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Geometric Properties of a Linear Operator Involving Lambert Series and Rabotnov Function

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Abstract: In this study, we consider a Lambert series whose coefficients are the sum of divisors function. Utilizing the Lambert series in the sequel we introduce a normalized linear operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ by applying the convolution with Rabotnov function. We then, acquire sufficient conditions for $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ to be Univalent, Starlike and Convex respectively. In each component of this study, we expand the derived results by applying two Robin's inequalities, one of which is equivalent to the Riemann hypothesis.

Keywords: Univalent; Starlike; Convex; Hadamard product; Lambert series; Sum of divisors function; Robin's inequalities; Riemann hypothesis; Rabotnov function

Mathematics Subject Classification: 30C45; 30C50; 00A27

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1. Introduction

A series introduced by Johann Heinrich Lambert, commonly known as Lambert series is expressed as follows:

$$S(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}, \tag{1}$$

It is a type of series that is well-known in both number theory and analytic function theory. Lambert (see [1,2]) considered it in the context of the convergence of power series. Lambert series given by (1) converges either everywhere except at $x = \pm 1$ when $\sum_1^\infty a_n$ converges, or at every x such that $\sum_1^\infty a_n x^n$ converges.

In number theory, (see [3–6]), Lambert series is used for certain problems due to its connection to the well-known arithmetic functions such as

$$\sum_{n=1}^{\infty} \sigma_0(n) x^n = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}, \tag{2}$$

where $\sigma_0(n) = d(n)$ is the number of positive divisors of n.

$$\sum_{n=1}^{\infty} \sigma_\alpha(n) x^n = \sum_{n=1}^{\infty} \frac{n^\alpha x^n}{1-x^n}, \tag{3}$$

where $\sigma_\alpha(n)$ is the higher-order sum of divisors function of n.

We restrict our attention to the series given by (3). In particular, when $\alpha = 1$, we write $\sigma_1(n) = \sigma(n)$, here $\sigma(n)$ is the sum of divisors function that appears in one of the elementary equivalent statements to the well-known Riemann Hypothesis.

We distinguish at the outset between Lambert series and Lambert W function that appears naturally in the solution of a wide range of problems in science and engineering [7].

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In 1984, Guy Robin [8] proved that

$$\sigma(n) < e^{\gamma} n \log\log n + \frac{0.6483n}{\log\log n}, \qquad n \geq 3 \tag{4}$$

Moreover, he proved that Riemann hypothesis is equivalent to

$$\sigma(n) < e^{\gamma} n\, log(log\, n), \quad n > 5040, \tag{5}$$

where $\gamma = 0.7721 \cdots$, is the Euler-Mascheroni constant.

This article makes no attempt to prove or refute the Robin's inequality (5) or the Riemann hypothesis. For more details, we refer the interested readers to read the articles listed in the references [9–14].

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \tag{6}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent (or one-to-one) functions on $\mathbb{D}$. The importance of the coefficients given by the power series in (6) emerged in the early stage of the theory of univalent functions. Earlier in 1916, Bieberbach [15] proved that the second coefficient $|a_2| \leq 2$, with equality holding if and only if f is a rotation of the Köebe's function: $f(z) = z + \sum_{n=2}^{\infty} nz^n$.

In the same work, Bieberbach conjectured the general coefficient bound that $|a_n| \leq n$, $n \geq 2$, while the equality holds if and only if f is a rotation of the Köebe's function. This conjecture came to be known as the famed Bieberbach Conjecture and resisted a rigorous proof for about seven decades until Louis de Branges proved it in 1985 [16], and the result came to be known as de Branges's Theorem. Geometrically this amounts to shrinking or expanding the domain $\mathbb{D}$, and possibly rotating $\mathbb{D}$ but does not disturb the univalence of the function. Later on, new concepts were introduced in the theory of univalent functions including, but not limited to, starlike, convex, spiral-like and uniformly starlike (convex).

In fact, the study on introducing new subclasses of analytic functions goes on by means of various applications, such as fractional calculus, quantum calculus or by involving some special functions like Mittag-Leffler function, Faber polynomial functions etc., see for details [17–24]. The most common concern in such a study is the inclusion conditions. Alternatively, it means that for *a given new subclass (say) $\mathcal{H}$, seek a set of useful conditions on the sequence $\{a_n\}$ that are both necessary and sufficient for $f(z)$ to be a member of $\mathcal{H}$.*

The Rabotnov function defined as follows (See [25])

$$R_{\alpha,\beta}(z) = z^{\alpha} \sum_{n=0}^{\infty} \frac{\beta^n}{\Gamma\big((n+1)(\alpha+1)\big)} z^{n(\alpha+1)}, \qquad \alpha, \beta, z \in \mathbb{C}.$$

Clearly, $R_{0,\beta}(z) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} z^n = e^{\beta z}$

Rabotnov function is the particular case of the familiar Mittag-Leffler function widely used in the solution of fractional order integral equations or fractional order differential equations. The relation between the Rabotnov function and Mittag-Leffler function can be written as follows

$$R_{\alpha,\beta}(z) = z^{\alpha} E_{\alpha+1,\alpha+1}(\beta z^{\alpha+1}),$$

where $E_{\alpha+1,\alpha+1}$ is the two parameters Mittag-Leffler function. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found in [26–30].

It is clear that the Rabotnov function does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Rabotnov function

$$\mathbb{R}_{\alpha,\beta}(z):=z^{\frac{1}{\alpha+1}}\Gamma(\alpha+1)R_{\alpha,\beta}\left(z^{\frac{1}{\alpha+1}}\right)=z+\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}z^{n} \quad (7)$$

Geometric properties including starlikeness, convexity and close-to-convexity for the normalized Rabotnov function were recently studies in [31].

This work is an attempt to apply Lambert series in the theory of univalent functions. This may open relevant studies if one considers the Lambert series associated to other special functions such as Mittag-Leffler or any of its generalizations. Hence, we can investigate various topics rather than the geometric properties, that can evoke Hankel determinant, subordination properties and Fekete-Szegö. Besides, these results are extendable to multivalent functions and meromorphic functions.

Here, we recall the definition of Hadamard product (convolution): For a given function $f\in\mathcal{A}$ of the form (6) and $g\in\mathcal{A}$ of the form

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n, \qquad z\in\mathbb{D}, \quad (8)$$

then the convolution $(*)$ of the two functions f and g becomes,

$$(f*g)(z):=z+\sum_{n=2}^{\infty}a_nb_nz^n, \quad z\in\mathbb{D}. \quad (9)$$

Subsequently, we utilize the Lambert series whose coefficients are the sum of divisors function $\sigma(n)$. The mathematical form is as under:

$$\mathcal{L}(z)=\sum_{n=1}^{\infty}\frac{n\,z^n}{1-z^n}=\sum_{n=1}^{\infty}\sigma(n)z^n=z+\sum_{n=2}^{\infty}\sigma(n)z^n, \quad z\in\mathbb{D}.$$

For function $f\in\mathcal{A}$ of the form (6), we define the linear operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z):\mathcal{A}\longrightarrow\mathcal{A}$ as follows:

$$\mathcal{J}\mathbb{R}_{\alpha,\beta}(z):=\left(\mathbb{R}_{\alpha,\beta}*\mathcal{L}\right)(z)=z+\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\sigma(n)z^n, \quad \in\mathbb{D}. \quad (10)$$

Now, for short hand we denote the coefficient of $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ by

$$a_n=\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\sigma(n).$$

From Robin's inequalities we obtain

Remark 1. *Unconditionally, from Robin's inequality (3). For $n\geq 3$*

$$a_n<\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\left\{e^{\gamma}n\log\log n+\frac{0.6483n}{\log\log n}\right\}$$

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Remark 2. *If Riemann hypothesis (4) holds true, then for* $n > 5040$

$$a_n < \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\{e^{\gamma} n \log\log n\}$$

Next, we provide sufficient conditions for the operator (10) to be starlike, convex and closed-to-convex, respectively. We also evoke the consequence of Robin's inequalities or Riemann hypothesis in each derived result and vice versa.

Firstly, we recall some relevant definitions and Lemmas that we consider in this study.

Definition 1. *Function* $f \in \mathcal{A}$ *of the form (6) is said to be starlike or* $f \in \mathcal{S}^*$ *if*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

Definition 2. *Function* $f \in \mathcal{A}$ *of the form (6) is said to be convex or* $f \in \mathcal{C}$ *if*

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, \quad z \in \mathbb{D}.$$

Definition 3. *Function* $f \in \mathcal{A}$ *of the form (6) is said to be closed-to-convex or* $f \in \mathcal{K}$ *if*

$$Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad g \in \mathcal{C}, z \in \mathbb{D}.$$

The above definitions have been investigated in different studies, see for example [32–35]. Moreover, Noshiro-Warschawski [36,37] provided the following inclusion result: $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$.

Lemma 1. ([38]) *Function* $f \in \mathcal{A}$ *of the form (6) is univalent in* $\mathbb{D}$ *if*

$$1 \geq 2a_2 \geq \cdots \geq 2a_n \geq \cdots \geq 0,$$

or

$$1 \leq 2a_2 \leq \cdots \leq 2a_n \leq \cdots \leq 2.$$

Furthermore, f *is closed-to-convex with respect to the convex function* $-Log\,(1-z)$.

Lemma 2. [39] *Function* $f \in \mathcal{A}$ *of the form (6) is starlike in* $\mathbb{D}$ *if* $a_n \geq 0$, $\{na_n\}$ *and* $\{na_n - (n+1)a_{n+1}\}$ *both are non-increasing.*

2. Main Results

Theorem 1. *The operator* $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ *defined in (10) is close-to-convex with respect to* $-\log(1-z)$ *and therefore univalent in* $\mathbb{D}$ *if for every consecutive natural numbers* n *and* $n+1$, *with* $\alpha \geq 0$ *and* $\beta > 0$.

$$(\alpha+1)\sigma(n) \geq 2\beta\sigma(n+1) \tag{11}$$

Proof. We utilize Lemma 1. First, we need to prove by induction that

$$(\alpha+1)^{n-1}(n-1)!\,\Gamma(\alpha+1) \leq \Gamma\big(n(\alpha+1)\big), n = 1,2,\ldots \tag{12}$$

For $n = 1$, (11) obviously holds true. Assuming (11) is true for $n-1$, we conclude

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$$\begin{aligned}(\alpha+1)^n n!\,\Gamma(\alpha+1) &= (\alpha+1)n(\alpha+1)^{n-1}(n-1)!\,\Gamma(\alpha+1)\\ &\le (\alpha+1)n\,\Gamma\big(n(\alpha+1)\big)\\ &= \Gamma\big((\alpha+1)n+1\big)\\ &\le \Gamma\big((\alpha+1)(n+1)\big).\end{aligned}$$

Recall

$$a_n = \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\sigma(n), \qquad n \ge 2,\ a_1 = 1.$$

From (12) when $n = 2$ we obtain

$$\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma\big(2(\alpha+1)\big)} \le 1$$

Using condition (11)

$$2a_2 = \frac{2\beta\sigma(2)\Gamma(\alpha+1)}{\Gamma\big(2(\alpha+1)\big)} \le \frac{2\beta\sigma(2)}{\alpha+1} \le 1.$$

To verify that the first condition of Lemma 1 holds, we need to show that the sequence $\{na_n\}$ is decreasing:

$$\begin{aligned}na_n-(n+1)a_{n+1} &= \frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\sigma(n) - \frac{(n+1)\beta^n\Gamma(\alpha+1)}{\Gamma\big((n+1)(\alpha+1)\big)}\sigma(n+1)\\ &\ge \frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\sigma(n) - \frac{(n+1)\beta^n\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n+1)\\ &= \frac{n^2(\alpha+1)\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n) - \frac{(n+1)\beta^n\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n+1)\\ &= \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}X(n),\end{aligned}$$

where $X(n) = n^2(\alpha+1)\sigma(n) - (n+1)\beta\sigma(n+1)$ considering $n^2 \ge n+1, n > 1$ we receive

$$\begin{aligned}X(n) &= n^2(\alpha+1)\sigma(n) - (n+1)\beta\sigma(n+1)\\ &\ge (n+1)(\alpha+1)\sigma(n) - (n+1)\beta\sigma(n+1)\\ &\ge (n+1)\big((\alpha+1)\sigma(n) - \beta\sigma(n+1)\big).\\ &\ge (n+1)\big((\alpha+1)\sigma(n) - 2\beta\sigma(n+1)\big) \ge 0.\end{aligned}$$

□

From Theorem 1. Using the fact that the coefficients of a univalent function satisfy the inequality $a_n \le n$, we derive the following results.

Corollary 1. *If the conditions of Theorem 1 hold true, then*

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$$\sigma(n) \leq \frac{n\Gamma\big(n(\alpha+1)\big)}{\beta^{n-1}\Gamma(\alpha+1)}, \quad n \geq 2.$$

Corollary 2. *If the conditions of Theorem 1 hold true and* $\frac{\Gamma(n(\alpha+1))}{\beta^{n-1}\Gamma(\alpha+1)} < e^{\gamma} \log\log n\,, n > 5040$ *then Riemann hypothesis holds true.*

Theorem 2. *The operator* $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ *defined in (10) is starlike in* $\mathbb{D}$ *if for every consecutive natural numbers* n *and* $n+1$, *with* $\alpha \geq 0$ *and* $\beta > 0$. $(\alpha+1)\sigma(n) \geq 2\beta\sigma(n+1)$

Proof. In here, we use Lemma 2. We omit the proof that the sequence $\{na_n\}$ is non-increasing since it is similar to the one in Theorem 1. Therefore, we need to show that $\{na_n - (n+1)a_{n+1}\}$ is also non-increasing. For the sake of simplicity, we let $b_n = na_n - (n+1)a_{n+1}$and we have

$$b_n - b_{n+1} = na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2}$$

$$= \frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\sigma(n) - \frac{2(n+1)\beta^{n}\Gamma(\alpha+1)}{\Gamma\big((n+1)(\alpha+1)\big)}\sigma(n+1) + \frac{(n+2)\beta^{n+1}\Gamma(\alpha+1)}{\Gamma\big((n+2)(\alpha+1)\big)}\sigma(n+2)$$

$$\geq \frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\sigma(n) - \frac{2(n+1)\beta^{n}\Gamma(\alpha+1)}{\Gamma\big((n+1)(\alpha+1)\big)}\sigma(n+1)$$

$$\geq \frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma\big(n(\alpha+1)\big)}\sigma(n) - \frac{2(n+1)\beta^{n}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n+1)$$

$$= \frac{n^2(\alpha+1)\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n) - \frac{2(n+1)\beta^{n}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n+1)$$

$$= \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}Y(n),$$

Where $Y(n) = n^2(\alpha+1)\sigma(n) - 2(n+1)\beta\sigma(n+1)$. We use the fact that $n^2 \geq n+1, n > 1$,

$$Y(n) = n^2(\alpha+1)\sigma(n) - 2(n+1)\beta\sigma(n+1)$$
$$\geq (n+1)(\alpha+1)\sigma(n) - 2(n+1)\beta\sigma(n+1)$$
$$\geq (n+1)\big((\alpha+1)\sigma(n) - 2\beta\sigma(n+1)\big) \geq 0.$$

□

Theorem 3. *The operator* $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ *defined in (10) is convex in* $\mathbb{D}$ *if for every consecutive natural numbers* n *and* $n+1$, *with* $\alpha \geq 0$ *and* $\beta > 0$.

$$\sum_{n=2}^{\infty} \frac{n^2\beta^{n-1}\sigma(n)}{(\alpha+1)^{n-1}(n-1)!} < 1.$$

Proof. Let

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$$p(z) = 1 + \frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}, z \in \mathbb{D},$$

$p(z)$ is analytic in $\mathbb{D}$ and $p(0) = 1$. We need to prove that $|p(z) - 1| < 1$,

$$\left|z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)\right| = \left|\sum_{n=2}^{\infty} \frac{n(n-1)\beta^{n-1}\Gamma(\alpha+1)\sigma(n)}{\Gamma\big(n(\alpha+1)\big)} z^{n-1}\right|$$

$$< \sum_{n=2}^{\infty} \frac{n(n-1)\beta^{n-1}\Gamma(\alpha+1)\sigma(n)}{\Gamma\big(n(\alpha+1)\big)}$$

$$\leq \sum_{n=2}^{\infty} \frac{n(n-1)\beta^{n-1}\sigma(n)}{(\alpha+1)^{n-1}(n-1)!} \quad (13)$$

$$\left|\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)\right| = \left|1 + \sum_{n=2}^{\infty} \frac{n\beta^{n-1}\Gamma(\alpha+1)\sigma(n)}{\Gamma\big(n(\alpha+1)\big)} z^{n-1}\right|$$

$$> 1 - \sum_{n=2}^{\infty} \frac{n\beta^{n-1}\Gamma(\alpha+1)\sigma(n)}{\Gamma\big(n(\alpha+1)\big)}$$

$$\geq 1 - \sum_{n=2}^{\infty} \frac{n\beta^{n-1}\sigma(n)}{(\alpha+1)^{n-1}(n-1)!} \quad (14)$$

From (13) and (14) we obtain

$$\left|\frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}\right| < \frac{\sum_{n=2}^{\infty} \frac{n(n-1)\beta^{n-1}\sigma(n)}{(\alpha+1)^{n-1}(n-1)!}}{1 - \sum_{n=2}^{\infty} \frac{n\beta^{n-1}\sigma(n)}{(\alpha+1)^{n-1}(n-1)!}} < 1$$

□

From Theorem 3. Using the fact that the coefficients of a univalent function satisfy the inequality $a_n \leq 1$, we derive the following results.

Corollary 3. *If the conditions of Theorem 3 hold true, then*

$$\sigma(n) \leq \frac{\Gamma\big(n(\alpha+1)\big)}{\beta^{n-1}\Gamma(\alpha+1)}, \quad n \geq 2.$$

Corollary 4. *If the conditions of Theorem 3 hold true, and* $\frac{\Gamma(n(\alpha+1))}{\beta^{n-1}\Gamma(\alpha+1)} < e^{\gamma} n \log\log n, n > 5040$ *then Riemann hypothesis holds true.*

3. Conclusions

By the means of Lambert series whose coefficients are the sum of divisors function, we considered the normalized linear operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ by applying the convolution with Rabotnov function. We provided sufficient conditions for $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ to be Univalent, Starlike and Convex respectively. When applicable, we expand the derived results by applying two Robin's inequalities (3) and (4).

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References

1. Lambert J H, *Anlage Zür Architektonik.*, Vol. **II** (1971) Riga p. 507.
2. R. P. Agarwal. Lambert series and Ramanujan. *Proc. Indian Acad. Sci.,* **103** (1993), 269–293.
3. L. Lóczi, Guaranteed- and high-precision evaluation of the Lambert W function, *Appl. Math. Comput.*, **433** (2022), 1-22.
4. Schmidt, M. D. A catalog of interesting and useful Lambert series identities. *arXiv preprint arXiv:2004.02976*, (2020).
5. Postnikov A G, *Introduction to analytical number theory* (Moscow) 1971.
6. T. M. Apostol. *Introduction to Analytic Number Theory*. Springer–Verlag, 1976.
7. Kesisoglou I, Singh G, Nikolaou M. The Lambert Function Should Be in the Engineering Mathematical Toolbox. *Comput Chem Eng*. (2021) May; 148:107259. doi: 10.1016/j.compchemeng.2021.107259. Epub 2021 Feb 17. PMID: 34267408; PMCID: PMC8276860.
8. G. Robin, Grande valeurs de la function somme des diviseurs et hypothese de Riemann, *J. Math. Pures App.*, **63** (1984), 187 – 213.
9. Axler, C. On Robin's inequality. *Ramanujan J* (2022). doi.org/10.1007/s11139-022-00683-0
10. YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé. On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux.*, **19** (2007), 357–372. doi:10.5802/jtnb.591.
11. Jamal Salah, Hameed Ur Rehman, and, Iman Al- Buwaiqi, The Non-Trivial Zeros of the Riemann Zeta Function through Taylor Series Expansion and Incomplete Gamma Function, *Mathematics and Statistics.*, **10** (2022), 410-418. doi: 10.13189/ms.2022.100216.
12. Jamal Salah, Some Remarks and Propositions on Riemann Hypothesis, *Mathematics and Statistics.*, **9** (2021), 159-165. doi: 10.13189/ms.2021.090210.
13. K Eswaran. The pathway to the Riemann hypothesis. Open Reviews of the Proof of The Riemann Hypothesis, page 189, 2021.
14. Enrico Bombieri. The Riemann hypothesis – official problem description (pdf). Clay Mathematics Institute, 2000.
15. L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Sitzungsber. Preuss. Akad. Wiss. Phys-Math. Kl. 940, (1916).
16. L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154**, 137 (1985), doi:10.1007/BF02392821.
17. G. Murugusundaramoorthy and N. Magesh, Linear operators associated with a subclass of uniformly convex functions, *Internat. J. Pure Appl. Math. Sci.,* **3** (2) (2006), 113–125.
18. B.A. Frasin, On certain subclasses of analytic functions associated with Poisson distribution series, *Acta Univ. Sapientiae, Mathematica.*, **11** (1) (2019), 78–86.
19. B.A. Frasin, Subclasses of analytic functions associated with Pascal distribution series, *Adv. Theory Nonlinear Anal. Appl.*, **4** (2) (2020), 92–99.
20. B. A. Frasin, S. R. Swamy and A. K. Wanas, Subclasses of starlike and convex functions associated with Pascal distribution series, *Kyungpook Math. J.*, **61** (2021), 99-110.
21. A. Amourah, B.A. Frasin and G. Murugusundaramoorthy, On certain subclasses of uniformly spirallike functions associated with Struve functions, *J. Math. Comput. Sci.,* **11** (4) (2021), 4586-4598.
22. El-Deeb S. M.; Murugusundaramoorthy, G. Applications on a subclass of β−uniformly starlike functions connected with q−Borel distribution. *Asian-Eur. J. Math.,* **15** (2022), 1–20. doi.org/10.1142/S1793557122501583_

 doi:10.20944/preprints202312.0676.v1

23. Jamal Salah, Hameed Ur Rehman, Iman Al Buwaiqi, Ahmad Al Azab and Maryam Al Hashmi, Subclasses of spiral-like functions associated with the modified Caputo's derivative operator, *AIMS Mathematics*. **8** (8) (2023), 18474-18490. doi: 10.3934/math.2023939.
24. Hameed Ur Rehman, Maslina Darus, Jamal Salah, Generalizing Certain Analytic Functions Correlative to the n-th Coefficient of Certain Class of Bi-Univalent Functions, *Journal of Mathematics*, (**2021)**, Article ID 6621315, 14 pages, 2021. doi.org/10.1155/2021/6621315.
25. Y. Rabotnov, Equilibrium of an Elastic Medium with After-Effect, *Prikladnaya Matematika I Mekhanika*, **12** (1) (1948), 53-62 (in Russian), Reprinted: *Fractional Calculus and Applied Analysis*, **17** (3) (2014), 684-696.
26. Y. Kao, Y. Li, J. H. Park, X. Chen, Mittag-Leffler synchronization of delayed fractional memristor neural networks via adaptive control, *IEEE T. Neur. Net. Lear.*, **32** (2021), 2279–2284. http://doi: 10.1109/TNNLS.2020.2995718
27. Y. Cao, Y. Kao, J. H. Park, H. Bao, Global Mittag-Leffler stability of the delayed fractional-coupled reaction-diffusion system on networks without strong connectedness, *IEEE T. Neur. Net. Lear.*, **33** (2022), 6473–6483. https://doi.org/10.1109/TNNLS.2021.3080830
28. Y. Kao, Y. Cao, X. Chen, Global Mittag-Leffler synchronization of coupled delayed fractional reaction-diffusion Cohen-Grossberg neural networks via sliding mode control, *Chaos*, **32** (2022), 113123. https://doi.org/10.1063/5.0102787
29. H. U. Rehman, M. Darus, J. Salah, Coefficient properties involving the generalized k-Mittag-Leffler functions, *Transylv. J. Math. Mech.*, **9** (2017), 155–164. Available from: http://tjmm.edyropress.ro/journal/17090206.pdf
30. J. Salah, M. Darus, A note on generalized Mittag-Leffler function and Application, *Far East. J. Math. Sci.*, **48** (2011), 33–46. Available from: http://www.pphmj.com/abstract/5478.htm.
31. S. S. Eker and S. Ece. Geometric properties of normalized Rabotnov function, *Hacet. J. Math. Stat.*, **51** (5) (2022), 1248-1259.
32. S. Ponnusamy and A. Baricz, Starlikeness and convexity of generalized Bessel functions. *Integral Transforms Spec. Funct.*, **21** (9) (2010), 641-653.
33. J. K. Prajapat, Certain geometric properties of the Wright functions, *Integral Transforms Spec. Funct.*, **26** (3) (2015), 203-212.
34. D. Răducanu, *Geometric properties of Mittag-Leffler functions, Models and Theories in Social Systems*, Springer: Berlin, Germany, (2019), 403-415.
35. S. Sumer Eker, S. Ece, Geometric Properties of the Miller-Ross Functions, *Iran. J. Sci. Technol. Trans.*, (2022).
36. PL. Duren, *Univalent Functions, Grundlehren der Mathematischen Wissenschaften*, New York, NY, USA: Springer-Verlag, (1983).
37. A. W. Goodman, *Univalent Functions*, New York, NY, USA: Mariner Publishing Company, (1983).
38. S. Ozaki, On the theory of multivalent functions, *Science Reports of the Tokyo Bunrika Daigaku, Section A.*, **2**(40) (1935), 167-188.
39. L. Fejér, *Untersuchungen uber Potenzreihen mit mehrfach monotoner Koeffizientenfolge*, *Acta Litt. Sci. Szeged.*, **8** (1936), 89-115.