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Pricing Contingent Claims in a Two-Interest Rate Multi-Dimensional Jump-diffusion Model via Market Completion

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Abstract: In this paper, we investigate a financial market in which asset prices evolve based on a multi-dimensional Brownian motion process and a multi-dimensional Poisson process with different credit and deposit rates We proceed to evaluate European options by establishing upper and lower hedging prices through a transition to a suitable auxiliary market. Additionally, we address the minimization of shortfall risk and no-arbitrage price bounds in incomplete markets within this framework.

Keywords: jump-diffusion; different interest rates; shortfall risk minimization; completion; multi-dimensional

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1. Introduction

Pricing contingent claims in complete markets has garnered significant attention since the seminal work of Black & Scholes in 1973. Efficient hedging of contingent claims is well-established in complete markets characterized by the same interest rate for the credit and deposit accounts. (refer to Karatzas and Shreve [6] for detailed insights). However, our focus shifts to a more realistic financial market scenario, introducing a two-interest rate model where the credit rate surpasses the deposit rate, aligning more closely with real-world financial markets (as discussed by Kane and Melnikov [5]). In this paper, we consider a multi-dimensional model featuring $m+2$ securities, encompassing two risk-free assets, d stocks driven by a d-dimensional Brownian motion, and $m-d$ stocks influenced by an $(m-d)$-dimensional Poisson process.

Given the incompleteness of the market with two interest rates, we transform it into a suitable auxiliary market using a multi-dimensional jump-diffusion model incorporating two interest rates (see Korn [7] for details).

The structure of this paper unfolds as follows: Section 2 provides an overview of the market model. Section 3 delves into contingent claim valuation within complete markets, accompanied by a theorem presenting a comprehensive solution to the contingent claim problem in such markets. In Section 4, we establish a martingale measure for the new auxiliary market characterized by a higher interest rate for the credit account. Additionally, in section 5, we explore the concept of shortfall risk, acknowledging situations where achieving a perfect hedge might be infeasible, yet it remains possible to minimize the expected shortfall risk, as demonstrated towards the end of this section. The final section will explore pricing contingent claims via market completion in (B_1, B_2, S_m)-market, where we study no-arbitrage price bounds in incomplete markets.

2. The Market Model

Let $(\Omega, \mathcal{F}_t, P, \mathbb{F})$ be a filtered probability space with a complete and right-continuous filtration $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T}$. Assume there are $m+2$ continuously traded securities, including two risk-free assets, d stocks driven by an $\mathbb{R}^d$-valued Brownian motion $W(t) = (W_1(t), \dots, W_d(t))^\intercal$, and a $(m-$

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d)-dimensional multivariate Poisson process $N(t) = (N_1(t), \ldots, N_{m-d}(t))^\intercal$ with a positive intensity λ. This intensity is independent of W and is denoted by $\lambda^{(k)}(t)$, representing the rate of the jump process at time t. The process $\lambda^{(k)}(t)$ is $\{\mathcal{F}_t\}$-predictable, positive, and uniformly bounded over $[0, T]$.

The price of the i^{th} stock, $S_i(t)$, is determined by the following equation

$$dS_i(t) = S_i(t-)\left(\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t) + \sum_{k=1}^{m-d} \nu_{ik}(t)d\tilde{N}_k(t)\right), \quad (1)$$

where $\tilde{N}_k(t) = N_k(t) - \int_0^t \lambda^{(k)}(s)\, ds$, $\nu_{ik}(t) > -1$ for all i, k, and $t \in [0, T]$, $\sigma_{ij} > 0$, and $\mu_i \in \mathbb{R}$. σ and ν are matrix-valued processes such that i^{th} row is given by $\sigma_{ij} = (\sigma_{i1}, \ldots, \sigma_{id})$, and $\nu_{ik} = (\nu_{i1}, \ldots, \nu_{i(m-d)})$ for $i = 1, \ldots, m$, respectively. We assume μ, σ, and ν are uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$. Henceforth, the dynamics of the price in 1 possess a unique solution under these assumptions. We also define the volatility coefficients $\tilde{\sigma}(t) = [\sigma(t)\ \nu(t)]$, forming an $m \times m$ full-rank matrix, ensuring that $\det(\tilde{\sigma}(t)\, \sigma^\intercal(t)) \neq 0$ a.s. for all $t \in [0, T]$.

In our model, the market incompleteness arises from denoting two different interest rates, as described below.

Let us consider one deposit account B_1 with the interest rate r_1 and one credit account B_2 with the interest rate r_2 satisfying

$$\begin{aligned} dB_1(t) &= B_1(t)r_1dt, \\ dB_2(t) &= B_2(t)r_2dt. \end{aligned} \quad (2)$$

Given that, in reality, the credit rate is always higher than the deposit rate, we assume constant values for r_1 and r_2 such that

$$r_2 > r_1, \quad (3)$$

and investors are not allowed to borrow and lend money simultaneously.

The market described above is denoted as the (B_1, B_2, S_m)-market.

In the (B_1, B_2, S_m)-market, we denote $\beta_1(t)$ and $\beta_2(t)$ as the number of units invested in the B_1 and B_2 accounts, respectively, and $\gamma(t) = (\gamma_1(t), \ldots, \gamma_m(t))$, where γ_i represents the number of units invested in the i^{th} stock. The portfolio process is then denoted as follows

$$\pi(t) = (\beta_1(t), \beta_2(t), \gamma_1(t), \ldots, \gamma_m(t)). \quad (4)$$

The value of the portfolio π is given by

$$X^\pi(t) = \beta_1(t)B_1(t) + \beta_2(t)B_2(t) + \sum_{i=1}^{m} \gamma_i(t)S_i(t) \quad \text{a.s.} \quad (5)$$

with $\beta_1 \geq 0$, $\beta_2 \leq 0$, and

$$X^\pi(0) = x, \quad (6)$$

where x is the initial value (initial capital) of the portfolio. This portfolio is self-financing (SF) if

$$dX^\pi(t) = \beta_1(t)dB_1(t) + \beta_2(t)dB_2(t) + \sum_{i=1}^{m} \gamma_i(t)dS_i(t). \quad (7)$$

The portfolio is admissible if

$$X^\pi(t) > 0 \quad \text{a.s.} \quad \text{for} \quad t \in [0, T]. \quad (8)$$

Denote the class of admissible portfolio strategies with initial capital x by

$$\mathcal{A}(x) = \{\pi \in \mathbb{R}^{m+2} : X^\pi(0) = x, X^\pi \geq -m \text{ for all } t \in [0, T]\}. \quad (9)$$

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Any non-negative $\mathcal{F}_t$-measurable random variable f_T is called a contingent claim with maturity time T. A market is complete if and only if any contingent claim f_T can be replicated. Namely, there exists an initial capital x and $\pi \in$ SF such that:

$$X_T^{\pi}(x) = x + \sum_{i=1}^{m} \int_0^T \pi_i(t) dS_i(t) = f_T \quad \text{P-a.s.} \tag{10}$$

Let us consider $X(t)$ (or $Y(t)$) as the investor's wealth (or debt) at time t and call it the wealth process (or debt process) if $X(t)$ (or $-Y(t)$) is generated by a self-financing and admissible strategy.

Since the (B_1, B_2, S_m)-market is not a complete market, standard methods for pricing and investing do not work. To address this, we transform the market into an auxiliary market $(B_z, S_m)_{z\in[0,r_2-r_1]}$. In this market, B_z is the bank account with the interest rate

$$r_z = r_1 + z. \tag{11}$$

Note that the (B_z, S_m)-market is complete for every r_z satisfying $r_z \in [0, r_2 - r_1]$ for any $t \in [0, T]$.

Now, we derive the dynamics of the wealth and debt processes in the (B_1, B_2, S_m)-market.

By the self-financing wealth process $X(T) \geq 0$,

$$dX(t) = \beta_1(t) dB_1(t) + \beta_2(t) dB_2(t) + \sum_{i=1}^{m} \gamma_i(t) dS_i(t), \tag{12}$$

where $\beta_1 > 0$, $\beta_2 < 0$. Then,

$$\begin{aligned} \frac{dX(t)}{X(t-)} =& \frac{\beta_1(t)B_1(t)}{X(t-)}\frac{dB_1(t)}{B_1(t)} + \frac{\beta_2(t)B_2(t)}{X(t-)}\frac{dB_2(t)}{B_2(t)} \\ &+ \sum_{i=1}^{m} \frac{\gamma_i(t)S_i(t-)}{X(t-)}\frac{dS_i(t)}{S_i(t-)}. \end{aligned} \tag{13}$$

Denoting

$$\begin{aligned} \zeta =& \zeta_1 + \cdots + \zeta_m \\ =& \frac{\gamma_1(t)S_1(t-)}{X(t-)} + \cdots + \frac{\gamma_m(t)S_m(t-)}{X(t-)} \\ =& \sum_{i=1}^{m} \frac{\gamma_i(t)S_i(t-)}{X(t-)}, \end{aligned} \tag{14}$$

we obtain

$$\frac{dX(t)}{X(t-)} = (1-\zeta(t))^+ r_1 dt - (1-\zeta(t))^- r_2 dt + \sum_{i=1}^{m} \zeta_i(t) \frac{dS_i(t)}{S_i(t-)}. \tag{15}$$

Taking the same steps, one can observe that the stochastic differential equation (SDE) of the seller is as follows

$$\frac{dY(t)}{Y(t-)} = (1-\zeta(t))^+ r_2 dt - (1-\zeta(t))^- r_1 dt + \sum_{i=1}^{m} \zeta_i(t) \frac{dS_i(t)}{S_i(t-)}. \tag{16}$$

A hedging strategy against f in the (B, S_m)-market is not necessarily a hedging strategy against f in the (B_1, B_2, S_m)-market. In this regard, we first pay attention to contingent claim valuation in the complete markets and then in the (B_1, B_2, S_m)-market.

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3. Contingent Claim Valuation in Complete Markets

As mentioned in the previous section, any non-negative $\mathcal{F}_T$-measurable random variable f_T is called a contingent claim with maturity T. The (B, S_m)-market is complete if and only if any contingent claim f_T can be replicated. This means that there exists an initial wealth x and a strategy $\pi \in$ SF such that $X_T^{\pi}(x) = f_T$. We show that this is the only price for a contingent claim, preventing any arbitrage opportunities. To do that, we define a unique equivalent martingale measure. Let us consider

$$\theta(t) := (\tilde{\sigma}(t))^{-1}[\mu(t) - r(t)] = \begin{bmatrix} \theta_W(t) \\ \theta_N(t) \end{bmatrix}, \tag{17}$$

where $\theta_W(t)$ is an $\mathbb{R}^d$-valued process, $\theta_N(t)$ is an $\mathbb{R}^{m-d}$-valued process, and $\tilde{\sigma}(t) := [\sigma(t)\ \nu(t)]$ is the $m \times m$ volatility matrix process. Let us define the following processes

$$\tilde{W}(t) := W(t) + \int_0^t \theta_W(s)ds,$$
$$\tilde{N}(t) := N(t) - \int_0^t \theta_N(s)ds.$$

and

$$Z_W(t) := \exp\left\{-\int_0^t \theta_W^T(s)dW(s) - \frac{1}{2}\int_0^t \|\theta_W(s)\|^2 ds\right\}, \tag{18}$$

$$Z_N(t) := \prod_{1\le k\le m-d}\left(\prod_{n\ge 1}((\psi^{(k)}(t_n^{(k)}) + 1)\mathbf{1}_{\{t_n^{(k)}\le t\}} + \mathbf{1}_{\{t_n^{(k)}>t\}}) \times \exp\{-\int_0^t \psi^{(k)}(s)\lambda^{(k)}(s)ds\}\right), \tag{19}$$

where

$$\psi^{(k)}(t) := -\theta_N^{(k)}(t)/\lambda^{(k)}(t),$$

$t_n^{(k)}$ is the time of the n-th jump, and $N_k(t) = \sup\{n : t_n^{(k)} \le t\}$ is the number of type k random jumps to the market by time t.

Lemma 1. *The process Z defined by*

$$Z(t) := Z_W(t)Z_N(t), \tag{20}$$

is a P-martingale with $E[Z(T)] = 1$. Define an auxiliary probability measure on $(\Omega, \mathcal{F}_T)$ as

$$\hat{P}(A) := E[Z(T)\mathbf{1}_A], \quad A \in \mathcal{F}_T.$$

Then, $\tilde{W}$ and $\tilde{N}$ are martingales under P. In particular, the jump process N_k admits $(P, \mathcal{F}_t)$-stochastic intensity

$$\tilde{\lambda}^{(k)}(t) = (\psi^{(k)}(t) + 1)\lambda^{(k)}(t).$$

See Bardhan and Chao [3].

Theorem 1. *Let f be a given contingent claim. The fair price of f is given by*

$$p = E(\gamma(T)f),$$

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and there exists a unique (up to equivalence) corresponding hedging strategy π with corresponding wealth process $X(t)$ satisfying

$$X(0) = p.$$

Here, E means expectation with respect to P.
The discount process $\gamma(t)$ is defined as

$$\gamma(t) = \exp\Big(-\int_0^t r(s)ds\Big) \quad \text{for } t \in [0, T].$$

4. Contingent Claim Valuation when the Interest Rate for the Credit Account is Higher than the Interest Rate for the Deposit Account

Now, we transform the problem of contingent claim valuation in the (B_1, B_2, S_m)-market to a suitable complete market (B_z, S_m). By substituting $r(t)$ with $r_z(t)$ and defining $\hat{\theta}(t)$, $\hat{Z}(t)$, $\hat{W}(t)$, $\hat{P}$, and $\hat{\gamma}(t)$ as in Section 3, one can obtain the same results. Then, the fair price $\hat{p}$ of the contingent claim f in the (B_z, S_m)-market is given by

$$\hat{p} = \hat{E}(\hat{\gamma}(T)f), \tag{21}$$

where $\hat{E}$ is the expectation with respect to the probability measure $\hat{P}$.

The following lemma relates the (B_z, S_m)-market to the (B_1, B_2, S_m)-market. In other words, we obtain a condition under which the wealth processes corresponding to a portfolio process π coincide in the (B_1, B_2, S_m)-market and (B_z, S_m)-market, respectively.

Lemma 2. *Let π be a portfolio process, $X(t)$ and $\hat{X}(t)$ be the wealth processes in the market (B_1, B_2, S_m) and (B_z, S_m), respectively. Denote*

$$\hat{X}(0) = X(0), \tag{22}$$

then

$$\hat{X}(t) = X(t) \quad \textit{for } t \in [0, T] \quad a.s. \tag{23}$$

if and only if

$$\begin{aligned}&(r_2(t) - r_1(t) - r_z(t))(1 - \zeta(t))^- + r_z(1 - \zeta(t))^+ = 0\\ &\textit{for } t \in [0, T] \quad a.s.\end{aligned} \tag{24}$$

Proof: $\hat{X}(t)$ follows the stochastic differential equation

$$\begin{aligned}d\hat{X}(t) =& \hat{X}(t)\Big[(1 - \zeta(t))r_z(t)dt\\ &+ \zeta(t)\Big(\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t) + \sum_{k=1}^{m-d} \nu_{ik}(t)d\tilde{N}_k(t)\Big)\Big].\end{aligned} \tag{25}$$

By comparing equation 25 to the stochastic differential equation for $X(t)$ as

$$\begin{aligned}dX(t) =& X(t)\Big[(1 - \zeta(t))^+ r_1(t)dt + (1 - \zeta(t))^- r_2(t)dt\\ &+ \zeta(t)\Big(\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t) + \sum_{k=1}^{m-d} \nu_{ik}(t)d\tilde{N}_k(t)\Big)\Big],\\ X(0) =& x,\end{aligned} \tag{26}$$

and by the assumption

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$$\hat{X}(0) = X(0),$$

then

$$\hat{X}(t) = X(t), \quad \text{for } t \in [0,T] \quad \text{a.s.}$$

is equivalent to

$$(1-\zeta(t))^{+} r_1(t) - (1-\zeta(t))^{-} r_2(t) = (1-\zeta(t)) r_z(t). \quad \text{for } t \in [0,T] \quad \text{a.s.} \tag{27}$$

By recalling the relation $a = a^{+} - a^{-}$ for $a \in \mathbb{R}$, one can find the equivalence of equation 27 and equation 24. □

Statement 1. *Let $r_z = (r_z(t))$ be a predictable process with values in the interval*

$$[0, r_2 - r_1]. \tag{28}$$

Assume that $\zeta(t)$ is the optimal hedging strategy against the claim f_T in the (B_z, S_m)-market and satisfies the condition in 24.
Then $C_{r_z}(0)$ (resp. $P_{r_z}(0)$), the initial price of the minimal hedge in (B_z, S_m) against f_T, is equal to C_+ (resp. P_+), the initial price of the minimal hedging strategy in (B_1, B_2, S_m).
Namely,

$$C_{r_z}(0) = C_+ \quad (resp. P_{r_z}(0) = P_+). \tag{29}$$

Proof. First, we demonstrate that the minimal hedging strategy ξ in the (B_z, S_m)-market is also a hedging strategy in the (B_1, B_2, S_m)-market under relation 24. Let C_{r_z} be the initial capital associated with that hedge in the (B_z, S_m)-market.

If ξ satisfies 24, then the stochastic differential equations of the wealth processes $\hat{X}_t$ and X_t in the markets (B_z, S_m) and (B_1, B_2, S_m), respectively, coincide. By taking C_{r_z} as the initial price in both markets, we establish the equality between the two processes at any time $t \in [0,T]$. Consequently,

$$\hat{X}_t = X_t = f(S_T^1). \tag{30}$$

Now, let's show that, under the assumption of Statement 1, the strategy ξ is minimal among the hedges against $f(S_T^1)$ in the (B_1, B_2, S_m)-market. To achieve this aim, it is sufficient to establish

$$\hat{E}[f(S_T^1) e^{-r_z T}] \leq x, \tag{31}$$

where x represents the initial capital of ξ^*, an arbitrary strategy in the (B_1, B_2, S_m)-market. $\hat{E}$ denotes the expected value under the martingale measure in Lemma 1. Let $X_t^{\xi^*}$ be the wealth process corresponding to the arbitrary strategy ξ^*. We show

$$\hat{E}[X_T^{\xi^*} e^{-r_z T}] \leq x. \tag{32}$$

Let us consider the discounted wealth process $\tilde{X}_t := X_t^{\xi^*} e^{-r_z t}$, then by using Ito's formula

$$\begin{aligned} d\tilde{X}_t = & X_{t-}^{\xi^*} e^{-r_z t} \Big([(1-\xi_t^*)^{-}(r_1 - r_2) - z(1-\xi_t^*)] dt \\ & + (\sum_{j=1}^{d} \xi_j^*(t) \sigma_{ij}) d\tilde{W}_t - (\sum_{k=1}^{m-d} \xi_k^*(t) \nu_{ik}) d(N_t - \tilde{\lambda} t) \Big). \end{aligned} \tag{33}$$

Note that

$$(1-\xi_t^*)^-(r_1-r_2)-z(1-\xi_t^*)\leq 0, \tag{34}$$

and

$$(\sum_{j=1}^{d}\xi_j^*(t)\sigma_{ij})d\tilde{W}_t-(\sum_{k=1}^{m-d}\xi_k^*(t)\nu_{ik})d(N_t-\tilde{\lambda}t) \tag{35}$$

is a $\hat{P}$ local martingale.
From integrating the relation 33 and taking the $\hat{P}$ expectation, we obtain

$$\hat{E}[\tilde{X}_t]=\hat{E}[X_t^{\xi^*}e^{-r_zt}]\leq x. \quad \text{for any } t\in[0,T] \tag{36}$$

Since ξ^* is a hedge for f_T, that yields

$$X_T^{\xi^*}e^{-r_zT}=\tilde{X}_T\geq f_Te^{-r_zT}. \tag{37}$$

Hence

$$C_{r_z}=\hat{E}[f_Te^{-r_zT}]\leq\hat{E}[\tilde{X}_T]=\hat{E}[X_T^{\xi^*}e^{-r_zT}]\leq x. \tag{38}$$

Given that the relation 24 satisfied, C_{r_z} is an initial price of a hedge for f_T in (B_1,B_2,S_m)-market. Therefore

$$C_{r_z}=C_+. \tag{39}$$

For the case of put, the proof is similar. □

Following the above statement and Lemma 2, the wealth process in the (B_z,S_m)-market, denoted as $\hat{X}_t(C_{r_z})$, coincides with the wealth process in the (B_1,B_2,S_m)-market, denoted as $X_t(C_{r_z})$, and

$$\hat{X}_T(C_{r_z})=X_T(C_{r_z})=f(S_T^1). \tag{40}$$

Therefore, we assert that the minimal hedge ζ in the (B_z,S_m)-market against f_T is also a hedge in the (B_1,B_2,S_m)-market if the relation 24 holds.

Statement 2. *Let f be a given contingent claim, and let $r_z(t),t\in[0,T]$, be a progressively measurable process satisfying the condition $r_z(t)\in[0,r_2(t)-r_1(t)]$ for $t\in[0,T]$ a.s.. If the minimal hedging strategy ξ^* corresponding to the solution of the contingent claim valuation problem for f in the (B_z,S_m)-market satisfies the equation*

$$(r_2(t)-r_1(t)-r_z(t))(1-\zeta(t))^-+r_z(1-\zeta(t))^+=0, \quad \textit{for } t\in[0,T] \quad \textit{a.s.} \tag{41}$$

then ξ^* is also a hedge against $-f_T$ in the (B_1,B_2,S_m)-market. Furthermore, if C_{r_z} (resp. P_{r_z}), the fair price of the claim in the (B_z,S_m)-market, verifies $C_{r_z}=\inf_{k\in[0,r_2-r_1]}C_{r_k}$ (resp. $P_{r_z}=\inf_{k\in[0,r_2-r_1]}P_{r_k}$), then

$$C_{r_z}=C_-\ (\textit{resp. } P_{r_z}=P_-), \tag{42}$$

where C_- (resp. P_-) is the initial debt of the minimal hedge (i.e., the seller's price). Namely,

$$-C_-(\textit{resp. } -P_-)=\sup\{y\leq 0/\exists\xi\in\mathcal{A}(x)\ s.t.\ T_T\leq -f_T\}. \tag{43}$$

Before proving this statement, we state the following lemma.

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Lemma 3. *The minimal hedging strategy against f_T in the (B_z, S_m) market (for the buyer) is also the minimal hedging strategy against $-f_T$ (for the seller) in the same market.*

Proof. The stochastic differential equations of the debt and wealth processes coincide in the (B_z, S_m) market. Therefore, if ξ^* is a hedge against f_T in the (B_z, S_m) market, we have

$$X_T^{\xi^*,x} = f_T. \tag{44}$$

By taking $y = -x$ as the initial price for the debt process,

$$Y_T = -X_T^{\xi^*,x} = -f_T. \tag{45}$$

Hence, ξ^* is a hedge against $-f_T$ in the (B_z, S_m) market.
□

Now let us return to the proof of Statement 2.

Proof of Statement 2. Provided that relation 24 verifies ξ^* as a hedge in (B_1, B_2, S_m) against $-f_T$, with an initial price of $-C_{r_z}$, it is sufficient to find a minimal hedge in the latter market. Assume $C_{r_z} = \inf_{k\in[0,r_2-r_1]} C_{r_k}$, and let y be the initial value for the debt process generated by ξ, an arbitrary strategy in the (B_1, B_2, S_m)-market. We aim to show that

$$y \leq \sup_{k\in[0,r_2-r_1]} (-C_{r_k}) := -C_{r_z}. \tag{46}$$

Accordingly, any hedging strategy against $-f_T$ has an initial value less than $-C_{r_z}$. However, $-C_{r_z}$ is the initial debt of the hedge ξ^* against $-f_T$ in the (B_1, B_2, S_m)-market. Therefore, $-C_{r_z}$ provides the lowest initial debt in (B_1, B_2, S_m). Any hedging strategy against $-f_T$ in (B_1, B_2, S_m) is a hedging strategy against the same claim in (B_{z^*}, S_m) where

$$z^* = \begin{cases} r_2 - r_1, & \text{if } 1 - \xi_t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{47}$$

However, by definition, $-C_{r_{z^*}} \leq -C_{r_z}$. Therefore, $y \leq -C_{r_z}$.
The proof holds for both Call and Put options. □

Now, let us provide an approximation of the arbitrage-free prices for the claim $f_T = (S_T^1 - K)^+$. In this scenario, we calculate the supremum and infimum over auxiliary markets to find approximations for the upper and lower hedging prices of the claim. Therefore, the arbitrage-free interval of prices can be approximated as follows

$$\left[\inf_{z\in[0,r_2-r_1]} C_{r_z}, \sup_{z\in[0,r_2-r_1]} C_{r_z}\right] \tag{48}$$

Example 1. *Consider the European call option on Stock 1 with maturity T, exercise price K, volatility σ_{11}, and interest rate r_z. The value of the option $f_T = (S_1(T) - K)^+$ can be expressed as follows*

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$$
\begin{aligned}
C_{r_z}(t) =& e^{-\hat{\lambda}(T-t)} \sum_{n=0}^{\infty} \frac{(\hat{\lambda}(T-t))^n}{n!} C_{BS}(T-t, S_1(0)(1-\nu_{11})^n e^{\nu_{11}\hat{\lambda}(T-t)}, r, \sigma_{11}, K) \\
=& S_1(t)(1-\nu_{11})^n e^{\nu_{11}\hat{\lambda}(T-t)} \left(\sum_{0}^{\infty} \frac{(\hat{\lambda}(T-t)^n}{n!} e^{\hat{\lambda}(T-t)} \Phi(d_1) \right) \\
& -Ke^{-r_z(T-t)} \left(\sum_{0}^{\infty} \frac{(\hat{\lambda}(T-t)^n}{n!} e^{\hat{\lambda}(T-t)} \Phi(d_2) \right).
\end{aligned} \tag{49}
$$

Here, C_{BS} represents the price of a call option driven by the Black-Scholes formula

$$
C_{BS}(S_t, t, K, \sigma, r) = S_t \Phi(d_1) - Ke^{-rt} \Phi(d_2) \tag{50}
$$

where

$$
\begin{aligned}
d_1 =& \frac{\ln \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}}, \\
d_2 =& d_1 - \sigma\sqrt{t}.
\end{aligned}
$$

Here, $\Phi(.)$ is the standard normal distribution function. In Example 1, $\tilde{\lambda}(t)$ represents the total jump intensity:

$$
\tilde{\lambda}(t) = \sum_{k=1}^{m-d} \tilde{\lambda}^{(k)}(t). \tag{51}
$$

One can approximate the upper and lower hedging prices for Example 1 within the interval:

$$
\left[\inf_{z\in[0,r_2-r_1]} C_{r_z}, \sup_{z\in[0,r_2-r_1]} C_{r_z} \right]. \tag{52}
$$

By the call-put parity, a similar method can be applied to

$$
f_T = (K - S_1(T))^+. \tag{53}
$$

This completes Example 1.

5. The Shortfall Risk Minimization Problem

In this section, we study the case where the initial wealth x is less than the required expected value of $e^{-r_z T} f_T$ denoted by $\hat{E}[e^{-r_z T} f_T]$. In this case, it is unlikely to apply a perfect hedge; however, it is possible to minimize the risk of shortfall corresponding to the initial cost constraint by considering the following optimization problem:

$$
u(x) = \inf_{\substack{\xi\in\mathcal{A} \\ x<\hat{E}[f(S_T^1)e^{-r_z T}]}} E[l_p((f_T - X_T^{\pi,z}(x))^+)] \tag{54}
$$

Here, $l^p(x) = \frac{x^p}{p}$ is the loss function with $p > 1$, and $\mathcal{A} = \{\xi \quad \text{s.t} \quad E[\sup_{0\le t\le T} |X_T^{\xi}(.)|] < \infty\}$, i.e., the set of all admissible portfolios with initial capital x. $f_T \in [L^{p+\epsilon}(\Omega, \mathcal{F}_T, P)]$ is the contingent claim with the maturity time T for some ϵ. $X_T^{\pi,d}$ is the wealth process.

In this problem set, if x is greater than the replication cost f_T, the completeness of the market allows the investor to hedge the contingent claim f_T without taking risks. On the other hand, if x is

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strictly less than the replication cost of f_T, there is a potential for a shortfall. We have the option to divide this problem into a perfect hedging problem of f_T and a utility minimization problem.

Denote $\mathcal{A}_0(\alpha)$ as the set of portfolio processes $\pi(t)$ for $0 \le t \le T$ and $\alpha > 0$, satisfying

$$X^{\alpha,\pi}(t) \ge 0 \quad t \in [0,T] \quad \text{a.s.,} \tag{55}$$

and

$$E[\sup_{0 \le t \le T} |X^{0,\pi}(t)|^p] < \infty. \tag{56}$$

Now, we introduce another optimization problem with the goal of

$$J(\alpha) := \inf_{\pi \in \mathcal{A}_0(\alpha)} E[l_p(X^{\alpha,\pi}(T))], \quad \alpha > 0. \tag{57}$$

Theorem 2. *(i) Let $\xi_0(t)$ be the optimal portfolio proportions for $\xi_0 \in \mathcal{A}_0(\alpha)$ for every $\alpha \in (0,\infty)$. The optimal portfolio, denoted by $\xi_0 = (\xi_0^1, \xi_0^2)$, obtained from $J(\alpha)$ is given by the system of equations:*

$$\sigma_1\xi_0^1 + \sigma_2\xi_0^2 = \frac{\phi}{p-1},$$
$$\nu_1\xi_0^1 + \nu_2\xi_0^2 = -\left(\frac{\lambda^*}{\lambda}\right)^{q-1}.$$

Solving for ξ_0^1 and ξ_0^2, we get

$$\xi_0^1 = \frac{\frac{\phi\nu^2}{p-1} + \sigma^2\left(\frac{\lambda^*}{\lambda}\right)^{q-1}}{\nu^2\sigma^1 - \nu^1\sigma^2},$$
$$\xi_0^2 = \frac{\frac{\phi\nu^1}{p-1} + \sigma^1\left(\frac{\lambda^*}{\lambda}\right)^{q-1}}{\nu^1\sigma^2 - \nu^2\sigma^1}.$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) The cost function $u(x)$ is given by

$$u(x) = l_p(x_{f_T} - x)e^{(-(p-1)aT)}, \tag{58}$$

where x_{f_T} is the replication cost of f_T, and

$$a = -qr_z + \frac{1}{2}q(q-1)\phi^2 - \lambda\left((q-1) - q\left(\frac{\lambda^*}{\lambda}\right) + \left(\frac{\lambda^*}{\lambda}\right)^q\right).$$

(iii) The optimal wealth is given by

$$X_T^{\pi_{f_T} - \xi_0, z}(x) = f_T - (x_{f_T} - x)(Z_T)^{q-1}e^{(-(a + \frac{r_z}{p-1})T)}. \tag{59}$$

See Kane and Melnikov [4].

Now, we present a solution to the problem 54 in a two-interest rate market.

Theorem 3. *Let $\hat{X}_t^{\pi}(x)$ be the wealth process in the (B_z, S_m) satisfying (25), and $X_t^{\pi}(x)$ the wealth process in the (B_1, B_2, S_m)-market satisfying (26) with initial capital x. Assume $\pi(t)$, the optimal proportion for problem (54) in the (B_z, S_m)-market verifies (24), and α_{f_T}, the optimal strategy hedging f_T in the (B_z, S_m)-market,*

satisfies the conditions in Statement 4. Then, in the (B_1, B_2, S_m)-market:
(i) The cost function (54) is given by (58). (ii) The optimal proportions invested are

$$\xi_t^i = \frac{\xi_f^i X_{t-}^{\pi_{f_T}}(x_{f_T}) - \xi_0^i X_{t-}^{\pi_0}(x_{f_T} - x)}{X_{t-}^{\pi_{f_T}-\pi_0}(x)} \quad on\ S_i, \tag{60}$$

and $(1-\xi)^+$ on the deposit account and $(1-\xi)^-$ on the credit account.

Proof. The proof follows a similar structure to the one presented by Kane and Melnikov [4] in the multi-dimensional case. □

6. Pricing Contingent Claims via Market Completion in (B_1, B_2, S_m)-market

In this section, our aim is to study no-arbitrage price bounds in incomplete markets. To initiate our analysis, we examine the market (B, S_m), which is characterized by multi-dimensional risky assets and one non-risky asset and results in a single interest rate. Our objective is to price contingent claims in incomplete markets, prompting a transition to a market with two different interests later on, resulting in market incompleteness.

Assuming the dynamics of the risky assets follow the equation 1, with parameters and assumptions identical, we introduce a non-risky asset governed by

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1. \tag{61}$$

Let $\pi = (\beta(t), \gamma_1(t), \ldots, \gamma_m(t))$ be a $\mathbb{R}^{(m+1)}$-valued process for $t \in [0, T]$, representing a portfolio. We assume that $\int_0^T \|\pi(t)\|^2 dt < \infty$ almost surely under the probability measure P.

The value of the portfolio, denoted by $X^\pi(t)$, is given by

$$X^\pi(t) = \beta(t)B(t) + \sum_{i=1}^{m} \gamma_i(t)S_i(t), \quad \text{for all } t \in [0, T].$$

It suffices to assume that our market is arbitrage-free if there exists an equivalent martingale measure, i.e., a measure equivalent to P under which the value of any self-financing strategy is a local martingale. The existence of this measure can be inferred by assuming at least one predictable process $\kappa = (\kappa_W, \kappa_N)^\intercal$, where κ_W and κ_N are defined on $\mathbb{R}^m$-valued Brownian motion and a $(m-d)$-dimensional Poisson process, respectively, such that the process κ satisfies

$$\sigma(t)\kappa_W(t) + \nu(t) \cdot (\mathbf{1} - \kappa_N) = \mu(t) = \tilde{\sigma}(t)\theta(t), \tag{62}$$

where $\mathbf{1}$ represents a vector of ones. Henceforth, we assume the existence of at least one process κ as described above. Let us define the probability measure such that

$$Z_\kappa^W(t) := \exp\left\{-\int_0^t \kappa_W(s)^\intercal dW(s) - \frac{1}{2}\int_0^t \|\kappa_W(s)\|^2 ds\right\},$$
$$Z_\kappa^N(t) := \exp\left\{-\int_0^t \lambda(s) \cdot (1 - \kappa_N(s))ds\right\} \prod_{k=1}^{m-d} \prod_{s \le t} \kappa_N^{(k)}(s)\Delta N_k(s), \quad \text{for } t \le T.$$

Define

$$Z_\kappa = Z_\kappa^W Z_\kappa^N, \tag{63}$$

a non-negative local martingale (See [8]) with $E[Z_\kappa(t)] = 1$ for all $t \in [0, T]$. The sufficient condition for market completeness is the uniqueness of the equivalent martingale measure. Therefore,

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our market is complete if Z_κ is a martingale and 62 has only one solution such that $\kappa_N^{(k)} > 0$ for $k \in \{1, \ldots, m-d\}$.

Assume Ξ represents the set of all possible equivalent martingale measures in this market, i.e., Ξ is the set of all κ which solve relation 62 with $\kappa_N^{(k)} > 0$ for $k \in \{1, \ldots, m-d\}$, and Z_κ is a martingale in this set. Therefore, the unique parameters of this are given by

$$\begin{aligned} \kappa_W(t) =& \theta_W(t), \\ \kappa_N(t) =& \lambda^{-1}(t) \cdot (\lambda(t) - \theta_N(t)). \end{aligned} \tag{64}$$

Proposition 1 ([2, Theorem 4.2]). *Let Ξ denote the set of all equivalent martingale measures in the (B,S)-market, and let $\frac{dQ_\kappa}{dP}\Big|_{\mathbb{F}_t} = Z_\kappa(t)$. Then, $Q_\kappa \in \Xi$ if and only if $\kappa \in \Xi$.*

Let us denote an $\mathcal{F}_T$-measurable random variable f_T as a contingent claim such that $E_Q[f_T] \leq \infty$ for all $Q \in \Xi$.
Consider the case where the financial market has the same deposit and credit rates, i.e., $r_1 = r_2$. This assumption leads to considering the same deposit and credit account $B_1 = B_2$. Finally, with this assumption, we are describing the (B, S_m)-market with a portfolio process $\pi = (\beta, \gamma_1, \ldots, \gamma_m)$. In this case, the capital follows

$$X(t) = \beta(t)B(t) + \sum_{i=1}^{m} \gamma_i(t)S_i(t).$$

Assuming $r_1 = r_2 = r$

$$\frac{dX(t)}{X(t-)} = \frac{dY(t)}{Y(t-)} = (1 - \xi(t))rdt + \sum_{i=1}^{m} \xi_i(t)\frac{dS_i(t)}{S_i(t-)}.$$

In such a market, the unique element of Ξ is given by

$$\begin{aligned} \kappa_W(t) =& \theta_W(t), \\ \kappa_N(t) =& \lambda^{-1}(t) \cdot (\lambda(t) - \theta_N(t)), \end{aligned}$$

where $\kappa_N(t) < \lambda(t)$ for all $t \in [0, T]$.

Let us return to the (B_1, B_2, S_m)-market where the credit rate is higher than the deposit rate. This market is incomplete due to differing borrowing and lending rates. We establish a no-arbitrage price bound over the set of equivalent martingale measures in this incomplete market. When a market is incomplete, replicating all contingent claims becomes impossible. However, by introducing specific sets of assets, we can achieve market completeness.
We broaden the set of admissible strategies to include investment strategies with consumption, represented by a $(m+3)$-dimensional $\mathcal{F}$-adapted portfolio process $(\pi, c) = (\beta_1(t), \beta_2(t), \gamma_1(t), \ldots, \gamma_m(t), c(t))$, where $c(t) \geq 0$ for $t \in [0, T]$.
The value of such a portfolio is given by

$$X^{\pi,c}(t) = \beta_1(t)B_1(t) + \beta_2(t)B_2(t) + \sum_{i=1}^{m} \gamma_i(t)S_i(t) - \int_0^t c(s)ds.$$

We then determine the upper and lower hedging prices as follows

$$C^*(f_T) = \inf\{x \geq 0 : \exists(\pi, c) \in \mathcal{A}(x) : X^{\pi,c}(T) \geq f_T, \text{P-a.s.}\} \tag{65}$$
$$C_*(f_T) = \inf\{x \geq 0 : \exists(\pi, c) \in \mathcal{A}(-x) : X^{\pi,c}(T) \geq -f_T, \text{P-a.s.}\} \tag{66}$$

The seller price, $C^*(f_T)$, represents the smallest initial capital required for the investor to establish their portfolio. The buyer price, $C_*(f_T)$, is the largest initial capital required for the investor to pay, ensuring they would not want to pay more than this amount. The upper and lower hedging prices are determined by taking the infimum and supremum over the set of all equivalent martingale measures accommodated in an incomplete market as follows

$$C^*(f_T) = \sup_{Q \in \Xi} E_Q[f_T],$$
$$C_*(f_T) = \inf_{Q \in \Xi} E_Q[f_T].$$

Now, we consider the case discussed in Section 4 and introduce the interest rate r_z as defined in Statement 1, ensuring that the assumption for market completeness is satisfied. We have previously denoted by $\hat{Z}$ the unique equivalent martingale measure on the complete market (B_z, S_m). We provide an approximate price by defining the upper and lower completion prices $\hat{C}^*(f_T; r_z)$ and $\hat{C}_*(f_T; r_z)$ as follows

$$\hat{C}^*(f_T; r_z) = \sup_{r_z \in [0, r_2 - r_1]} E[\hat{Z} f_T],$$
$$\hat{C}_*(f_T; r_z) = \inf_{r_z \in [0, r_2 - r_1]} E[\hat{Z} f_T].$$

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