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Article

# Inhomogeneous Boundary Value Problems for Generalized Boussinesq Model of Mass Transfer

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**Abstract:** We consider boundary value problems for a nonlinear mass transfer model, which generalizes the classical Boussinesq approximation, under inhomogeneous Dirichlet boundary conditions for the velocity and the substance's concentration. It is assumed that the leading coefficients of viscosity and diffusion and the buoyancy force in the model equations depend on concentration. We develop mathematical apparatus of studying inhomogeneous boundary value problems under consideration. It is based on using of a weak solution of the boundary value problem and construction of liftings of the inhomogeneous boundary data. They remove the inhomogeneity of the data and reduce original problems to equivalent homogeneous boundary value problems. Based on this apparatus we will prove the theorem of the global existence of a weak solution to boundary value problem under study and establish important properties of the solution. In particular, we will prove the validity of the maximum principle for the substance's concentration. We will also establish sufficient conditions for the problem data, ensuring the local uniqueness of weak solutions.

**Keywords:** generalized Boussinesq model of mass transfer; binary fluid; inhomogeneous boundary conditions; global solvability; maximum principle; local uniqueness

**MSC:** 35Q35

## 1. Introduction and statement of the boundary problem

A large number of works are devoted to the issues concerning mathematical modeling the processes of fluid flows taking into account convective effects. These papers explore both applied questions, concerning the derivation and justification of basic convection models, and purely mathematical questions. These include methods for constructing exact and approximate solutions, theoretic-group analysis (or Li-Ovsyannikov symmetry) method of investigating the qualitative properties of solutions to differential equations underlying the convection models under study, and theoretical analysis of the solvability and uniqueness of solutions of boundary value problems for basic convection models.

Among the various convection models for binary and/or thermally conducting liquids, an important role is played by so-called Oberbeck–Boussinesq models [1,2], which are derived from the exact Navier–Stokes equations of fluid dynamics taking into account the following assumptions (see [1], Section 54):

1. Convective motion is similar to that of an incompressible fluid with constant density  $\rho_0$ , but the possible deviation of the true density  $\rho$  from  $\rho_0$  is taken into account in the momentum conservation equation in the form of a term describing the additional volumetric force – the buoyancy (Archimedes) force.

2. The change in density is caused by changes in temperature and concentration of dissolved substance, but not changes in the pressure.

3. The velocity gradients are small enough so that the process of transition of work to heat during movement does not lead to a change in the temperature of the medium.

The model that results from these simplifications is called the Oberbeck-Boussinesq model. In turn, the Oberbeck-Boussinesq model allows for further simplification, often called the simplified or classical Boussinesq model.

When modeling heat transfer processes in a viscous heat-conducting liquid, by (classical) Boussinesq model one usually understands the model in which the bulk buoyancy force included in the momentum conservation equation linearly depends on temperature, and, besides, the main parameters of the fluid, namely: the viscosity coefficient and the thermal conductivity coefficient are positive constants. When modeling mass transfer processes in a binary fluid, by (classical) Boussinesq model one usually understands the model in which the bulk buoyancy force (included in the momentum conservation equation) linearly depends on the concentration of dissolved substance while the main parameters, namely: the viscosity coefficient, diffusion coefficient, as well as another parameter, called the reaction coefficient, are the positive constants. The latter parameter is responsible for the possible decay of the dissolved substance in the main medium due to the chemical reaction. When all or some of the above conditions are not fulfilled, the corresponding model is often referred to as a generalized Boussinesq model.

Let us emphasize that for the classical Boussinesq model, many theoretical issues are quite fully studied. This, in particular, holds for the study of the correctness of boundary or initial-value problems for stationary or non-stationary models of heat and mass transfer. Among many works in this area, we note the cycle [3–8] of works by the first author and his coauthors on the study of the correctness of boundary value problems for stationary equations of heat and/or mass transfer. We emphasize that in these works, in addition to the study of correctness a boundary value problems, a theoretical analysis of control problems for the models of heat and mass transfer was performed. The analysis of the results obtained in [3–8] made it possible to identify interesting regularities related to the interaction of hydrodynamic and thermal fields in binary and/or heat-conducting media and, in particular, to establish the most effective mechanisms for controlling thermohydrodynamic processes in viscous liquids. The close problems of boundary or distributed control for the heat transfer equations in the Boussinesq approximation have also been investigated in the works [9–14].

Theoretical questions for the generalized Boussinesq model have been studied to a much lesser extent. However, significant progress has been made in recent years in this area as well. Over the past decades, a large number of papers have been published regarding the study of heat and mass transfer equations with variable transfer coefficients and with variable buoyancy force depending on temperature and/or concentration of dissolved substance. These works can be divided into several groups. The first group contains papers that develop methods for finding exact solutions to these equations (see, for example, [15–19] and review [20]). The second group contains works devoted to application of Li-Ovsyannikov symmetry method to study qualitative properties of solutions of equations of heat and mass transfer in viscous binary and/or heat-conducting liquids. This group includes very large quantity of works (see e.g. [21–25], monograph [26] and reviews [27,28]). Another group of works is that in which mathematical modeling of fluid motion processes takes into account thermodiffusion effects (or Sorét effects) and/or concentration diffusion effects (or Dufort effects). A detailed list and analysis of these works can be found in the reviews [27,28].

At the same time, the authors know only a few papers in which the solvability of boundary value problems for equations of heat and mass transfer with variable coefficients is investigated. The mentioned works can be divided into several groups. The first group includes works [29–31] or [32], devoted to the study of the solvability of boundary value problems for stationary equations of heat or mass transfer. The second group is joined by works [33–38], devoted to the study of the solvability of boundary value and control problems for non-stationary Boussinesq equations of heat (or mass) transfer. The works [39–41] form once more group in which the solvability of boundary value and control problems is studied for the stationary mass transfer model in the case where the reaction coefficient can depend on the concentration of matter and spatial variables.

Close questions on the study of correctness of boundary value or control problems for stationary equations of magnetic hydrodynamics of viscous incompressible or heat-conducting liquid in the Boussinesq approximation were investigated in [42–46]. In [47], the solvability of the initial-boundary problem for the non-stationary MHD-Boussinesq system, considered under mixed boundary conditions for velocity, magnetic field, and temperature, in the case when the viscosity coefficient, magnetic permeability, electrical conductivity, thermal conductivity and specific heat of the fluid depend on the temperature.

Finally, we mention papers [48–55] that touch upon issues close to the subject of this paper from nonlinear diffusion, viscoelasticity, engineering mechanics, complex heat exchange, acoustics and oceanology.

The purpose of this work is to analyze the global solvability and local uniqueness of solutions of the boundary value problem for a generalized Boussinesq mass transfer model describing the flow of binary fluid in which the diffusion, viscosity and reaction coefficients and the buoyancy force depend on the substance concentration.

The paper is organized as follows. In Section 2, we will formulate the main boundary value problem, to which we will refer below as Problem 1. Besides, we introduce functional spaces and formulate a number of auxiliary results in the form of Lemmas 1, 2 and 3 which will be used when studying the solvability and uniqueness of Problem 1. In Section 3, we will formulate and prove the theorem on the global existence of a weak solution to Problem 1 and establish the maximum principle for substance concentration  $\varphi$ . In Section 4, we will establish sufficient conditions on the data of Problem 1 that provide conditional uniqueness of the weak solution having an additional property of smoothness for concentration. The last Section 5 (Conclusion) contains a brief summary of the results obtained in our paper.

## 2. Statement of the main problem. Functional spaces

Let  $\Omega$  be a bounded domain in the space  $\mathbb{R}^3$  with a Lipschitz boundary  $\Gamma$ . Below, we will consider the following boundary value problem describing the motion of binary fluid within the framework the generalized Boussinesq model of mass transfer:

$$-\operatorname{div}(v(\varphi)\nabla\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} + b(\varphi)\varphi\mathbf{G}, \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \quad (1)$$

$$-\operatorname{div}(\lambda(\varphi)\nabla\varphi) + (\mathbf{u} \cdot \nabla)\varphi + k(\varphi)\varphi = f \text{ in } \Omega, \quad (2)$$

$$\mathbf{u} = \mathbf{g} \text{ and } \varphi = \psi \text{ on } \Gamma. \quad (3)$$

Here  $\mathbf{u}$  is the velocity vector,  $\varphi$  is the concentration of dissolved substance,  $p = P/\rho_0$ , where  $P$  is the pressure,  $\rho = \text{const}$  is the fluid density,  $v = v(\varphi) > 0$  is the kinematic (molecular) viscosity coefficient,  $\lambda = \lambda(\varphi) > 0$  is the diffusion coefficient,  $b \equiv b(\varphi)$  is the mass expansion factor,  $k = k(\varphi)$  is the reaction coefficient,  $\mathbf{G} = -(0, 0, G)$  is the gravitational acceleration,  $\mathbf{f}$  or  $f$  is the bulk density of external forces or of the external sources of matter, respectively. Below the problem (1) – (3) for the given functions  $v(\varphi)$ ,  $\lambda(\varphi)$ ,  $b(\varphi)$ ,  $k(\varphi)$ ,  $\mathbf{f}$ , and  $f$  will be referred to as Problem 1.

When studying Problem 1, we will use the Sobolev functional spaces  $H^s(D)$ ,  $s \in \mathbb{R}$ . Here,  $D$  means either domain  $\Omega$  or some subset  $Q \subset \Omega$ , or the boundary  $\Gamma$ . By  $\|\cdot\|_{s,Q}$ ,  $|\cdot|_{s,Q}$  and  $(\cdot, \cdot)_{s,Q}$  the norm, half-norm and inner product in  $H^s(Q)$  will be denoted, respectively. The norms and scalar product in  $L^2(Q)$  or in  $L^2(\Omega)$  are denoted by  $\|\cdot\|_Q$ ,  $(\cdot, \cdot)_Q$  or by  $\|\cdot\|_\Omega$ , and  $(\cdot, \cdot)$ , respectively.  $X^*$  denotes the dual space of Hilbert space  $X$ , while the duality relation for the pair of dual spaces  $X$  and  $X^*$  is written as  $\langle \cdot, \cdot \rangle_{X^* \times X}$ , or simply as  $\langle \cdot, \cdot \rangle$ .

An important role in our analysis will be played by the following functional spaces:

$$H(\operatorname{div}, \Omega) = \{\mathbf{v} \in L^2(\Omega) : \operatorname{div}\mathbf{v} \in L^2(\Omega)\},$$

$$H^0(\operatorname{div}, \Omega) = \{\mathbf{v} \in H(\operatorname{div}, \Omega) : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\},$$

$$\begin{aligned}
H_{\text{div}}^1(\Omega) &= \{\mathbf{v} \in H^1(\Omega) : \text{div} \mathbf{v} = 0\}, \\
L_0^2(\Omega) &= \{h \in L^2(\Omega) : (h, 1) = 0\}, \\
\mathcal{D}(\Omega) &= \{\mathbf{v} \in C_0^\infty(\Omega)^3 : \text{div} \mathbf{v} = 0 \text{ in } \Omega\}, \\
H &\text{ is the closure } \mathcal{D}(\Omega) \text{ in } L^2(\Omega)^3, \\
V &\text{ is the closure } \mathcal{D}(\Omega)^3 \text{ in } H^1(\Omega)^3, \\
L_+^p(\Omega) &= \{k \in L^p(\Omega) : k \geq 0\}, \quad p \geq 1, \\
H_0^1(\Omega) &= \{h \in H^1(\Omega) : h|_\Gamma = 0\}.
\end{aligned}$$

Note that each of the spaces  $H^1(\Omega)$  and  $H_0^1(\Omega)$  is Hilbert at norm  $\|\cdot\|_{1,\Omega}$  which is equivalent to  $|\cdot|_{1,\Omega}$  for  $\varphi \in H_0^1(\Omega)$  due to the Friedrichs-Poincaré inequality

$$\|\varphi\|_{1,\Omega} \leq C_P |\varphi|_{1,\Omega} \quad \forall \varphi \in H_0^1(\Omega), \quad C_P = \text{const} > 1. \quad (4)$$

It is well known (see, for example, [56,57]) that for the domain  $\Omega$  with the Lipschitz boundary the spaces  $H$  and  $V$  are characterized as follows:

$$\begin{aligned}
H &= \{\mathbf{v} \in H^0(\text{div}, \Omega) : \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0 \text{ in } H^{-1/2}(\Gamma)\}, \\
V &= \{\mathbf{v} \in H_0^1(\Omega)^3 : \text{div} \mathbf{v} = 0 \text{ in } \Omega\}.
\end{aligned}$$

Let us define the products of spaces  $X = H_0^1(\Omega)^3 \times H_0^1(\Omega)$  and  $W = V \times H_0^1(\Omega) \subset X$  with the norm

$$\|\mathbf{x}\|_X^2 = \|\mathbf{u}\|_{1,\Omega}^2 + \|\varphi\|_{1,\Omega}^2 \quad \forall \mathbf{x} \equiv (\mathbf{u}, \varphi) \in X \text{ (or } \forall (\mathbf{u}, \varphi) \in W)$$

and denote by  $X^*$  the space  $(H^{-1}(\Omega))^3 \times H^{-1}(\Omega)$  which is dual of  $X$ .

Let the following conditions be met:

**2.1.**  $\Omega$  is a bounded domain in  $\mathbb{R}^3$  with a boundary  $\Gamma \in C^{0,1}$  consisting of  $N$  component  $\Gamma^{(i)}$ ,  $i = 1, 2, \dots, N$ .

**2.2.**  $\mathbf{f} \in H^{-1}(\Omega)^3$ ,  $f \in H^{-1}(\Omega)$ ,  $\psi \in H^{1/2}(\Gamma)$ .

**2.3.**  $\mathbf{g} \in H^{1/2}(\Gamma)^3$ ,  $(\mathbf{g}, \mathbf{n})_{\Gamma^{(i)}} = 0$ ,  $i = 1, 2, \dots, N$ .

**2.4.** For any function  $\varphi \in H^1(\Omega)$ , the embedding  $b(\varphi) \in L^p(\Omega)$  is valid where  $p \geq 3/2$  is a fixed number independent of  $\varphi$ , and the vector-function  $\mathbf{b}(\varphi) \equiv b(\varphi)\mathbf{G}$  satisfy

$$\|\mathbf{b}(\varphi)\|_{L^p(\Omega)^3} \leq \hat{\beta}_p \|\varphi\|_{1,\Omega} \quad \forall \varphi \in H^1(\Omega), \quad p \geq 3/2 \quad (\mathbf{b}(\varphi) \equiv b(\varphi)\mathbf{G}). \quad (5)$$

Here  $\hat{\beta}_p$  is a positive constant dependent on  $p$ . In addition, for any pair of functions  $\varphi_1, \varphi_2 \in H^1(\Omega)$  belonging to sphere  $B_r = \{\varphi \in H^1(\Omega) : \|\varphi\|_{1,\Omega} \leq r\}$  of radius  $r$ , the following inequality is true:

$$\|\mathbf{b}(\varphi_1) - \mathbf{b}(\varphi_2)\|_{L^p(\Omega)} \leq L_b \|\varphi_1 - \varphi_2\|_{L^4(\Omega)} \quad \forall \varphi_1, \varphi_2 \in B_r. \quad (6)$$

Here  $L_b$  is a constant that depends on  $b$  and on  $r$ , but does not depend on  $\varphi_1, \varphi_2 \in B_r$ .

**2.5.** For any function  $\varphi \in H^1(\Omega)$ , the embedding  $k(\varphi) \in L_+^p(\Omega)$  is valid, where  $p \geq 3/2$  is a fixed number independent of  $\varphi$ , and the following estimate for  $\|k(\varphi)\|_{L^p(\Omega)}$  is valid

$$\|k(\varphi)\|_{L^p(\Omega)} \leq \hat{\gamma}_p. \quad (7)$$

Here  $\hat{\gamma}_p$  is a positive constant dependent on  $p$ . Also for any sphere  $B_r = \{\varphi \in H^1(\Omega) : \|\varphi\|_{1,\Omega} \leq r\}$  the following inequality:

$$\|k(\varphi_1) - k(\varphi_2)\|_{L^p(\Omega)} \leq L_k \|\varphi_1 - \varphi_2\|_{L^4(\Omega)} \quad \forall \varphi_1, \varphi_2 \in B_r \quad (8)$$

holds. Here  $L_k$  is a constant that depends on  $k$  and on  $r$ , but does not depend on  $\varphi_1, \varphi_2 \in B_r$ .

**2.6.**  $\nu \in C^0(\mathbb{R}), \lambda \in C^0(\mathbb{R})$  and there are positive constants  $\nu_{\min}, \nu_{\max}, \lambda_{\min}$  and  $\lambda_{\max}$  such that

$$0 < \nu_{\min} \leq \nu(\tau) \leq \nu_{\max}, 0 < \lambda_{\min} \leq \lambda(\tau) \leq \lambda_{\max} \quad \forall \tau \in \mathbb{R}. \quad (9)$$

Consider the function  $\mu \in C^0(\mathbb{R})$ , satisfying the following condition:

$$0 < \mu_{\min} \leq \mu(\tau) \leq \mu_{\max} < \infty \quad \forall \tau \in \mathbb{R}. \quad (10)$$

Clearly  $\mu(h) \in L^\infty(\Omega)$  for any  $h \in H^1(\Omega)$  and the following estimates:

$$\mu_{\min} \leq \mu(h) \leq \mu_{\max} \text{ a.e. in } \Omega, \|\mu(h)\|_{L^\infty(\Omega)} \leq \mu_{\max} \quad \forall h \in H^1(\Omega)$$

hold. In addition,  $\mu(h_n) \rightarrow \mu(h)$  a.e. in  $\Omega$  if  $h_n \rightarrow h$  a.e. in  $\Omega$  as  $n \rightarrow \infty$ .

Let  $\mu_n = \mu(h_n)$ . Since  $|\mu(h_n)| \leq \mu_{\max}$  a.e. in  $\Omega$ , the Lebesgue's theorem on majorant convergence implies that

$$\int_{\Omega} \mu_n f dx \rightarrow \int_{\Omega} \mu f dx \text{ as } n \rightarrow \infty \quad \forall f \in L^1(\Omega). \quad (11)$$

The property (11) will play an important role when proving the solvability of Problem 1 which contains the variable leading coefficients  $\nu(\varphi)$  and  $\lambda(\varphi)$ .

Below, we will often use the following inequalities:

$$\|\varphi\|_{L^s(\Omega)} \leq C_s \|\varphi\|_{1,\Omega} \quad \forall \varphi \in H^1(\Omega), 1 \leq s \leq 6, \quad (12)$$

$$\|\mathbf{u}\|_{L^s(\Omega)^3} \leq C_s \|\mathbf{u}\|_{1,\Omega} \quad \forall \mathbf{u} \in H^1(\Omega)^3, 1 \leq s \leq 6, \quad (13)$$

$$|(\varphi \mathbf{q}, \mathbf{v})| \leq C^p \|\mathbf{q}\|_{L^p(\Omega)} \|\varphi\|_{L^6(\Omega)} \|\mathbf{v}\|_{L^6(\Omega)^3} \quad \forall \mathbf{q} \in L^p(\Omega)^3, \varphi \in H^1(\Omega), \mathbf{v} \in H^1(\Omega)^3. \quad (14)$$

Here  $C_s$  is a constant dependent on  $\Omega$  and  $s \in [1, 6]$ ,  $C^p$  is a constant dependent on  $\Omega$  and  $p$  at  $p \geq 3/2$ . The inequality (12) is a consequence of Sobolev's embedding theorem, according to which the space  $H^1(\Omega)$  is imbedded into  $L^s(\Omega)$  continuously at  $s \leq 6$  and compactly at  $s < 6$ . The inequality (14) is a consequence of the Hölder inequality for the three functions. In turn, the following inequality:

$$|(\varphi \mathbf{q}, \mathbf{v})| \leq \hat{C}_p \|\mathbf{q}\|_{L^p(\Omega)} \|\varphi\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{q} \in L^p(\Omega)^3, \varphi \in H^1(\Omega), \quad (15)$$

is the consequence of inequalities (12) and (14) where

$$\mathbf{v} \in H^1(\Omega)^3, \hat{C}_p = C^p C_6^2.$$

A similar inequality holds for the scalar functions  $q, \varphi$ , and  $h$ . It has the form

$$|(q\varphi, h)| \leq \hat{C}_p \|q\|_{L^p(\Omega)} \|\varphi\|_{1,\Omega} \|h\|_{1,\Omega} \quad \forall q \in L^p(\Omega), \varphi, h \in H^1(\Omega). \quad (16)$$

Along with inequalities (12)–(16), we will use a number of other important inequalities and properties of bilinear and trilinear forms, which we will write as the following Lemma.

**Lemma 1.** *Let conditions 2.1, 2.4, 2.5 and 2.6 be met and let  $\mathbf{u} \in H_{\text{div}}^1(\Omega)$  be a given function. Then there are the positive constants  $\delta_0, \delta_1, \gamma_1, \gamma'_1, \gamma_2, \gamma'_2, \gamma_3$ , and  $\beta$ , which depend on  $\Omega$ , and the constants  $\beta_p$  and  $\gamma_p$  depending on  $\Omega$  and  $p$ , such that the following relationships are fulfilled:*

$$|(v(\varphi) \nabla \mathbf{u}, \nabla \mathbf{v})| \leq \nu_{\max} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{u}, \mathbf{v} \in H^1(\Omega)^3 \text{ and } \varphi \in H^1(\Omega), \quad (17)$$

$$\begin{aligned} (\nabla \mathbf{v}, \nabla \mathbf{v}) &\geq \delta_0 \|\mathbf{v}\|_{1,\Omega}^2, \quad (v(\varphi) \nabla \mathbf{v}, \nabla \mathbf{v}) \geq \nu_* \|\mathbf{v}\|_{1,\Omega}^2 \\ \forall \mathbf{v} \in H_0^1(\Omega)^3, \quad \varphi \in H^1(\Omega), \quad \nu_* &= \delta_0 \nu_{\min}, \end{aligned} \quad (18)$$

$$\begin{aligned} |((\mathbf{w} \cdot \nabla) \mathbf{u}, \mathbf{v})| &\leq \gamma'_1 \|\mathbf{w}\|_{L^4(\Omega)^3} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \leq \\ &\leq \gamma_1 \|\mathbf{w}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{w}, \mathbf{u}, \mathbf{v} \in H^1(\Omega)^3, \end{aligned} \quad (19)$$

$$((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w}) = -((\mathbf{u} \cdot \nabla) \mathbf{w}, \mathbf{v}), \quad ((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega)^3, \quad \mathbf{w} \in H^1(\Omega)^3, \quad (20)$$

$$\sup_{\mathbf{v} \in H_0^1(\Omega)^3, \mathbf{v} \neq 0} -(\operatorname{div} \mathbf{v}, p) / \|\mathbf{v}\|_{1,\Omega} \geq \beta \|p\|_{\Omega} \quad \forall p \in L_0^2(\Omega), \quad (21)$$

$$|(\lambda(\eta) \nabla \varphi, \nabla h)| \leq \lambda_{\max} \|\eta\|_{1,\Omega} \|\varphi\|_{1,\Omega} \|h\|_{1,\Omega} \quad \forall \varphi, \eta, h \in H^1(\Omega), \quad (22)$$

$$\begin{aligned} (\nabla h, \nabla h) &\geq \delta_1 \|h\|_{1,\Omega}^2, \quad (\lambda(\varphi) \nabla h, \nabla h) \geq \lambda_* \|h\|_{1,\Omega}^2 \\ \forall \varphi \in H^1(\Omega), \quad h &\in H_0^1(\Omega), \quad \lambda_* \equiv \delta_1 \lambda_{\min}, \end{aligned} \quad (23)$$

$$\begin{aligned} |(\mathbf{w} \cdot \nabla \varphi, h)| &\leq \gamma'_2 \|\mathbf{w}\|_{L^4(\Omega)^3} \|\varphi\|_{1,\Omega} \|h\|_{1,\Omega} \leq \\ &\leq \gamma_2 \|\mathbf{w}\|_{1,\Omega} \|\varphi\|_{1,\Omega} \|h\|_{1,\Omega} \quad \forall \mathbf{w} \in H^1(\Omega)^3, \quad \varphi, h \in H^1(\Omega), \end{aligned} \quad (24)$$

$$(\mathbf{u} \cdot \nabla \varphi, h) = -(\mathbf{u} \cdot \nabla h, \varphi), \quad (\mathbf{u} \cdot \nabla h, h) = 0 \quad \forall \varphi \in H^1(\Omega), \quad h \in H_0^1(\Omega), \quad (25)$$

$$|(\mathbf{u} \cdot \nabla \varphi, h)| \leq \|\mathbf{u}\|_{1,\Omega} \|\varphi\|_{L^4(\Omega)} \|h\|_{1,\Omega} \quad \forall \varphi \in H^1(\Omega), \quad h \in H_0^1(\Omega) \quad (26)$$

$$|(\mathbf{b}(\eta) \varphi, \mathbf{v})| \leq \beta_p \|\varphi\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \quad \forall \eta, \varphi \in H^1(\Omega), \quad \mathbf{v} \in H_0^1(\Omega)^3, \quad (27)$$

$$|(k(\eta) \varphi, h)| \leq \gamma_p \|\varphi\|_{1,\Omega} \|h\|_{1,\Omega} \quad \forall \eta, \varphi \in H^1(\Omega), \quad h \in H_0^1(\Omega). \quad (28)$$

Here  $\delta_1 = 1/C_p^2$ ,  $\beta_p = \hat{\beta}_p \hat{C}_6$ ,  $\gamma_p = \hat{\gamma}_p \hat{C}_6$ , where the constants  $C_p$ ,  $\hat{\beta}_p$ ,  $\hat{\gamma}_p$  and  $\hat{C}_6$  were defined in (4), (5), (7) and (15) respectively;  $\nu_{\min}$ ,  $\nu_{\max}$ ,  $\lambda_{\min}$  and  $\lambda_{\max}$  – constants defined in (9).

We prove, for example, (27) and establish a relationship between the constant  $\beta_p$  and the constant  $\hat{\beta}_p$  defined in (5). To do this we use the inequality (15) at  $\mathbf{q} = \mathbf{b}(\varphi)$  and property (5). Using these relations successively we have :

$$\begin{aligned} |(\mathbf{b}(\eta) \varphi, \mathbf{v})| &\leq \hat{C}_p \|\mathbf{b}(\eta)\|_{L^p(\Omega)} \|\varphi\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \leq \hat{C}_p \hat{\beta}_p \|\varphi\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \equiv \\ &\beta_p \|\varphi\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}. \end{aligned}$$

Proof of the remaining statements constituting Lemma 1 we leave to the reader. They can also be found in [56–59].

**Remark 1.** From the estimate (15) and the property (6) of the function  $\mathbf{b}(\varphi)$  the following estimate follows for the difference  $\mathbf{b}(\varphi_1) - \mathbf{b}(\varphi_2)$ :

$$\begin{aligned} |((\mathbf{b}(\varphi_1) - \mathbf{b}(\varphi_2)) \varphi, \mathbf{v})| &\leq \\ &\leq \hat{C}_p \|\mathbf{b}(\varphi_1) - \mathbf{b}(\varphi_2)\|_{L^p(\Omega)} \|\varphi\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \leq \\ &\leq \hat{C}_p L_b \|\varphi_1 - \varphi_2\|_{L^4(\Omega)} \|\varphi\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \quad \forall \varphi, \varphi_1, \varphi_2 \in H^1(\Omega), \quad \mathbf{v} \in H_0^1(\Omega)^3. \end{aligned} \quad (29)$$

Similarly, the following estimate for the difference  $k(\varphi_1) - k(\varphi_2)$  follows from the estimate (25) and the property (8) of the function  $k(\varphi)$ :

$$\begin{aligned} |((k(\varphi_1) - k(\varphi_2)) \varphi, h)| &\leq \\ &\leq \hat{C}_p \|k(\varphi_1) - k(\varphi_2)\|_{L^p(\Omega)} \|\varphi\|_{1,\Omega} \|h\|_{1,\Omega} \leq \end{aligned}$$

$$\leq \hat{C}_p L_k \|\varphi_1 - \varphi_2\|_{L^4(\Omega)} \|\varphi\|_{1,\Omega} \|h\|_{1,\Omega} \forall \varphi, \varphi_1, \varphi_2, h \in H^1(\Omega). \quad (30)$$

The following lemmas about the existence of liftings for the velocity and concentration will play an important role below.

**Lemma 2.** *Let under the assumption 2.1 the boundary vector  $\mathbf{g}$  satisfy the condition 2.3.. Then for any arbitrary number  $\varepsilon > 0$  there exists a function (velocity lifting)  $\mathbf{u}_\varepsilon \in H^1(\Omega)^3$  such that  $\operatorname{div} \mathbf{u}_\varepsilon = 0$  in  $\Omega$ ,  $\mathbf{u}_\varepsilon = \mathbf{g}$  on  $\Gamma$  and*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega} \leq C_\varepsilon \|\mathbf{g}\|_{1/2,\Gamma}, |(\mathbf{v} \cdot \nabla) \mathbf{u}_\varepsilon, \mathbf{v}| \leq \varepsilon \|\mathbf{g}\|_{1/2,\Gamma} \|\mathbf{v}\|_{1,\Omega}^2 \forall \mathbf{v} \in V.$$

Here  $C_\varepsilon$  is a constant dependent on  $\varepsilon$  and  $\Omega$ .

**Lemma 3.** *Let assumption 2.1 be fulfilled. Then there is a family of continuous non-decreasing functions  $M_\varepsilon : \mathbb{R}_+ \equiv (0, \infty) \rightarrow \mathbb{R}$  with  $M_\varepsilon(0) = 0$  depending on the parameter  $\varepsilon \in (0, 1]$  as well as from  $\Omega$ , such that for any function  $\psi \in H^{1/2}(\Gamma)$  that is not equal to zero, there exists a function  $\varphi_\varepsilon \in H^1(\Omega)$  that satisfies the conditions*

$$\varphi_\varepsilon|_\Gamma = \psi, \|\varphi_\varepsilon\|_{L^4(\Omega)} \leq \varepsilon, \|\varphi_\varepsilon\|_{1,\Omega} \leq M_\varepsilon^\psi \equiv M_\varepsilon(\|\psi\|_{1/2,\Gamma}) \forall \varepsilon \in (0, 1].$$

**Proof of Lemma 3.** Detailed proof of Lemmas 2 and 3 can be found in book [58] (appendix 2).  $\square$

### 3. Global solvability of Problem 1

Our nearest goal is to derive the weak formulation of Problem 1 and to prove the existence of its solution. To this end, we multiply the first equation in (1) by the function  $\mathbf{v} \in H_0^1(\Omega)^3$ , equation (2) by the function  $h \in H_0^1(\Omega)$  and integrate the result over  $\Omega$  using Green's formulas. As a result, we will get a weak formulation of Problem 1. It consists in finding a trio of functions  $(\mathbf{u}, \varphi, p) \in H^1(\Omega)^3 \times H^1(\Omega) \times L_0^2(\Omega)$  that satisfy the relations:

$$\begin{aligned} (v(\varphi) \nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) &= \\ &= \langle \mathbf{f}, \mathbf{v} \rangle + (\mathbf{b}(\varphi) \varphi, \mathbf{v}) \forall \mathbf{v} \in H_0^1(\Omega)^3, \end{aligned} \quad (31)$$

$$(\lambda(\varphi) \nabla \varphi, \nabla h) + (k(\varphi) \varphi, h) + (\mathbf{u} \cdot \nabla \varphi, h) = \langle f, h \rangle \forall h \in H_0^1(\Omega), \quad (32)$$

$$\operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \mathbf{u} = \mathbf{g} \text{ and } \varphi = \psi \text{ on } \Gamma. \quad (33)$$

The specified three functions  $(\mathbf{u}, \varphi, p)$  satisfying (31) – (33), will be called below a weak solution to Problem 1.

Consider the restriction of the identity (31) by the space  $V$  which, taking into account the condition  $\operatorname{div} \mathbf{v} = 0$  for  $\mathbf{v} \in V$ , becomes :

$$(v(\varphi) \nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle + (\mathbf{b}(\varphi) \varphi, \mathbf{v}) \forall \mathbf{v} \in V. \quad (34)$$

It is well known that for the proof of existence of a weak solution of Problem 1 it is enough to prove existence of the solution  $(\mathbf{u}, \varphi) \in H^1(\Omega)^3 \times H^1(\Omega)$  of problem (32)–(34), and then, using the standard scheme, to restore the pressure  $p \in L_0^2(\Omega)$  so that the identity (31) is fulfilled. One can read more about pressure recovery in [57], p. 134, [58], Section 3.2.

We will look for the solution  $(\mathbf{u}, \varphi) \in H^1(\Omega)^3 \times H^1(\Omega)$  of problem (32)–(34) in a form

$$\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}, \varphi = \varphi_0 + \tilde{\varphi}. \quad (35)$$

Here  $\mathbf{u}_0 \equiv \mathbf{u}_{\varepsilon_0}$  and  $\varphi_0 = \varphi_{\varepsilon_0}$  are the velocity and concentration liftings defined above, and  $\tilde{\mathbf{u}} \in V$  and  $\tilde{\varphi} \in H_0^1(\Omega)$  are the new unknown functions which we are looking for. The value  $\varepsilon_0$  of the parameter  $\varepsilon$  will be chosen so that the following conditions are fulfilled:

$$\|\mathbf{u}_{\varepsilon_0}\|_{1,\Omega} \leq C_{\varepsilon_0} \|\mathbf{g}\|_{1/2,\Gamma}, |((\mathbf{v} \cdot \nabla) \mathbf{u}_{\varepsilon_0}, \mathbf{v})| \leq (\nu_*/2) \|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \mathbf{v} \in V. \quad (36)$$

The corresponding value  $\varepsilon_0$  of the parameter  $\varepsilon$  will be selected below.

Substituting (35) in (32), (34), we come to the following relations with respect to the pair  $(\tilde{\mathbf{u}}, \tilde{\varphi})$ :

$$\begin{aligned} & (\nu(\varphi_0 + \tilde{\varphi}) \nabla \tilde{\mathbf{u}}, \nabla \mathbf{v}) + ((\mathbf{u}_0 \cdot \nabla) \tilde{\mathbf{u}}, \mathbf{v}) + ((\tilde{\mathbf{u}} \cdot \nabla) \mathbf{u}_0, \mathbf{v}) + ((\tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}}, \mathbf{v}) = \\ & = \langle \mathbf{f}, \mathbf{v} \rangle - (\nu(\varphi_0 + \tilde{\varphi}) \nabla \mathbf{u}_0, \nabla \mathbf{v}) - ((\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0, \mathbf{v}) + (\mathbf{b}(\varphi_0 + \tilde{\varphi})(\varphi_0 + \tilde{\varphi}), \mathbf{v}) \quad \forall \mathbf{v} \in V, \end{aligned} \quad (37)$$

$$\begin{aligned} & (\lambda(\varphi_0 + \tilde{\varphi}) \nabla \tilde{\varphi}, \nabla h) + (k(\varphi_0 + \tilde{\varphi}) \tilde{\varphi}, h) + \\ & + (\mathbf{u}_0 \cdot \nabla \tilde{\varphi}, h) + (\tilde{\mathbf{u}} \cdot \nabla \varphi_0, h) + (\tilde{\mathbf{u}} \cdot \nabla \tilde{\varphi}, h) = \\ & = \langle f, h \rangle - (\lambda(\varphi_0 + \tilde{\varphi}) \nabla \varphi_0, \nabla h) - (\mathbf{u}_0 \cdot \nabla \varphi_0, h) - (k(\varphi_0 + \tilde{\varphi}) \varphi_0, h) \quad \forall h \in H_0^1(\Omega). \end{aligned} \quad (38)$$

To prove the existence of the solution  $(\tilde{\mathbf{u}}, \tilde{\varphi}) \in V \times H_0^1(\Omega)$  of problem (37), (38), we apply Schauder's fixed point theorem according to the scheme proposed in [58], Section 4.2. To this end we define the pairs  $\mathbf{z} = (\mathbf{w}, \eta) \in W$ ,  $\mathbf{y} = (\tilde{\mathbf{u}}, \tilde{\varphi}) \in W$  and construct the operator  $F : W \rightarrow W$  acting by:  $F(\mathbf{z}) = \mathbf{y}$ , where  $\mathbf{y} = (\tilde{\mathbf{u}}, \tilde{\varphi}) \in W$  is the solution of the linear problem

$$\begin{aligned} a_1^{\mathbf{w},\eta}(\tilde{\mathbf{u}}, \mathbf{v}) & \equiv (\nu(\varphi_0 + \eta) \nabla \tilde{\mathbf{u}}, \nabla \mathbf{v}) + ((\mathbf{u}_0 \cdot \nabla) \tilde{\mathbf{u}}, \mathbf{v}) + ((\tilde{\mathbf{u}} \cdot \nabla) \mathbf{u}_0, \mathbf{v}) + ((\mathbf{w} \cdot \nabla) \tilde{\mathbf{u}}, \mathbf{v}) = \\ & = \langle \mathbf{f}_1, \mathbf{v} \rangle + (\mathbf{b}(\varphi_0 + \eta)(\varphi_0 + \tilde{\varphi}), \mathbf{v}) \quad \forall \mathbf{v} \in V, \end{aligned} \quad (39)$$

$$\begin{aligned} a_2^{\mathbf{w},\eta}(\tilde{\varphi}, h) & \equiv (\lambda(\varphi_0 + \eta) \nabla \tilde{\varphi}, \nabla h) + (k(\varphi_0 + \eta) \tilde{\varphi}, h) + (\mathbf{u}_0 \cdot \nabla \tilde{\varphi}, h) + \\ & + (\mathbf{w} \cdot \nabla \tilde{\varphi}, h) = \langle f_1, h \rangle - (\mathbf{w} \cdot \nabla \varphi_0, h) \quad \forall h \in H_0^1(\Omega). \end{aligned} \quad (40)$$

Here  $(\mathbf{w}, \eta) \in V \times H_0^1(\Omega)$  is a given pair of functions, functionals  $\mathbf{f}_1 : V \rightarrow \mathbb{R}$  and  $f_1 : H_0^1(\Omega) \rightarrow \mathbb{R}$  are defined by formulas

$$\langle \mathbf{f}_1, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle - (\nu(\varphi_0 + \eta) \nabla \mathbf{u}_0, \nabla \mathbf{v}) - ((\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0, \mathbf{v}), \quad (41)$$

$$\langle f_1, h \rangle = \langle f, h \rangle - (\lambda(\varphi_0 + \eta) \nabla \varphi_0, \nabla h) - (\mathbf{u}_0 \cdot \nabla \varphi_0, h) - (k(\varphi_0 + \eta) \varphi_0, h). \quad (42)$$

Using (17), (18), (19), (22), (23), (24), (25), (28), (27), (35), (36) and Lemmas 2, 3, we deduce that

$$|(\nu(\varphi_0 + \eta) \nabla \mathbf{u}_0, \nabla \mathbf{v})| \leq \nu_{\max} C_{\varepsilon_0} \|\mathbf{g}\|_{1/2,\Gamma} \|\mathbf{v}\|_{1,\Omega} \quad \forall \eta \in H_0^1(\Omega), \mathbf{v} \in V, \quad (43)$$

$$(\nu(\varphi_0 + \eta) \nabla \mathbf{v}, \nabla \mathbf{v}) \geq \nu_* \|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \eta \in H_0^1(\Omega), \mathbf{v} \in V, \quad (44)$$

$$(\nu(\varphi_0 + \eta) \nabla \mathbf{v}, \nabla \mathbf{v}) + ((\mathbf{v} \cdot \nabla) \mathbf{u}_0, \mathbf{v}) \geq (\nu_*/2) \|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \eta \in H_0^1(\Omega), \mathbf{v} \in V, \quad (45)$$

$$|\lambda(\varphi_0 + \eta) \nabla \varphi_0, \nabla h| \leq \lambda_{\max} \|\varphi_0\|_{1,\Omega} \|h\|_{1,\Omega} \leq \lambda_{\max} M_\varepsilon^\psi \|h\|_{1,\Omega} \quad \forall \eta, h \in H_0^1(\Omega), \quad (46)$$

$$|\lambda(\varphi_0 + \eta) \nabla h, \nabla h| \geq \lambda_* \|h\|_{1,\Omega}^2, \quad \forall \eta, h \in H_0^1(\Omega), \quad (47)$$

$$|(\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0, \mathbf{v}| \leq \gamma_1 \|\mathbf{u}_0\|_{1,\Omega}^2 \|\mathbf{v}\|_{1,\Omega} \leq \gamma_1 C_{\varepsilon_0}^2 \|\mathbf{g}\|_{1,\Gamma}^2 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V, \quad (48)$$

$$|(\mathbf{u}_0 \cdot \nabla \varphi_0, h)| \leq \gamma_2 \|\mathbf{u}_0\|_{1,\Omega} \|\varphi_0\|_{1,\Omega} \|h\|_{1,\Omega} \leq \gamma_2 \|\mathbf{u}_0\|_{1,\Omega} M_\varepsilon^\psi \|h\|_{1,\Omega}, \quad h \in H_0^1(\Omega), \quad (49)$$

$$\begin{aligned} |(\mathbf{w} \cdot \nabla \varphi_0, h)| & \leq \gamma_2' \|\mathbf{w}\|_{1,\Omega} \|\varphi_0\|_{L^4(\Omega)} \|h\|_{1,\Omega} \leq \\ & \gamma_2' \varepsilon \|\mathbf{w}\|_{1,\Omega} \|h\|_{1,\Omega}, \quad \forall \mathbf{w} \in V, h \in H_0^1(\Omega), \end{aligned} \quad (50)$$

$$|\mathbf{b}(\varphi_0 + \eta)(\varphi_0 + \tilde{\varphi}), \mathbf{v}| \leq \beta_p \|\varphi_0 + \tilde{\varphi}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \quad \forall \eta \in H_0^1(\Omega), \mathbf{v} \in V, \quad (51)$$

$$|k(\varphi_0 + \eta)\varphi_0, h| \leq \gamma_p \|\varphi_0\|_{1,\Omega} \|h\|_{1,\Omega} \leq \gamma_p M_\epsilon^\psi \|h\|_{1,\Omega} \quad \forall \eta, h \in H_0^1(\Omega). \quad (52)$$

From (43), (46), (48), (49), (51), (52) follows that  $\mathbf{f}_1 \in V^*$ ,  $f_1 \in H^{-1}(\Omega)$  and the following estimates are valid:

$$\|\mathbf{f}_1\|_{-1,\Omega} \leq M_{\mathbf{f}_1} \equiv \|\mathbf{f}\|_{-1,\Omega} + M_{\mathbf{g}}, \quad M_{\mathbf{g}} = \nu_{\max} C_{\epsilon_0} \|\mathbf{g}\|_{1/2,\Gamma} + \gamma_1 C_{\epsilon_0}^2 \|\mathbf{g}\|_{1/2,\Gamma}^2, \quad (53)$$

$$\|f_1\|_{H^{-1}(\Omega)} \leq M_{f_1} = \|f\|_{-1,\Omega} + \lambda_{\max} M_\epsilon^\psi + \gamma_2 C_{\epsilon_0} \|\mathbf{g}\|_{1/2,\Gamma} M_\epsilon^\psi + \gamma_p M_\epsilon^\psi. \quad (54)$$

In addition, for each fixed pair  $(\mathbf{w}, \eta) \in V \times H_0^1(\Omega)$  bilinear form  $a_2^{\mathbf{w},\eta} : H_0^1(\Omega) \times H_0^1(\Omega)$  defined in (40) is continuous and coercive with the constant  $\lambda_*$  defined in (23), while the right-hand side of the identity (40) defines a linear continuous functional over  $H_0^1(\Omega)$ , and the following estimates hold:

$$a_2^{\mathbf{w},\eta}(h, h) \geq \lambda_{\min}(\nabla h, \nabla h) \geq \lambda_* \|h\|_{1,\Omega}^2 \quad \forall h \in H_0^1(\Omega), \quad (55)$$

$$|\langle f_1, h \rangle - (\mathbf{w} \cdot \nabla \varphi_0, h)| \leq (M_{f_1} + \gamma'_2 \epsilon \|\mathbf{w}\|_{1,\Omega}) \|h\|_{1,\Omega} \quad \forall h \in H_0^1(\Omega). \quad (56)$$

In this case, it follows from the Lax-Milgram theorem that for any pair  $(\mathbf{w}, \eta) \in V \times H_0^1(\Omega)$  a solution  $\tilde{\varphi} = \tilde{\varphi}_{\mathbf{w},\eta} \in H_0^1(\Omega)$  of problem (40) exists and is unique. Besides, by (55), (56) and Lemma 3 the following estimates are performed for  $\tilde{\varphi}$  and  $\varphi \equiv \varphi_0 + \tilde{\varphi}$ :

$$\|\tilde{\varphi}\|_{1,\Omega} \leq c_* \gamma'_2 \epsilon \|\mathbf{w}\|_{1,\Omega} + c_* M_{f_1} \quad (c_* = 1/\lambda_*), \quad (57)$$

$$\|\varphi_0 + \tilde{\varphi}\|_{1,\Omega} \leq c_* \gamma'_2 \epsilon \|\mathbf{w}\|_{1,\Omega} + c_* M_{f_1} + M_\epsilon^\psi. \quad (58)$$

Let us turn to the problem (39) where we put  $\tilde{\varphi} = \tilde{\varphi}_{\mathbf{w},\eta}$ . From (51), (53) and (58) it follows that the right-hand side in (39) is a value on the element  $\mathbf{v} \in V$  of the linear continuous functional of  $V^*$  and we have

$$\begin{aligned} |\langle \mathbf{f}_1, \mathbf{v} \rangle - (\mathbf{b}(\varphi_0 + \eta)(\varphi_0 + \tilde{\varphi}), \mathbf{v})| &\leq M_{\mathbf{f}_1} + \beta_p \|\varphi_0 + \tilde{\varphi}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \leq \\ &\leq [M_{\mathbf{f}_1} + \beta_p (c_* \gamma'_2 \epsilon \|\mathbf{w}\|_{1,\Omega} + c_* \tilde{M}_{f_1} + M_\epsilon^\psi)] \|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{v} \in V. \end{aligned}$$

In turn, from the estimates (43), (44), (45) and from the second identity in (20) it follows that the bilinear form  $a_1^{\mathbf{w},\eta} : V \times V \rightarrow \mathbb{R}$  defined in (39), is continuous and coercive with constant  $(\nu_*/2)$ , where  $\nu_*$  is defined in (18). From Lax-Milgram's theorem, it then follows that for any pair  $(\mathbf{w}, \eta) \in V \times H_0^1(\Omega)$  there is a single solution  $\tilde{\mathbf{u}} \in V$  of the problem (39) and the following estimate is performed:

$$(\nu_*/2) \|\tilde{\mathbf{u}}\|_{1,\Omega} \leq \beta_p c_* \gamma'_2 \epsilon \|\mathbf{w}\|_{1,\Omega} + \beta_p c_* M_{f_1} + \beta_p M_\epsilon^\psi + M_{\mathbf{f}_1}. \quad (59)$$

Let us assume that  $\nu_* \leq 2\beta_p c_* \gamma'_2$  and choose the parameter  $\epsilon$  from the condition

$$\beta_p c_* \gamma'_2 \epsilon = (\nu_*/4), \quad \epsilon = \frac{\nu_*}{4\beta_0 \gamma'_2 c_*}. \quad (60)$$

From (59) we infer that

$$(\nu_*/2) \|\tilde{\mathbf{u}}\|_{1,\Omega} \leq (\nu_*/4) \|\mathbf{w}\|_{1,\Omega} + \beta_p c_* M_{f_1} + \beta_p M_\epsilon^\psi + M_{\mathbf{f}_1}. \quad (61)$$

Considering (60), from (57) we conclude that

$$\|\tilde{\varphi}\|_{1,\Omega} \leq (\nu_*/4\beta_p) \|\mathbf{w}\|_{1,\Omega} + c_* M_{f_1}. \quad (62)$$

An important feature of the estimates (61), (62) is the fact that their right-hand parts depend only on  $\|\mathbf{w}\|_{1,\Omega}$ , but do not depend on  $\|\eta\|_{1,\Omega}$ . This will allow us to select below the convex closed set  $K \subset W$  which the operator  $F$  maps into itself.

Still it was assumed that  $(\mathbf{w}, \eta)$  is an arbitrary pair of functions from  $W$ . Suppose now that  $\eta$  is still arbitrary, while  $\mathbf{w}$  satisfies the condition

$$\|\mathbf{w}\|_{1,\Omega} \leq r_1 \equiv (4/\nu_*)(\beta_p c_* M_{f_1} + \beta_p M_\epsilon^\psi + M_{f_1}). \quad (63)$$

By this assumption, it follows from (61) that

$$\|\tilde{\mathbf{u}}\|_{1,\Omega} \leq (2/\nu_*)[(\nu_*/4)r_1 + \beta_p c_* M_{f_1} + \beta_p M_\epsilon^\psi + M_{f_1}]. \quad (64)$$

According to (63) we have that

$$(\nu_*/4)r_1 = \beta_p c_* M_{f_1} + \beta_p M_\epsilon^\psi + M_{f_1}.$$

In this case, from (64) we conclude that

$$\|\tilde{\mathbf{u}}\|_{1,\Omega} \leq M_{\tilde{\mathbf{u}}} \equiv (4/\nu_*)(\beta_p c_* M_{f_1} + \beta_p M_\epsilon^\psi + M_{f_1}) = r_1. \quad (65)$$

Similarly, considering (63), from (62) we get

$$\begin{aligned} \|\tilde{\varphi}\|_{1,\Omega} \leq \frac{\nu_*}{4\beta_p} \cdot \frac{4(\beta_p c_* M_{f_1} + \beta_p M_\epsilon^\psi + M_{f_1})}{\nu_*} + c_* M_{f_1} = M_{\tilde{\varphi}} \equiv \\ 2c_* M_{f_1} + M_\epsilon^\psi + \beta_p^{-1} M_{f_1}. \end{aligned} \quad (66)$$

Thus, we showed that the operator  $F : W \rightarrow W$  maps a pair  $(\mathbf{w}, \eta) \in W$ , where  $\mathbf{w}$  satisfies (63), and  $\eta$  is an arbitrary function, into the pair  $(\tilde{\mathbf{u}}, \tilde{\varphi}) \in W$ , for which the next estimates are fulfilled:

$$\|\mathbf{u}\|_{1,\Omega} \leq M_{\tilde{\mathbf{u}}}, \|\tilde{\varphi}\|_{1,\Omega} \leq M_{\tilde{\varphi}}, \|\varphi_0 + \tilde{\varphi}\|_{1,\Omega} \leq M_\varphi \equiv M_{\tilde{\varphi}} + M_{\epsilon^*}^\psi. \quad (67)$$

Therefore, if we choose a convex closed set

$$K = \{(\mathbf{w}, \eta) \in W : \|\mathbf{w}\|_{1,\Omega} \leq M_{\tilde{\mathbf{u}}}, \|\eta\|_{1,\Omega} \leq M_{\tilde{\varphi}}\} \quad (68)$$

in the space  $W$ , then the obtained estimates (67) mean that the operator  $F$  maps the set  $K$  to itself.

To apply Schauder's theorem, we have to prove that the operator  $F$  is continuous and compact on the set  $K$  defined in (68). To this end, denote by  $\mathbf{z}_n = (\mathbf{w}_n, \eta_n)$ ,  $n = 1, 2, \dots$  an arbitrary sequence from  $K$ . Let us put  $\mathbf{y}_n \equiv (\tilde{\mathbf{u}}_n, \tilde{\varphi}_n) = F(\mathbf{z}_n)$ ,  $n = 1, 2, \dots$ , and show that from the sequence  $\mathbf{y}_n$  we can extract a subsequence converging in the norm defined in  $X$  to some element  $\mathbf{y} \in K$ .

Due to reflexivity of spaces  $H^1(\Omega)$  and  $H^1(\Omega)^3$  and compactness of embeddings of  $H^1(\Omega) \subset L^4(\Omega)$  and  $H^1(\Omega)^3 \subset L^4(\Omega)^3$  there exists the subsequence of sequence  $\{\mathbf{z}_n\} = \{(\mathbf{w}_n, \eta_n)\}$  which we again designate through  $\{\mathbf{z}_n\}$ , and there is  $\mathbf{z} = (\mathbf{w}, \eta) \in K$  such that

$$\mathbf{w}_n \rightarrow \mathbf{w} \text{ weakly in } H^1(\Omega)^3 \text{ and strongly in } L^4(\Omega)^3 \text{ as } n \rightarrow \infty, \quad (69)$$

$$\eta_n \rightarrow \eta \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^4(\Omega) \text{ as } n \rightarrow \infty. \quad (70)$$

Let  $\mathbf{y} = F(\mathbf{z})$ . By construction, the element  $\mathbf{y} \equiv (\tilde{\mathbf{u}}, \tilde{\varphi}) \in W$  is a solution to the problem (39), (40) corresponding to the pair  $(\mathbf{w}, \eta) = \mathbf{z}$ , while the element  $\mathbf{y}_n \equiv (\tilde{\mathbf{u}}_n, \tilde{\varphi}_n) \in W$  is solving the problem

$$a_1^{\mathbf{w}_n, \eta_n}(\tilde{\mathbf{u}}_n, \mathbf{v}) \equiv (\nu(\varphi_0 + \eta_n) \nabla \tilde{\mathbf{u}}_n, \nabla \mathbf{v}) + ((\mathbf{u}_0 \cdot \nabla) \tilde{\mathbf{u}}_n, \mathbf{v}) + ((\tilde{\mathbf{u}}_n \cdot \nabla) \mathbf{u}_0, \mathbf{v}) +$$

$$\begin{aligned} +((\mathbf{w}_n \cdot \nabla) \tilde{\mathbf{u}}_n, \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle - \nu(\varphi_0 + \eta_n) \nabla \mathbf{u}_0, \nabla \mathbf{v} - ((\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0, \mathbf{v}) + \\ &+ (\mathbf{b}(\varphi_0 + \eta_n)(\varphi_0 + \tilde{\varphi}_n), \mathbf{v}) \quad \forall \mathbf{v} \in V, \end{aligned} \quad (71)$$

$$\begin{aligned} a_2^{\mathbf{w}_n, \eta_n}(\varphi_n, h) &\equiv (\lambda(\varphi_0 + \eta_n) \nabla \tilde{\varphi}_n, \nabla h) + (k(\varphi_0 + \eta_n) \tilde{\varphi}_n, h) + (\mathbf{u}_0 \cdot \nabla \tilde{\varphi}_n, h) + \\ &+ (\mathbf{w}_n \cdot \nabla \tilde{\varphi}_n, h) = \langle f, h \rangle - (\lambda(\varphi_0 + \eta_n) \nabla \varphi_0, \nabla h) - \\ &- (\mathbf{u}_0 \cdot \nabla \varphi_0, h) - (\mathbf{w}_n \cdot \nabla \varphi_0, h) - (k(\varphi_0 + \eta_n) \varphi_0, h) \quad \forall h \in H_0^1(\Omega), \end{aligned} \quad (72)$$

which is obtained from (39), (40) by replacing  $\mathbf{z} = (\mathbf{w}, \eta)$  on  $\mathbf{z}_n = (\mathbf{w}_n, \eta_n)$ .

Let its show that  $\mathbf{y}_n \rightarrow \mathbf{y}$  strongly in  $X$  or, equivalently,

$$\tilde{\varphi}_n \rightarrow \tilde{\varphi} \text{ strongly in } H^1(\Omega) \text{ and } \tilde{\mathbf{u}}_n \rightarrow \tilde{\mathbf{u}} \text{ strong by in } H^1(\Omega)^3 \text{ as } n \rightarrow \infty.$$

To do this, we need to subtract (39), (40) from (71), (72). Taking into account the following equalities:

$$\begin{aligned} &(\lambda(\varphi_0 + \eta_n) \nabla(\varphi_0 + \tilde{\varphi}_n), \nabla h) - (\lambda(\varphi_0 + \eta) \nabla(\varphi_0 + \tilde{\varphi}), \nabla h) = \\ &(\lambda(\varphi_0 + \eta_n) \nabla(\tilde{\varphi}_n - \tilde{\varphi}), \nabla h) + ((\lambda(\varphi_0 + \eta_n) - \lambda(\varphi_0 + \eta)) \nabla(\varphi_0 + \tilde{\varphi}), \nabla h), \\ &(\nu(\varphi_0 + \eta_n) \nabla(\mathbf{u}_0 + \tilde{\mathbf{u}}_n), \nabla \mathbf{v}) - (\nu(\varphi_0 + \eta) \nabla(\mathbf{u}_0 + \tilde{\mathbf{u}}), \nabla \mathbf{v}) = \\ &= (\nu(\varphi_0 + \eta_n) \nabla(\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}), \nabla \mathbf{v}) + ((\nu(\varphi_0 + \eta_n) - \nu(\varphi_0 + \eta)) \nabla(\mathbf{u}_0 + \tilde{\mathbf{u}}), \nabla \mathbf{v}), \\ &((\mathbf{w} \cdot \nabla) \tilde{\mathbf{u}}_n, \mathbf{v}) - ((\mathbf{w} \cdot \nabla) \tilde{\mathbf{u}}, \mathbf{v}) = ((\mathbf{w}_n \cdot \nabla)(\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}), \mathbf{v}) + ((\mathbf{w}_n - \mathbf{w}) \cdot \nabla \tilde{\mathbf{u}}, \mathbf{v}) \\ &(\mathbf{w}_n \cdot \nabla \tilde{\varphi}_n, h) - (\mathbf{w} \cdot \nabla \tilde{\varphi}, h) = (\mathbf{w}_n \cdot \nabla(\tilde{\varphi}_n - \tilde{\varphi}), h) + ((\mathbf{w}_n - \mathbf{w}) \cdot \nabla \tilde{\varphi}, h), \\ &(k(\varphi_0 + \eta_n)(\varphi_0 + \tilde{\varphi}_n), h) - (k(\varphi_0 + \eta)(\varphi_0 + \tilde{\varphi}), h) = \\ &= (k(\varphi_0 + \eta_n)(\tilde{\varphi}_n - \tilde{\varphi}), h) + ((k(\varphi_0 + \eta_n) - k(\varphi_0 + \eta))(\varphi_0 + \tilde{\varphi}), h), \\ &(\mathbf{b}(\varphi_0 + \eta_n)(\varphi_0 + \tilde{\varphi}_n), \mathbf{v}) - (\mathbf{b}(\varphi_0 + \eta)(\varphi_0 + \tilde{\varphi}), \mathbf{v}) = \\ &= (\mathbf{b}(\varphi_0 + \eta_n)(\tilde{\varphi}_n - \tilde{\varphi}), \mathbf{v}) + ((\mathbf{b}(\varphi_0 + \eta_n) - \mathbf{b}(\varphi_0 + \eta))(\varphi_0 + \tilde{\varphi}), \mathbf{v}), \end{aligned}$$

we come to the relations:

$$\begin{aligned} a_2^{\mathbf{w}_n, \eta_n}(\tilde{\varphi}_n - \tilde{\varphi}, h) &\equiv (\lambda(\varphi_0 + \eta_n) \nabla(\tilde{\varphi}_n - \tilde{\varphi}), \nabla h) + (k(\varphi_0 + \eta_n)(\tilde{\varphi}_n - \tilde{\varphi}), h) + \\ &+ (\mathbf{w}_n \cdot \nabla(\tilde{\varphi}_n - \tilde{\varphi}), h) + (\mathbf{u}_0 \cdot \nabla(\tilde{\varphi}_n - \tilde{\varphi}), h) = \\ &= -((\lambda(\varphi_0 + \eta_n) - \lambda(\varphi_0 + \eta)) \nabla(\varphi_0 + \tilde{\varphi}), \nabla h) - ((\mathbf{w}_n - \mathbf{w}) \cdot \nabla(\varphi_0 + \tilde{\varphi}), h) - \\ &- ((k(\varphi_0 + \eta_n) - k(\varphi_0 + \eta))(\varphi_0 + \tilde{\varphi}), h) \quad \forall h \in H_0^1(\Omega), \end{aligned} \quad (73)$$

$$\begin{aligned} a_1^{\mathbf{w}_n, \eta_n}(\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}, \mathbf{v}) &\equiv (\nu(\varphi_0 + \eta_n) \nabla(\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}), \nabla \mathbf{v}) + (\mathbf{u}_0 \cdot \nabla)(\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}, \mathbf{v}) + \\ &(((\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}) \cdot \nabla) \mathbf{u}_0, \mathbf{v}) + (((\mathbf{w}_n \cdot \nabla)(\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}_n), \mathbf{v}) = \\ &= -((\nu(\varphi_0 + \eta_n) - \nu(\varphi_0 + \eta)) \nabla(\mathbf{u}_0 + \tilde{\mathbf{u}}), \nabla \mathbf{v}) - (((\mathbf{w}_n - \mathbf{w}) \cdot \nabla) \tilde{\mathbf{u}}, \mathbf{v}) + \\ &+ (\mathbf{b}(\varphi_0 + \eta_n)(\tilde{\varphi}_n - \tilde{\varphi}), \mathbf{v}) + ((\mathbf{b}(\varphi_0 + \eta_n) - \mathbf{b}(\varphi_0 + \eta))(\varphi_0 + \tilde{\varphi}), \mathbf{v}) \quad \forall \mathbf{v} \in V. \end{aligned} \quad (74)$$

Using the estimate (30) with  $\varphi_1 = \varphi_0 + \eta_n$ ,  $\varphi_2 = \varphi_0 + \eta$ , the estimate in (67) for  $\|\varphi_0 + \tilde{\varphi}\|_{1,\Omega}$  and (70), we deduce that

$$\begin{aligned} &|((k(\varphi_0 + \eta_n) - k(\varphi_0 + \eta))(\varphi_0 + \tilde{\varphi}), h)| \leq \\ &\leq \hat{C}_p L_k \|\eta_n - \eta\|_{L^4(\Omega)} \|\varphi_0 + \tilde{\varphi}\|_{1,\Omega} \|h\|_{1,\Omega} \leq \\ &\hat{C}_p L_k M_\varphi \|\eta_n - \eta\|_{L^4(\Omega)} \|h\|_{1,\Omega} \rightarrow 0 \text{ as } n \rightarrow \infty \quad \forall h \in H_0^1(\Omega). \end{aligned} \quad (75)$$

Similarly, using (24), (67), and (69), we have

$$\begin{aligned} \|(\mathbf{w}_n - \mathbf{w}) \cdot \nabla(\varphi_0 + \tilde{\varphi}), h\| &\leq \gamma'_2 \|\mathbf{w}_n - \mathbf{w}\|_{L^4(\Omega)} \|\varphi_0 + \tilde{\varphi}\|_{1,\Omega} \|h\|_{1,\Omega} \leq \\ &\leq \gamma'_2 M_\varphi \|\mathbf{w}_n - \mathbf{w}\|_{L^4(\Omega)^3} \|h\|_{1,\Omega} \rightarrow 0 \text{ as } n \rightarrow \infty \forall h \in H_0^1(\Omega). \end{aligned} \quad (76)$$

Also, from (11) at  $f = \nabla(\varphi_0 + \tilde{\varphi}) \cdot \nabla h$ ,  $\mu = \lambda(\varphi_0 + \eta)$ ,  $\mu_n = \lambda(\varphi_0 + \eta_n)$  it follows that

$$\begin{aligned} &((\lambda(\varphi_0 + \eta_n) - \lambda(\varphi_0 + \eta)) \nabla(\varphi_0 + \tilde{\varphi}), \nabla h) \equiv \\ &\int_{\Omega} [\lambda(\varphi_0 + \eta_n) - \lambda(\varphi_0 + \eta)] \nabla(\varphi_0 + \tilde{\varphi}) \cdot \nabla h dx \rightarrow 0 \text{ as } n \rightarrow \infty \forall h \in H_0^1(\Omega). \end{aligned} \quad (77)$$

As mentioned above, for each pair  $(\mathbf{w}_n, \eta_n) \in V \times H_0^1(\Omega)$  the bilinear with respect to the difference  $\tilde{\varphi}_n - \tilde{\varphi}$  and  $h$  form  $a_2^{\mathbf{w}_n, \eta_n} : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$  defined in (73) is continuous and coercive with constant  $\lambda_* > 0$ . This means by (75), (76), (77) and by virtue of the Lax–Milgram theorem applied to the problem (73) that

$$\|\tilde{\varphi}_n - \tilde{\varphi}\|_{1,\Omega} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (78)$$

According to a similar scheme, it can be proved that

$$\|\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}\|_{1,\Omega} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (79)$$

Really, using (19), (67), and (69), we conclude that

$$\begin{aligned} |(((\mathbf{w}_n - \mathbf{w}) \cdot \nabla) \tilde{\mathbf{u}}, \mathbf{v})| &\leq \gamma'_1 \|\mathbf{w}_n - \mathbf{w}\|_{L^4(\Omega)} \|\tilde{\mathbf{u}}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \rightarrow 0 \\ &\text{as } n \rightarrow \infty \forall \mathbf{v} \in V. \end{aligned} \quad (80)$$

Similarly, using (27) at  $\eta = \varphi_0 + \eta_n$ ,  $\varphi = \tilde{\varphi}_n - \tilde{\varphi}$ , and (78) or (29) at  $\varphi_1 = \varphi_0 + \eta_n$ ,  $\varphi_2 = \varphi_0 + \eta$  and (70), we conclude that

$$|(\mathbf{b}(\varphi_0 + \eta_n)(\tilde{\varphi}_n - \tilde{\varphi}), \mathbf{v})| \leq \beta_p \|\tilde{\varphi}_n - \tilde{\varphi}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \rightarrow 0 \text{ as } n \rightarrow \infty \forall \mathbf{v} \in V \quad (81)$$

or

$$\begin{aligned} &|((\mathbf{b}(\varphi_0 + \eta_n) - \mathbf{b}(\varphi_0 + \eta))(\varphi_0 + \tilde{\varphi}), \mathbf{v})| \leq \\ &\leq \hat{C}_p L_b \|\eta_n - \eta\|_{L^4(\Omega)} \|\varphi_0 + \tilde{\varphi}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \rightarrow 0 \text{ as } n \rightarrow \infty \forall \mathbf{v} \in V. \end{aligned} \quad (82)$$

Finally, from (11) at  $f = \nabla(\mathbf{u}_0 + \tilde{\mathbf{u}}) \cdot \nabla \mathbf{v}$ ,  $\mu = \nu(\varphi_0 + \eta)$ ,  $\mu_n = \nu(\varphi_0 + \eta_n)$  follows that

$$\begin{aligned} &((\nu(\varphi_0 + \eta_n) - \nu(\varphi_0 + \eta)) \nabla(\mathbf{u}_0 + \tilde{\mathbf{u}}), \nabla \mathbf{v}) \equiv \\ &\equiv \int_{\Omega} [\nu(\varphi_0 + \eta_n) - \nu(\varphi_0 + \eta)] \nabla(\mathbf{u}_0 + \tilde{\mathbf{u}}) \cdot \nabla \mathbf{v} dx \rightarrow 0 \text{ as } n \rightarrow \infty \forall \mathbf{v} \in V. \end{aligned} \quad (83)$$

Recall that for each pair  $(\mathbf{w}_n, \eta_n) \in V \times H_0^1(\Omega)$  bilinear with respect to the difference  $\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}$  and  $\mathbf{v}$  form  $a_1^{\mathbf{w}_n, \eta_n} : V \times V \rightarrow \mathbb{R}$  defined in (74) is continuous and coercive with constant  $\nu^*/2 > 0$ . Again, this means by (80), (81), (82) and (83) and by virtue of the Lax–Milgram theorem applied to the problem (74) the validity of the estimate (79).

From (78) and (79) it follows that the operator  $F : W \rightarrow W$  is continuous and compact on the set  $K$ . In this case, Schauder's fixed point theorem implies that the operator  $F$  has a fixed point  $\mathbf{y} \equiv (\tilde{\mathbf{u}}, \tilde{\varphi}) = F(\mathbf{z}) \in K$ , which by construction is the desired solution to the problem (37), (38) and satisfies estimates in (67). From this important fact, in turn, it follows that the pair  $(\mathbf{u}, \varphi) \in H^1(\Omega)^3 \times H^1(\Omega)$  defined in (35) is the desired solution to the problem (32) – (34) and the following estimates are valid :

$$\|\mathbf{u}\|_{1,\Omega} \leq M_{\mathbf{u}} \equiv M_{\tilde{\mathbf{u}}} + C_{\varepsilon_0} \|\mathbf{g}\|_{1/2,\Gamma}, \quad (84)$$

$$\|\varphi\|_{1,\Omega} \leq M_\varphi \equiv M_{\tilde{\varphi}} + M_{\epsilon^*}(\|\psi\|_{1/2,\Gamma}). \quad (85)$$

We formulate the results obtained in the form of the following theorem.

**Theorem 1.** *Let the conditions 2.1–2.6 be fulfilled. Then there exists at least one solution  $(\tilde{\mathbf{u}}, \tilde{\varphi}) \in V \times H_0^1(\Omega)$  of problem (32), (33), (34) and, in addition, the solution  $(\tilde{\mathbf{u}}, \tilde{\varphi})$  meets the estimates (84), (85).*

**Remark 2.** *The theorem corresponding to the case when the condition  $\lambda_* \nu_* > 2\beta_p \gamma_2'$  takes place instead of  $\lambda_* \nu_* \leq 2\beta_p \gamma_2'$  can be proved in a similar way.*

The existence of pressure  $p \in L_0^2(\Omega)$ , which together with the specified pair  $(\mathbf{u}, \varphi)$  satisfies the relation (31), is proved as in [58], Section 3.2. It remains to derive the estimate for  $p$ . For this purpose, we will use the inf-sup condition (21), according to which for the above function  $p$  and any (arbitrarily small) number  $\delta > 0$  there exists a function  $\mathbf{v}_0 \in H_0^1(\Omega)^3$ ,  $\mathbf{v}_0 \neq 0$ , such that

$$-(\operatorname{div} \mathbf{v}_0, p) \geq \beta_* \|\mathbf{v}_0\|_{1,\Omega} \|p\|_\Omega, \quad \beta_* = (\beta - \delta) > 0.$$

Assuming  $\mathbf{v} = \mathbf{v}_0$  in (31) and using the last inequality and estimates (17), (19), (27), we deduce that

$$\begin{aligned} \beta_* \|\mathbf{v}_0\|_{1,\Omega} \|p\|_\Omega &\leq \nu_{\max} \|\mathbf{v}_0\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} + \gamma_1 \|\mathbf{v}_0\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega}^2 + \\ &+ \beta_p \|\varphi\|_{1,\Omega} \|\mathbf{v}_0\|_{1,\Omega} + \|\mathbf{f}\|_{-1,\Omega} \|\mathbf{v}_0\|_{1,\Omega}. \end{aligned} \quad (86)$$

Dividing by  $\|\mathbf{v}_0\|_{1,\Omega} \neq 0$  and taking into account the estimate (84), we derive from (86) that

$$\|p\|_\Omega \leq M_p = \beta_*^{-1}[(\nu_{\max} + \gamma_1 M_{\mathbf{u}})M_{\mathbf{u}} + \beta_p M_\varphi + \|\mathbf{f}\|_{-1,\Omega}]. \quad (87)$$

We formulate the result in the form of the following theorem.

**Theorem 2.** *Under conditions of Theorem 1 there exists a weak solution  $(\mathbf{u}, \varphi, p) \in H^1(\Omega)^3 \times H^1(\Omega) \times L_0^2(\Omega)$  of Problem 1, for which estimates (84), (85) and (87) are performed.*

In conclusion of this Section, we will establish sufficient conditions on the data of Problem 1, under which the maximum (or minimum) principle is valid for the concentration component  $\varphi$  of the solution  $(\mathbf{u}, \varphi, p)$  of Problem 1.

Let  $f_{\max}$  be a positive number and, in addition to 2.1 – 2.6, the following conditions hold:

**3.1.**  $f \in L^\infty(\Omega) : 0 \leq f \leq f_{\max}$  a.e. in  $\Omega$ ;  $\psi_{\min} \leq \psi \leq \psi_{\max}$  a.e. on  $\Gamma$ .

**3.2.** the nonlinearity  $k(\varphi)\varphi$  is monotonic in the following sense:

$$(k(\varphi_1)\varphi_1 - k(\varphi_2)\varphi_2, \varphi_1 - \varphi_2) \geq 0 \quad \forall \varphi_1, \varphi_2 \in H^1(\Omega).$$

**3.3.** it is assumed that every of functional for  $M_1$  or  $m_1$  equations

$$k(M_1)M_1 = f_{\max}, \text{ or } k(m_1)m_1 = f_{\min} \quad (88)$$

has at least one (positive) solution  $M_1$  or  $m_1$ .

We set

$$M = \max\{\psi_{\max}, M_1\}, \quad m = \min\{\psi_{\min}, m_1\}. \quad (89)$$

**Theorem 3.** *Let under conditions 2.1 – 2.3 and 3.1 – 3.3 the functions  $\nu(\tau)$  and  $\lambda(\tau)$  be continuous for  $\tau \in \mathbb{R}$ , and*

$$0 \leq \nu_{\min} \leq \nu(\tau) < \infty, \quad 0 \leq \lambda_{\min} \leq \lambda(\tau) < \infty \quad \forall \tau \in \mathbb{R}.$$

Then for the component  $\varphi$  of the weak solution  $(\mathbf{u}, \varphi, p) \in H^1(\Omega)^3 \times H^1(\Omega) \times L_0^2(\Omega)$  of Problem 1 the maximum and minimum principle holds having the form:

$$m \leq \varphi \leq M \text{ everywhere in } \Omega. \quad (90)$$

Here  $m$  and  $M$  are constants defined in relation (89) in which  $M_1$  is choosed as a minimum root of the first equation in (88) while  $m_1$  is choosed as a maximum root of the second equation in (88).

**Proof of Theorem 3.** Let  $M$  be constant defined in (89). First, we will prove that  $\varphi \leq M$  a.e. in  $\Omega$ . To this end, we will define the function  $\tilde{\varphi} = \max\{\varphi - M, 0\}$ . It is clear that the principle of maximum or estimate  $\varphi \leq M$  holds a.e. in  $\Omega$  if and only if  $\tilde{\varphi} = 0$  everywhere in  $\Omega$ .

Denote by  $\Omega_M \subset \Omega$  an open measurable subset of  $\Omega$  in which  $\varphi > M$ . From ([59, p. 152], ) it follows that  $\nabla \tilde{\varphi} = \nabla \varphi$  everywhere in  $\Omega_M$  and, besides,  $\tilde{\varphi} \in H_0^1(\Omega)$ . So the following equalities take place:

$$\begin{aligned} (\lambda(\varphi)\nabla\varphi, \nabla\tilde{\varphi}) &= (\lambda(\varphi)\nabla\tilde{\varphi}, \nabla\tilde{\varphi})_{\Omega_M} = (\lambda(\varphi)\nabla\tilde{\varphi}, \nabla\tilde{\varphi}), \\ (\mathbf{u} \cdot \nabla\varphi, \tilde{\varphi}) &= (\mathbf{u} \cdot \nabla\tilde{\varphi}, \tilde{\varphi}) = 0. \end{aligned} \quad (91)$$

Taking into account (91) let us set  $h = \tilde{\varphi}$  in (32). We get that

$$(\lambda(\varphi)\nabla\tilde{\varphi}, \nabla\tilde{\varphi}) + (k(\varphi, \cdot)\varphi, \tilde{\varphi}) = (f, \tilde{\varphi}). \quad (92)$$

The following equations follow from the properties of function  $\tilde{\varphi}$ :

$$\begin{aligned} (k(\varphi, \cdot)\varphi, \tilde{\varphi}) &= (k(\varphi, \cdot)\varphi, \tilde{\varphi})_{\Omega_M} = \\ &= (k(\tilde{\varphi} + M, \cdot)(\tilde{\varphi} + M), \tilde{\varphi})_{\Omega_M}. \end{aligned}$$

Due to the property 3.2 for the functions  $\varphi_1 = \tilde{\varphi} + M$  and  $\varphi_2 = M$  which belong of  $H^1(\Omega)$ , and by condition  $\tilde{\varphi} = 0$  in  $\Omega \setminus \overline{\Omega}_M$  the following relation takes place:

$$\begin{aligned} 0 \leq (k(\tilde{\varphi} + M, \cdot)(\tilde{\varphi} + M) - k(M, \cdot)M, \tilde{\varphi}) &= \\ = (k(\tilde{\varphi} + M, \cdot)(\tilde{\varphi} + M) - k(M, \cdot)M, \tilde{\varphi})_{\Omega_M}. \end{aligned} \quad (93)$$

Now we subtract the equality  $(k(M)M, \tilde{\varphi}) \equiv (k(M)M, \tilde{\varphi})_{\Omega_M}$  from the both parts of (92). As a result we arrive at

$$\begin{aligned} (\lambda(\varphi)\nabla\tilde{\varphi}, \nabla\tilde{\varphi}) + (k(\tilde{\varphi} + M)(\tilde{\varphi} + M) - k(M)M, \tilde{\varphi})_{\Omega_M} &= \\ = (f - k(M)M, \tilde{\varphi})_{\Omega_M}. \end{aligned} \quad (94)$$

Using (23) and (93), we deduce from (94) that

$$\lambda_* \|\tilde{\varphi}\|_{1,\Omega}^2 \leq (f_{\max} - k(M)M, \varphi)_{\Omega_M}. \quad (95)$$

By definition of  $M$  in (89), it follows from (95) that (if  $M$  is selected from the first condition in (88)) then  $\tilde{\varphi} = 0$  and therefore  $\varphi \leq M$  in  $\Omega$ .

The principle of minimum is proved in a similar way using the non-positive function  $\hat{\varphi} = \min\{\varphi - m, 0\}$  (see in more details in [60]).  $\square$

#### 4. Conditional uniqueness of solution of Problem

In this Section, we will prove the conditional uniqueness of the weak solution  $(\mathbf{u}, \varphi, p)$  of Problem 1 provided that the component  $\varphi$  has an additional property of smoothness, namely:  $\varphi \in H^2(\Omega)$ .

We will assume, in addition to the property 2.6, that the leading coefficients  $\nu(\cdot)$  and  $\lambda(\cdot)$  have the following properties:

4.1.  $\lambda(\cdot)$  belongs to the space  $C^1(\mathbb{R})$  and

$$\lambda'_{\min} \leq \lambda'(s) \leq \lambda'_{\max} \quad \forall s \in \mathbb{R}, \quad (96)$$

where  $\lambda'_{\min}$  and  $\lambda'_{\max}$  are positive constants.

4.2. Functions  $\nu$ ,  $\lambda$  and  $\lambda'$  are Lipschitz continuous, i.e.

$$\begin{aligned} |\nu(s_1) - \nu(s_2)| &\leq L_\nu |s_1 - s_2|, \quad |\lambda(s_1) - \lambda(s_2)| \leq L_\lambda |s_1 - s_2| \quad \text{and} \\ |\lambda'(s_1) - \lambda'(s_2)| &\leq L'_\lambda |s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{R}. \end{aligned} \quad (97)$$

Here  $L_\nu$ ,  $L_\lambda$  and  $L'_\lambda$  are some positive constants.

To achieve our goal, we will use as in [29,30,32] the equivalence between the standard  $\|\cdot\|_{2,\Omega}$  and  $L^2$ -norm  $\|\Delta \cdot\|_\Omega$  of the Laplace operator in the space  $H^2(\Omega) \cap H_0^1(\Omega)$  for domain with the boundary  $\Gamma \in C^2$  (see [56,57]). The mentioned equivalence is described by the following inequalities:

$$\|\Delta h\|_\Omega \leq \tilde{C}_1 \|h\|_{2,\Omega}, \quad \|h\|_{2,\Omega} \leq \tilde{C}_2 \|\Delta h\|_\Omega \quad \forall h \in H^2(\Omega) \cap H_0^1(\Omega). \quad (98)$$

Here and below  $\tilde{C}_i$ ,  $i = 1, 2, \dots$  are positive constants that depend on  $\Omega$  and, may be, also some indexes.

Recall that in [32] the proof of the local uniqueness of the solution to the boundary value problem under study was based precisely on the equivalence property of the norms  $\|\cdot\|_{2,\Omega}$  and  $\|\Delta \cdot\|_\Omega$ , which is valid for the components  $\varphi_1 \in H_0^1(\Omega)$  and  $\varphi_2 \in H_0^1(\Omega)$  of two possible solutions  $\mathbf{u}_i$ ,  $\varphi_i$ ,  $p_i$ ,  $i = 1, 2$ . Unlike [32], where the equations of the Boussinesq model are considered under homogeneous Dirichlet conditions, in our case, the specified equivalence property for components  $\varphi_1$  and  $\varphi_2$  does not work because they satisfy the inhomogeneous boundary condition  $\varphi_i|_\Gamma = \psi$ ,  $i = 1, 2$ . However, this property remains valid for the difference  $\varphi = \varphi_1 - \varphi_2$  which belongs to  $H_0^1(\Omega)$ . And, as will be shown below, this property is quite enough to prove the local uniqueness of the weak solution to Problem 1, which has an additional smoothness of the form  $\varphi \in H^2(\Omega)$ .

Below, we will also use the following estimates:

$$\|h\|_{L^r(\Omega)} \leq \tilde{C}_3 \|h\|_{2,\Omega} \quad \forall h \in H^2(\Omega) \quad 1 \leq r < \infty, \quad (99)$$

$$\|h\|_{L^\infty(\Omega)} \leq \tilde{C}_4 \|h\|_{2,\Omega}, \quad \forall h \in H^2(\Omega), \quad (100)$$

which result from the continuity of the embedding  $H^2(\Omega) \subset L^r(\Omega)$  at any  $r \in [1, \infty]$ , and known estimates

$$\|\nabla h\|_{L^4(\Omega)^3} \leq \tilde{C}_5 \|h\|_{2,\Omega} \quad \forall h \in H^2(\Omega), \quad (101)$$

$$\|\nabla h\|_{L^4(\Omega)^3} \leq \tilde{C}_6 \|\Delta h\|_\Omega \quad \forall h \in H^2(\Omega) \cap H_0^1(\Omega), \quad \tilde{C}_6 = \tilde{C}_2 \tilde{C}_5. \quad (102)$$

The latter is a consequence of estimates (98) and (101). In (99)  $\tilde{C}_3$  is a positive constant that depends on  $\Omega$  and  $r \in [1, \infty]$ .

The following result is valid regarding the uniqueness of a weak solution to Problem 1, which has a certain property of smallness:

**Theorem 4.** *Let, in addition to conditions 2.1-2.6 and 4.1, 4.2, the following conditions are met:  $\Gamma \in C^2$ ,  $f \in L^2(\Omega)$ , while the conditions 2.1 and 2.4 met for some  $p \geq 2$ . If there exists such a number  $\varepsilon > 0$  that there is a weak solution  $(\mathbf{u}, \varphi, p) \in H^1(\Omega)^3 \cap H^2(\Omega) \times L_0^2(\Omega)$  of Problem 1 satisfying the condition*

$$\|\mathbf{u}\|_{1,\Omega} + \|\varphi\|_{2,\Omega} < \varepsilon,$$

*then this solution is unique.*

**Proof of Theorem 4.** Suppose that there are two weak solutions  $(\mathbf{u}_i, \varphi_i, p_i) \in H^1(\Omega)^3 \times H^2(\Omega) \times L_0^2(\Omega)$ ,  $i = 1, 2$  of problem (31), (32), (33). Then, the differences

$$\varphi = \varphi_1 - \varphi_2 \in H^2(\Omega) \cap H_0^1(\Omega), \text{ and } \mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2 \in V$$

satisfy the relations

$$\begin{aligned} & -(\operatorname{div}(\lambda(\varphi_1)\nabla\varphi), h) + (k(\varphi_1)\varphi, h) + (\mathbf{u}_1 \cdot \nabla\varphi, h) = \\ & = (\operatorname{div}(\lambda(\varphi_1) - \lambda(\varphi_2))\nabla\varphi_2, h) - (k(\varphi_1) - k(\varphi_2), \varphi_2 h) - \\ & \quad - (\mathbf{u} \cdot \nabla\varphi_2, h) \quad \forall h \in L^2(\Omega), \end{aligned} \quad (103)$$

$$\begin{aligned} & (v(\varphi_2)\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}_2 \cdot \nabla)\mathbf{u}, \mathbf{v}) = -((v(\varphi_1) - v(\varphi_2))\nabla\mathbf{u}_1, \nabla\mathbf{v}) \\ & + (\mathbf{b}(\varphi_2)\varphi, \mathbf{v}) + (\mathbf{b}(\varphi_1) - \mathbf{b}(\varphi_2), \varphi_1\mathbf{v}) - ((\mathbf{u} \cdot \nabla)\mathbf{u}_1, \mathbf{v}) \quad \forall \mathbf{v} \in V. \end{aligned} \quad (104)$$

Using known formulas of vector analysis

$$\begin{aligned} \operatorname{div}(\lambda(\varphi_1)\nabla\varphi) &= \nabla\lambda(\varphi_1) \cdot \nabla\varphi + \lambda(\varphi_1)\Delta\varphi = \\ &= \lambda'(\varphi_1)\nabla\varphi_1 \cdot \nabla\varphi + \lambda(\varphi_1)\Delta\varphi \text{ in } \Omega, \end{aligned}$$

and

$$\begin{aligned} \operatorname{div}((\lambda(\varphi_1) - \lambda(\varphi_2))\nabla\varphi_2) &= (\lambda'(\varphi_1)\nabla\varphi_1 - \lambda'(\varphi_2)\nabla\varphi_2) \cdot \nabla\varphi_2 + \\ & \quad + (\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2 = \\ &= ((\lambda'(\varphi_1) - \lambda'(\varphi_2))\nabla\varphi_1 + \lambda'(\varphi_2)\nabla\varphi) \cdot \nabla\varphi_2 + \\ & \quad + (\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2, \end{aligned}$$

rewrite (103) as follows:

$$\begin{aligned} -(\lambda(\varphi_1)\Delta\varphi, h) &= (\lambda'(\varphi_1)\nabla\varphi_1 \cdot \nabla\varphi, h) - (k(\varphi_1)\varphi, h) - (\mathbf{u}_2 \cdot \nabla\varphi, h) + \\ & \quad + ((\lambda'(\varphi_1) - \lambda'(\varphi_2))\nabla\varphi_1 + \lambda'(\varphi_2)\nabla\varphi) \cdot \nabla\varphi_2, h) + \\ & \quad + ((\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2, h) - \\ & \quad - (k(\varphi_1) - k(\varphi_2), \varphi_2 h) - (\mathbf{u} \cdot \nabla\varphi_1, h) \quad \forall h \in L^2(\Omega). \end{aligned} \quad (105)$$

Let us set  $h = \Delta\varphi$  in (105) and  $\mathbf{v} = \mathbf{u}$  in (104). We arrive at

$$\begin{aligned} -(\lambda(\varphi_1)\Delta\varphi, \Delta\varphi) &= (\lambda'(\varphi_1)\nabla\varphi_1 \cdot \nabla\varphi, \Delta\varphi) - \\ & \quad - (k(\varphi_1)\varphi, \Delta\varphi) - (\mathbf{u}_2 \cdot \nabla\varphi, \Delta\varphi) + \\ & \quad + ((\lambda'(\varphi_1) - \lambda'(\varphi_2))\nabla\varphi_1 + \lambda'(\varphi_2)\nabla\varphi) \cdot \nabla\varphi_2, \Delta\varphi) + \\ & \quad + ((\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2, \Delta\varphi) - \\ & \quad - (k(\varphi_1) - k(\varphi_2), \varphi_2\Delta\varphi) - (\mathbf{u} \cdot \nabla\varphi_1, \Delta\varphi), \end{aligned} \quad (106)$$

$$\begin{aligned} (v(\varphi_2)\nabla\mathbf{u}, \nabla\mathbf{u}) &= -((v(\varphi_1) - v(\varphi_2))\nabla\mathbf{u}_1, \nabla\mathbf{u}) + \\ & + (\mathbf{b}(\varphi_2)\varphi, \mathbf{u}) + (\mathbf{b}(\varphi_1) - \mathbf{b}(\varphi_2), \varphi_1\mathbf{u}) - ((\mathbf{u} \cdot \nabla)\mathbf{u}_1, \mathbf{u}). \end{aligned} \quad (107)$$

Using the above estimates (97), (98) – (102), as well as the estimates defined in Section 2, we will estimate each term of the right-hand side of (106) successively. Let us start with the first term.

1. Using the Hölder inequality for the four functions and the estimates (96), (101), (102), we infer

$$\begin{aligned} |(\lambda'(\varphi)\nabla\varphi_1 \cdot \nabla\varphi, \Delta\varphi)| &\leq \lambda'_{\max}\|\nabla\varphi_1\|_{L^4(\Omega)}\|\nabla\varphi\|_{L^4(\Omega)}\|\Delta\varphi\|_{\Omega} \leq \\ &\leq \lambda'_{\max}\tilde{C}_5\tilde{C}_6\|\varphi_1\|_{2,\Omega}\|\Delta\varphi\|_{\Omega}^2. \end{aligned} \quad (108)$$

2. Using the Hölder inequality for three functions, the estimate (2.4) at  $p \geq 2$ , the estimates (98), (99) and arguing as in the derivation of the estimate (108) we infer

$$|(k(\varphi_1)\varphi, \Delta\varphi)| \leq \|k(\varphi_1)\|_{L^p(\Omega)}\|\varphi\|_{L^r(\Omega)}\|\Delta\varphi\|_{\Omega} \leq \hat{\gamma}_p\tilde{C}_2\tilde{C}_3\|\Delta\varphi\|_{\Omega}^2. \quad (109)$$

Here  $r > 1$  is the number associated with  $p$  by

$$\frac{1}{r} + \frac{1}{p} + \frac{1}{2} = 1.$$

3. Using the Hölder inequality for three functions, the estimate (13) at  $s = 4$  and the estimate (102) we derive

$$\begin{aligned} |(\mathbf{u}_2 \cdot \nabla\varphi, \Delta\varphi)| &\leq \|\mathbf{u}_2\|_{L^4(\Omega)}\|\nabla\varphi\|_{L^4(\Omega)}\|\Delta\varphi\|_{\Omega} \leq \\ &\leq C_4\tilde{C}_6\|\mathbf{u}_2\|_{1,\Omega}\|\Delta\varphi\|_{\Omega}^2. \end{aligned} \quad (110)$$

4. Using the Hölder inequality for four functions and estimates (97), (98), (100), and (101), we infer

$$\begin{aligned} |((\lambda'(\varphi_1) - \lambda'(\varphi_2))\nabla\varphi_1 \cdot \nabla\varphi_2, \Delta\varphi)| &\leq L'_\lambda|(\varphi\nabla\varphi_1 \cdot \nabla\varphi_2, \Delta\varphi)| \leq \\ &\leq L'_\lambda\|\varphi\|_{L^\infty(\Omega)}\|\nabla\varphi_1\|_{L^4(\Omega)}\|\nabla\varphi_2\|_{L^4(\Omega)}\|\Delta\varphi\|_{\Omega} \leq \\ &\leq \tilde{C}_2\tilde{C}_4\hat{C}_5^2L'_\lambda\|\varphi_1\|_{2,\Omega}\|\varphi_2\|_{2,\Omega}\|\Delta\varphi\|_{\Omega}^2, \leq \\ &\leq L'_\lambda\tilde{C}_2\tilde{C}_4\hat{C}_5^2\|\varphi_1\|_{2,\Omega}\|\varphi_2\|_{2,\Omega}\|\Delta\varphi\|_{\Omega}^2. \end{aligned} \quad (111)$$

5. Using the Hölder inequality for four functions and estimate (96), (101), (102) we output

$$\begin{aligned} |(\lambda'(\varphi_2)\nabla\varphi \cdot \nabla\varphi_2, \Delta\varphi)| &\leq \lambda'_{\max}\|\nabla\varphi\|_{L^4(\Omega)}\|\nabla\varphi_2\|_{L^4(\Omega)}\|\Delta\varphi\|_{\Omega} \leq \\ &\leq \lambda'_{\max}\tilde{C}_5\tilde{C}_6\|\varphi_2\|_{2,\Omega}\|\Delta\varphi\|_{\Omega}^2. \end{aligned} \quad (112)$$

6. Using the Hölder inequality for three functions and estimates (97), (98), (100) we output that

$$\begin{aligned} |((\lambda(\varphi_1) - \lambda(\varphi_2))\Delta\varphi_2, \Delta\varphi)| &\leq L_\lambda|(\varphi\Delta\varphi_2, \Delta\varphi)| \leq \\ L_\lambda\|\varphi\|_{L^\infty(\Omega)}\|\Delta\varphi_2\|_{\Omega}\|\Delta\varphi\|_{\Omega} &\leq L_\lambda\tilde{C}_4\|\varphi\|_{2,\Omega}\|\Delta\varphi_2\|_{\Omega}\|\Delta\varphi\|_{\Omega} \\ &\leq L_\lambda\tilde{C}_2\tilde{C}_4\|\Delta\varphi_2\|_{\Omega}\|\Delta\varphi\|_{\Omega}^2. \end{aligned} \quad (113)$$

7. Using the Hölder inequality for three functions, the property (8) of the function  $k(\cdot)$  and the estimates (98), (99) we output

$$\begin{aligned} |((k(\varphi_1) - k(\varphi_2))\varphi_2, \Delta\varphi)| &\leq \|k(\varphi_1) - k(\varphi_2)\|_{L^p(\Omega)}\|\varphi_2\|_{L^r(\Omega)}\|\Delta\varphi\|_{\Omega} \leq \\ &\leq L_k\|\varphi\|_{L^4(\Omega)}\|\varphi_2\|_{L^r(\Omega)}\|\Delta\varphi\|_{\Omega} \leq L_k\tilde{C}_3^2\|\varphi\|_{2,\Omega}\|\varphi_2\|_{2,\Omega}\|\Delta\varphi\|_{\Omega} \leq \\ &\leq L_k\tilde{C}_2\tilde{C}_3^2\|\varphi_2\|_{2,\Omega}\|\Delta\varphi\|_{\Omega}^2. \end{aligned} \quad (114)$$

8. Using the Hölder inequality for three functions and estimates (13) and (101) we infer

$$|(\mathbf{u} \cdot \nabla\varphi_1, \Delta\varphi)| \leq \|\mathbf{u}\|_{L^4(\Omega)^3}\|\nabla\varphi_1\|_{L^4(\Omega)^3}\|\Delta\varphi\|_{\Omega} \leq$$

$$\begin{aligned} &\leq C_4 \|\mathbf{u}\|_{1,\Omega} \|\nabla \varphi_1\|_{L^4(\Omega)} \|\Delta \varphi\|_{\Omega} \leq \\ &\leq C_4 \tilde{C}_5 \|\varphi_1\|_{2,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\Delta \varphi\|_{\Omega}. \end{aligned} \quad (115)$$

9. In addition, it follows from (9) that

$$(\lambda(\varphi_1) \Delta \varphi, \Delta \varphi) \geq \lambda_{\min} \|\Delta \varphi\|_{\Omega}^2. \quad (116)$$

Considering (108) – (116), we come from (106) to inequality

$$\begin{aligned} \lambda_{\min} \|\Delta \varphi\|_{\Omega}^2 &\leq (\lambda'_{\max} \tilde{C}_5 \tilde{C}_6 \|\varphi_1\|_{2,\Omega} + \\ &+ \hat{\gamma}_p \tilde{C}_2 \tilde{C}_3 + C_4 \tilde{C}_6 \|\mathbf{u}_2\|_{1,\Omega} + \\ &+ L'_\lambda \tilde{C}_2 \tilde{C}_4 \tilde{C}_5^2 \|\varphi_1\|_{2,\Omega} \|\varphi_2\|_{2,\Omega} + \lambda'_{\max} \tilde{C}_5 \tilde{C}_6 \|\varphi_2\|_{2,\Omega} + \\ &+ L_\lambda \tilde{C}_2 \tilde{C}_4 \|\Delta \varphi_2\|_{\Omega} + \\ &+ L_k \tilde{C}_2 \tilde{C}_3^2 \|\varphi_2\|_{2,\Omega}) \|\Delta \varphi\|_{\Omega}^2 + C_4 \tilde{C}_5 \|\varphi_1\|_{2,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\Delta \varphi\|_{\Omega}. \end{aligned} \quad (117)$$

Let us turn now to (107). Arguing, as above, we infer taking into account (18), (19), (27), (29) and (97), (98), (100) that

$$|(v(\varphi_2) \nabla \mathbf{u}, \nabla \mathbf{u})| \geq \nu_* \|\mathbf{u}\|_{1,\Omega}^2, \quad (118)$$

$$\begin{aligned} &|(v(\varphi_1) - v(\varphi_2)) \nabla \mathbf{u}_1, \nabla \mathbf{u})| \leq L_\nu |((\varphi \nabla \mathbf{u}_1, \nabla \mathbf{u}))| \\ &\leq L_\nu \|\varphi\|_{L^\infty(\Omega)} \|\nabla \mathbf{u}_1\|_{\Omega} \|\nabla \mathbf{u}\|_{\Omega} \leq L_\nu \tilde{C}_2 \tilde{C}_4 \|\mathbf{u}_1\|_{1,\Omega} \|\Delta \varphi\|_{\Omega} \|\mathbf{u}\|_{1,\Omega}, \end{aligned} \quad (119)$$

$$|((\mathbf{u} \cdot \nabla) \mathbf{u}_1, \mathbf{u})| \leq \gamma_1 \|\mathbf{u}_1\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega}^2 \leq \gamma_1 \|\mathbf{u}_1\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega}^2, \quad (120)$$

$$|(\mathbf{b}(\varphi_2) \varphi, \mathbf{u})| \leq \beta_p \|\varphi\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \leq \beta_p \tilde{C}_2 \|\Delta \varphi\|_{\Omega} \|\mathbf{u}\|_{1,\Omega}, \quad (121)$$

$$\begin{aligned} |(\mathbf{b}(\varphi_1) - \mathbf{b}(\varphi_2), \varphi_1 \mathbf{u})| &\leq \hat{C}_p L_b \|\varphi\|_{L^4(\Omega)} \|\varphi_1\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \leq \\ &\leq \hat{C}_p \tilde{C}_3 L_b \|\varphi_1\|_{2,\Omega} \|\varphi\|_{2,\Omega} \|\mathbf{u}\|_{1,\Omega} \leq \\ &\leq \hat{C}_p \tilde{C}_2 \tilde{C}_3 L_b \|\varphi_1\|_{2,\Omega} \|\Delta \varphi\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega}. \end{aligned} \quad (122)$$

Considering (118), (119), (120), (121), (122), from (107) we come to the following inequality:

$$\begin{aligned} \nu_* \|\mathbf{u}\|_{1,\Omega}^2 &\leq \gamma_1 \|\mathbf{u}_1\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega}^2 + L_\nu \tilde{C}_2 \tilde{C}_4 \|\mathbf{u}_1\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\Delta \varphi\|_{\Omega} + \\ &+ \beta_p \tilde{C}_2 \|\mathbf{u}\|_{1,\Omega} \|\Delta \varphi\|_{\Omega} + L_b \hat{C}_p \tilde{C}_2 \tilde{C}_3 \|\varphi_1\|_{2,\Omega} \|\Delta \varphi\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega}. \end{aligned} \quad (123)$$

Using Young's inequality, we estimate the terms in (117) and (123) containing the products  $\|\Delta \varphi\|_{\Omega}$  and  $\|\mathbf{u}\|_{1,\Omega}$ . Let us start with the last term in (117). We have

$$C_4 \tilde{C}_5 \|\varphi_1\|_{2,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\Delta \varphi\|_{\Omega} \leq (1/2) C_4 \tilde{C}_5 \|\varphi_1\|_{2,\Omega} (\|\Delta \varphi\|_{\Omega}^2 + \|\mathbf{u}\|_{1,\Omega}^2). \quad (124)$$

According to a similar scheme, we withdraw

$$\begin{aligned} &L_\nu \tilde{C}_2 \tilde{C}_4 \|\mathbf{u}_1\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\Delta \varphi\|_{\Omega} \leq \\ &\leq (1/2) L_\nu \tilde{C}_2 \tilde{C}_4 \|\mathbf{u}_1\|_{1,\Omega} (\|\Delta \varphi\|_{\Omega}^2 + \|\mathbf{u}\|_{1,\Omega}^2), \end{aligned} \quad (125)$$

$$\beta_p \tilde{C}_2 \|\Delta \varphi\|_{\Omega} \|\mathbf{u}\|_{1,\Omega} \leq (1/2) \beta_p \tilde{C}_2 (\|\Delta \varphi\|_{\Omega}^2 + \|\mathbf{u}\|_{1,\Omega}^2), \quad (126)$$

$$\begin{aligned} &L_b \hat{C}_p \tilde{C}_2 \tilde{C}_3 \|\varphi_1\|_{2,\Omega} \|\Delta \varphi\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \leq \\ &\leq (1/2) L_b \hat{C}_p \tilde{C}_2 \tilde{C}_3 \|\varphi_1\|_{2,\Omega} (\|\Delta \varphi\|_{\Omega}^2 + \|\mathbf{u}\|_{1,\Omega}^2). \end{aligned} \quad (127)$$

Adding the inequalities (117) and (123) and considering (124), (125), (126), (127), we come to the inequality

$$(\lambda_{\min} - a)\lambda_{\min}\|\Delta\varphi\|_{\Omega}^2 + (v_* - b)\|\mathbf{u}\|_{1,\Omega}^2 \leq 0. \quad (128)$$

Here

$$\begin{aligned} a = & \lambda'_{\max}\tilde{C}_5\tilde{C}_6\|\varphi_1\|_{2,\Omega} + \hat{\gamma}_p\tilde{C}_2\tilde{C}_3 + C_4\tilde{C}_5\|\mathbf{u}_2\|_{1,\Omega} + \\ & + L'_\lambda\tilde{C}_2\tilde{C}_4\tilde{C}_5^2\|\varphi_1\|_{2,\Omega}\|\varphi_2\|_{2,\Omega} + \lambda'_{\max}\tilde{C}_5\tilde{C}_6\|\varphi_2\|_{2,\Omega} + L_\lambda\tilde{C}_2\tilde{C}_4\|\Delta\varphi_2\|_{\Omega} + \\ & + L_k\tilde{C}_2\tilde{C}_3^2\|\varphi_2\|_{2,\Omega} + (1/2)C_4\tilde{C}_5\|\varphi_1\|_{2,\Omega} + \\ & + (1/2)L_\nu\tilde{C}_2\tilde{C}_4\|\mathbf{u}_1\|_{1,\Omega} + (1/2)\beta_p\tilde{C}_2 + (1/2)L_b\hat{C}_p\tilde{C}_2\tilde{C}_3\|\varphi_1\|_{2,\Omega}, \end{aligned} \quad (129)$$

$$\begin{aligned} b = & \gamma_1\|\mathbf{u}_1\|_{1,\Omega} + (1/2)C_4\tilde{C}_5\|\varphi_1\|_{2,\Omega} + (1/2)L_\nu\tilde{C}_2\tilde{C}_4\|\mathbf{u}_1\|_{1,\Omega} + \\ & + (1/2)\beta_p\tilde{C}_2 + (1/2)L_b\hat{C}_p\tilde{C}_2\tilde{C}_3\|\varphi_1\|_{2,\Omega}. \end{aligned} \quad (130)$$

Let the pairs  $(\mathbf{u}_1, \varphi_1)$  and  $(\mathbf{u}_2, \varphi_2)$  be such that the following smallness conditions are met:

$$a < \lambda_{\min}, \quad b < v_*. \quad (131)$$

If the conditions (131) are satisfied from the inequality (128), it follows that

$$\|\Delta\varphi\|_{\Omega} = 0 \quad \text{and} \quad \|\mathbf{u}\|_{1,\Omega} = 0. \quad (132)$$

From the second relation in (132) it follows that  $\mathbf{u}_1 = \mathbf{u}_2$ , and from the one it follows that  $\Delta\varphi = 0$ . Since the assumption  $\varphi_i \in H^2(\Omega)$  implies that  $\varphi \equiv \varphi_1 - \varphi_2 \in H^2(\Omega) \cap H_0^1(\Omega)$ , then using the second estimate in (98) we infer from (132) that

$$\|\varphi\|_{2,\Omega} \leq \tilde{C}_2\|\Delta\varphi\|_{\Omega} = 0.$$

This means that  $\varphi = 0$  or  $\varphi_1 = \varphi_2$ .

It remains to prove that the components  $p_1$  and  $p_2$  introduced at the beginning of the proof of Theorem 4 coincide. To this end, subtract the identity (31) for  $(\mathbf{u}_2, \varphi_2, p_2)$  from (31) for  $(\mathbf{u}_1, \varphi_1, p_1)$  and take into account that  $\mathbf{u}_1 = \mathbf{u}_2$  and  $\varphi_1 = \varphi_2$ . As a result, we get that the difference  $p = p_1 - p_2$  satisfies the identity

$$-(p, \operatorname{div}\mathbf{v}) = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega)^3. \quad (133)$$

From (133) it follows due to the inf-sup condition (21) that  $p = 0$  or  $p_1 = p_2$ . Thus, the local uniqueness property of the weak solution of Problem 1 from the space  $H^1(\Omega)^3 \times H^2(\Omega) \times L_0^2(\Omega)$  is proven.  $\square$

## 5. Conclusions

In this paper a boundary value problem for a nonlinear mass transfer model that generalizes the classical Boussinesq approximation was studied. It is assumed that the coefficients  $\nu$ ,  $\lambda$ ,  $k$  and  $b$  of the model depend on the concentration of the substance. We defined a concept of a weak solution of the boundary value problem under study and established the conditions on the coefficients  $\nu$ ,  $\lambda$ ,  $k$ ,  $b$  and on another data which provide the global solvability of the problem. We also found additional important properties of the weak solution and in particular the maximum principle for the substance's concentration. Also, we have proved conditional uniqueness of the weak solution having an additional smoothness property for the concentration.

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