{-2\sigma_2^2}) \\ &\quad \times \cos(2\phi) - 2(e^{-\sigma_1^2} + e^{-\sigma_2^2}) \cos^2(\phi)],\end{aligned}\quad (32)$$

and

$$\begin{aligned}\langle \hat{Y} \rangle_S &= -\sqrt{2}\alpha e^{-r} (e^{-\frac{1}{2}\sigma_1^2} + e^{-\frac{1}{2}\sigma_2^2}) \sin(\phi), \\ [\text{Var}(\hat{Y})]_S &= e^{-2r} + (\alpha e^{-r})^2 [2 - (e^{-2\sigma_1^2} + e^{-2\sigma_2^2}) \\ &\quad \times \cos(2\phi) - 2(e^{-\sigma_1^2} + e^{-\sigma_2^2}) \sin^2(\phi)].\end{aligned}\quad (33)$$

Substituting the above results into Equation (5), the optimized phase uncertainty is given by

$$\begin{aligned}
 (\Delta^2\phi)_{out} = & \frac{1}{[(e^{2r} + e^{-2r}) \sin^2(\phi) + e^{2r}]^2} \times \left\{ \frac{(\frac{1}{2}\Lambda_{\sigma}^{\frac{1}{2}})^{-2}}{8\alpha^2} \right. \\
 & + \frac{1}{2} [2(\Lambda_{\sigma}^{\frac{1}{2}})^{-2} - \Lambda_{\sigma}^2(\Lambda_{\sigma}^{\frac{1}{2}})^{-2} \cos^2(2\phi) \\
 & \left. - \Lambda_{\sigma}^1(\Lambda_{\sigma}^{\frac{1}{2}})^{-2} \sin^2(2\phi) \right\}. \quad (34)
 \end{aligned}$$

At the optimal point $\phi = 0$, the above Equation (34) becomes

$$\begin{aligned}
 (\Delta^2\phi)_{out} = & e^{-4r} \left[\frac{1}{8\alpha^2} (\frac{1}{2}\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2} \right. \\
 & \left. + \frac{2(\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2} - \Lambda_{\sigma_1, \sigma_2}^2 (\Lambda_{\sigma_1, \sigma_2}^{\frac{1}{2}})^{-2}}{2} \right]. \quad (35)
 \end{aligned}$$

When the squeezed intensity $r > 0$, the squeezing operation can always slow down the impact of noise on the measurement results. In particular, when the loss coefficient $\sigma_1 = \sigma_2 = \sigma$, Equation (35) becomes

$$(\Delta^2\phi)_{out} = e^{-4r} \left[\frac{1}{8\alpha^2} e^{\sigma^2} + \frac{1}{2} \sinh(\sigma^2) \right]. \quad (36)$$

The result shows that squeezing can be used as a powerful tool in quantum measurements to reduce the negative effects of phase diffusion. The numerical results are shown in Figure 3 and Figure 4. In Figure 3 we can see how a phase-diffused coherent state is modified by the evolution through the squeezing operation, and the phase diffusion noise is reduced as the squeezed degree r increases.

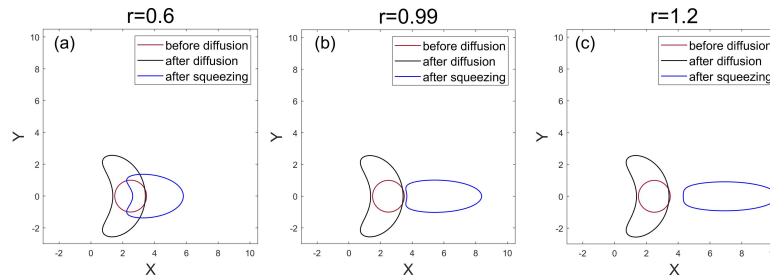


Figure 3. Phase-space representation of a coherent state $|\alpha\rangle$ (red), its phase-diffused counterpart with squeeze counteraction (blue) and without squeeze counteraction (black) for $\alpha = 2.5$ and $\sigma = 1.2$.

From the results in Figure 4, we can see that the effect of squeezing to counteract phase diffusion noise seems perfect and can even reach the shot noise limit in the presence of noise. However, there are still several issues that need further elaboration. Firstly, we have established a Gaussian noise model to describe the phase noise during the measurement process to know the effect of this noise on the measurement accurately. However, in actual measurements, the physical model of phase diffusion noise does not evolve precisely as we want, leading to a decrease in the ability of squeezing optimization. Secondly, during the experimental operation, to add the squeezing operation, some unavoidable adverse effects may be brought, which will also decrease the optimization effect. However, it is undeniable that even if the above practical operation problems exist, the squeezing operation can still reduce diffusion noise.

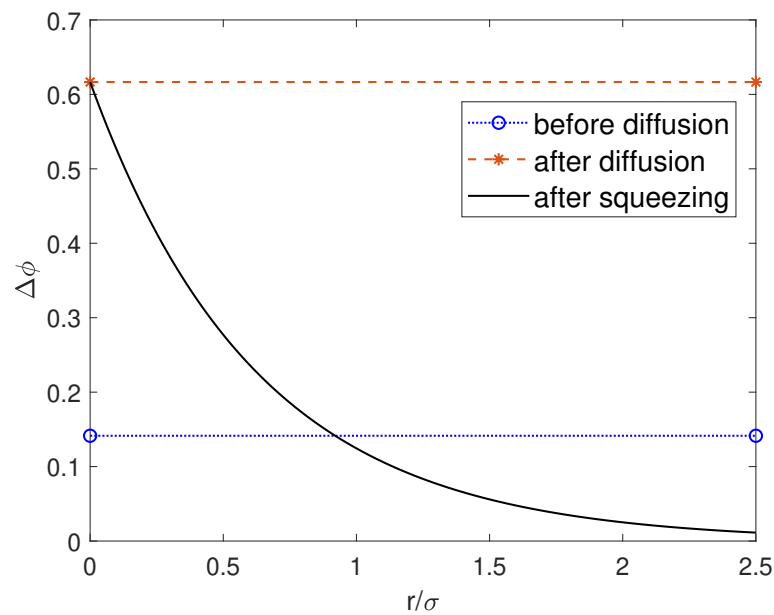


Figure 4. The phase uncertainty versus the normalized squeezed degree r/σ . While $\alpha = 2.5$ and $\sigma = 0.8$.

4. Conclusions

In the presence of phase diffusion, the measurement precision will be reduced. To counteract the adverse effects, we introduced a truncated MZ interferometer model and studied the phase uncertainty of it in the presence of phase diffusion. In this truncated MZ interferometer, combined with the relevant characteristics of the Gaussian state, we find that the squeezing operation can counteract this effect of phase diffusion, and the theoretical explanation is given. This strategy is beneficial to quantum precision measurement in lossy environments.

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Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MZ	Mach-Zehnder
BS	Beam splitter
HD	Homodyne detection
LO	Local oscillator
OPO	Optical parametric oscillator
LIGO	Laser interferometer gravitational-wave observatory
QCRB	Cramer-Rao bound
QFI	Quantum Fisher information

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