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Article

Applications of Supersymmetric Polynomials in Statistical Quantum Physics

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Abstract: We propose a correspondence between partition functions of ideal gases consisting of both bosons and fermions and algebraic bases of supersymmetric polynomials on the Banach space of absolutely summable two-sides sequences $\ell_1(\mathbb{Z}_0)$. Such an approach allows us to interpret some combinatorial identities for supersymmetric polynomials from a physical point of view. We consider a relation of equivalence on $\ell_1(\mathbb{Z}_0)$ induced by the supersymmetric polynomials, and semiring algebraic structures on the quotient set with respect to this relation. The quotient set is a natural model for the set of energy levels of a quantum system. We introduce two different topological semiring structures on this set and discuss their possible physical interpretations.

Keywords: quantum ideal gas; grand partition function; supersymmetric polynomials on Banach spaces; algebraic basis; topological semiring; tropical semiring

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1. Introduction

Symmetric polynomials variables and relations between bases of the algebra of symmetric polynomials are widely used in Algebra, Combinatorics (see [1]), and in particular, in Statistical Quantum Mechanics. In [2,3] Schmidt and Schnack proposed some correspondence between relations in the algebra of symmetric polynomials and partition functions of bosons and fermions. Under this correspondence, one basis of symmetric polynomials is responsible for bosons and another for fermions. Such an approach was applied and developed for different cases by many authors (see e.g. [4–8]). On the other hand, recently some new results for algebras of symmetric analytic functions on infinite-dimensional Banach spaces were obtained [9–14]. The infinite number of variables of the underlying space allows us to introduce some interesting algebraic operations on the spectra of such algebras that may have a physical meaning. In addition, in the infinite-dimensional case, we can consider the behavior of the ideal gas “at infinity” if, for example, the number of particles grows to infinity while the total energy of the system is bounded.

In [15–17] were considered supersymmetric polynomials and analytic functions on abstract Banach spaces. Supersymmetric polynomials of several variables were studied in [18–20]. It seems to be that some bases of supersymmetric polynomials give us a tool for the investigation of a quantum ideal gas consisting of both bosons and fermions. Also, supersymmetric polynomials define a relation of equivalence on the underlying vector space and the quotient set with respect to this relation looks like the most natural model for the set of energy levels of a given quantum system. Such a set admits some algebraic semiring structures, related, in particular, to tropical (idempotent) mathematics.

In this paper, we discuss relations between algebras of supersymmetric polynomials on Banach spaces and partition functions of bosons and fermions and consider some new algebraic structures on the set of energy levels of corresponding quantum systems.

In Section 2 we gather basically known information about algebraic bases of symmetric polynomials on the Banach space ℓ_1 and their relations to partition functions of ideal quantum gases. In Section 3 we consider algebraic bases of supersymmetric polynomials and discuss their relations to partition functions of ideal gases consisting simultaneously of bosons and fermions. In Section 4 we construct two different semiring structures on set of energy levels. The first one is related to algebraic operations that were introduced in [17] for a more general case. The second is related to the idempotent operation max and looks like an infinite-dimensional generalization of the tropical semiring $\mathbb{R} \cup +\infty$ (c.f. [21]).

General information on polynomials and analytic functions on abstract Banach spaces can be found in [22,23]. The idempotent analysis and tropical semirings are considered in [24,25].

2. Preliminaries results on symmetric polynomials and partition functions

2.1. Symmetric polynomials

Let \mathbb{N} be the set of all positive integers, and ℓ_1 be the Banach space of all absolutely summing complex sequences $x = (x_1, \dots, x_n, \dots)$ with norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$. A function f on ℓ_1 is called symmetric if

$$f((x_{\sigma(1)}, x_{\sigma(2)}, \dots)) = f((x_1, x_2, \dots))$$

for every $(x_1, x_2, \dots) \in \ell_1$ and every bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$.

Let us define following symmetric polynomials on ℓ_1 . Let the polynomial F_n be defined by

$$F_n((x_1, x_2, \dots)) = \sum_{i=1}^{\infty} x_i^n, \quad (1)$$

where $n \in \mathbb{N}$. Polynomials F_n are called power sum symmetric polynomials. Let us define polynomials B_n as

$$B_n((x_1, x_2, \dots)) = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n}, \quad (2)$$

where $n \in \mathbb{N}$. Polynomials B_n are called complete symmetric polynomials. Let the polynomial G_n be defined by

$$G_n((x_1, x_2, \dots)) = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}, \quad (3)$$

where $n \in \mathbb{N}$. Polynomials G_n are called elementary symmetric polynomials.

Definition 1. A linear combination of finite products of powers (zero powers are also allowed) of elements of an algebra is called an algebraic combination of these elements.

A subset of an algebra is called algebraically independent if zero element of the algebra cannot be represented as a nontrivial algebraic combination of elements of this subset.

An algebraically independent subset of an algebra is called an algebraic basis of this algebra if every element of the algebra can be represented as an algebraic combination of elements of the subset. Due to the algebraic independence every such a representation is unique.

Let $\mathcal{P}_s(\ell_1)$ denotes the algebra of all continuous symmetric complex-valued polynomials on ℓ_1 . Every set of polynomials $\{F_n: n \in \mathbb{N}\}$, $\{B_n: n \in \mathbb{N}\}$ and $\{G_n: n \in \mathbb{N}\}$ is an algebraic basis in $\mathcal{P}_s(\ell_1)$ (see e.g. [9,13]). There are so-called Newton recurrent formulas connecting different algebraic bases:

$$mG_m = \sum_{k=1}^m (-1)^{k-1} G_{m-k} F_k, \quad m \in \mathbb{N}, \quad (4)$$

$$mB_m = \sum_{k=1}^m B_{m-k} F_k, \quad m \in \mathbb{N}, \quad (5)$$

$$G_m = \sum_{k=1}^m (-1)^{k-1} G_{m-k} B_k, \quad m \in \mathbb{N}, \quad (6)$$

and

$$B_m = \sum_{k=1}^m (-1)^{k-1} B_{m-k} G_k, \quad m \in \mathbb{N}. \quad (7)$$

Let $\mathcal{B}(x)(t)$ and $\mathcal{G}(x)(t)$ be the so-called generating functions for polynomials B_n and G_n respectively, defined as the following formal series

$$\mathcal{B}(x)(t) = \sum_{n=0}^{\infty} t^n B_n(x), \quad B_0 = 1 \quad (8)$$

and

$$\mathcal{G}(x)(t) = \sum_{n=0}^{\infty} t^n G_n(x), \quad G_0 = 1. \quad (9)$$

The following relations are well-known ([1], p. 3)

$$\mathcal{G}(x)(t) = \exp\left(-\sum_{n=1}^{\infty} t^n \frac{F_n(-x)}{n}\right) \quad \text{and} \quad \mathcal{B}(x)(t) = \exp\left(\sum_{n=1}^{\infty} t^n \frac{F_n(x)}{n}\right), \quad (10)$$

and they immediately imply that

$$\mathcal{G}(x)(t)\mathcal{B}(-x)(t) = 1. \quad (11)$$

Here the equality holds for every $x \in \ell_1$ and for every t in the common domain of convergence. Note that $\mathcal{G}(x)(t)$ is a well-defined analytic function of $x \in \ell_1$ for every fixed $t \in \mathbb{C}$ and a function of exponential type of t for every fixed x [26].

2.2. Partition functions

The canonical partition function plays a fundamental role in statistical mechanics since most thermodynamic functions can be derived from it [3]. It is defined by

$$Z_N(\beta) = \text{Tr} \exp(-\beta H), \quad (12)$$

where H denotes the Hamiltonian of the system, N is the number of particles and

$$\beta = \frac{1}{k_B T}$$

denotes the inverse temperature (k_B is the Boltzmann constant, T is the temperature). In other words, H is a self adjoint operator such that $\exp(-\beta H)$ is a trace class operator for $\beta \in \mathbb{R}$.

The grand canonical partition function is defined by

$$Z(z, \beta) = \sum_{N=0}^{\infty} Z_N(\beta) z^N, \quad (13)$$

where the variable z is physically interpreted as the fugacity of the system, i.e., $z = \exp(\mu/(k_B T))$ (μ is the chemical potential). It describes the system in which the number of particles can be changed. The physical interpretation implies that z must be non-negative.

Note that the partition function completely defines all possible states of the system. Also it can be used for deriving the possibilities of states.

Consider the ideal gas consisting of noninteracting identical particles (bosons or fermions). In this case the Hamiltonian H is the sum of N identical single-particle Hamiltonians:

$$H = \sum_{n=1}^N h_n.$$

Let E_i be single-particle energy eigenvalues. In [27] it is shown that

$$Z_N(\beta) = B_N((x_1, x_2, \dots)) \quad (14)$$

for the system of bosons and

$$Z_N(\beta) = G_N((x_1, x_2, \dots)) \quad (15)$$

for the system of fermions, where B_N is defined by (2), G_N is defined by (3) and

$$x_i = \exp(-\beta E_i). \quad (16)$$

Note that Z_N is a symmetric function between energy levels, not between particles.

By (8), (9), (13), (14) and (15), the grand canonical partition function can be represented in the form

$$Z(z, \beta) = \mathcal{B}((x_1, x_2, \dots))(z)$$

for bosons and

$$Z(z, \beta) = \mathcal{G}((x_1, x_2, \dots))(z)$$

for fermions, where x_i are defined by (16). In addition, according to [2] the coordinates (x_1, x_2, \dots) of $x \in \ell_1$ correspond to abstract energy levels of the system, a monomial $x_1^{n_1} \cdots x_m^{n_m}$, $n_1 + \cdots + n_m = N$ in a partition function corresponds to possible occupation of levels x_1, \dots, x_m by N particles. Also, there exists so-called *fundamental symmetry* ω of $\mathcal{P}_s(\ell_1)$ which is defined as an algebra homomorphism from $\mathcal{P}_s(\ell_1)$ to itself such that $\omega(F_n) = (-1)^{n-1} F_n$, $n \in \mathbb{N}$. In other words, for every n , $(\omega(F_n))(x) = -F_n(x)$. Note that ω is an involution in the sense that ω^2 is the unity operator. It is known that $\omega G_n = B_n$ and $\omega B_n = G_n$ for every $n \in \mathbb{N}$ [1, p. 4]. In [2] was observed that Newton's identity (4) corresponds to Landsberg's identity in physics [28] and equation (11) is related to the Bose-Fermi symmetry.

2.3. Note about the Banach space ℓ_1

As we mentioned above, $\exp(-\beta H)$ is a trace class operator and so, its eigenvalues $x_i = \exp(-\beta E_i)$ are summable, that is, $x = (x_1, x_2, \dots) \in \ell_1$. On the other hand, in [2] it was observed that for the case $n = \infty$, the evaluations \mathcal{G} and \mathcal{B} leads to corresponding grand canonical partition functions only if these series converge. Since all $x_i \geq 0$, the vector x must be in ℓ_1 . Thus, the space of absolutely summable sequences ℓ_1 is a most natural domain for vectors $x = (x_1, x_2, \dots)$ and $\mathcal{P}_s(\ell_1)$ is a most natural algebra of symmetric polynomials for $n = \infty$. However, it is possible to consider symmetric polynomials in the general case ℓ_p , $1 \leq p < \infty$ and even for the case of "continual" number of variables if $x \in L_p$, $1 \leq p \leq \infty$ (see [13,29–31] and references therein).

Note that in [32] were considered some relations between a trace class operator A and the (infinite-dimensional) Fredholm determinant $\det(I - A)$, where I is the identity operator. In particular, if A is self-adjoint with eigenvalues x_i , then

$$\det(I - tA) = \mathcal{G}(x)(t) \quad \text{and} \quad (\det(I - A))^{-1} = \mathcal{B}(-x)(t).$$

Applications of determinants of the form $\det(I - A)$ to partition functions can be found in [33].

3. Supersymmetric polynomials and partition functions for mixed systems of bosons and fermions

Let \mathbb{Z} be the set of all integers and $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. We denote by $\ell_1(\mathbb{Z}_0)$ the Banach space of all absolutely summing complex sequences indexed by elements of \mathbb{Z}_0 (two-sides sequences). Every element of $\ell_1(\mathbb{Z}_0)$ can be represented in the form

$$(y|x) = (\dots, y_2, y_1 | x_1, x_2, \dots)$$

with

$$\|(y|x)\| = \sum_{i=1}^{\infty} (|x_i| + |y_i|),$$

where $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ belong to ℓ_1 .

For every $n \in \mathbb{N}$ we define the polynomials $T_n, n \in \mathbb{N}$ on $\ell_1(\mathbb{Z}_0)$ by

$$T_n((y|x)) = F_n(x) - F_n(y),$$

where F_n is defined by (1).

A polynomial on $\ell_1(\mathbb{Z}_0)$ is called *supersymmetric* (see [17]) if it can be represented as an algebraic combination of elements of the set $\{T_n : n \in \mathbb{N}\}$. Let us denote $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$ the algebra of all supersymmetric polynomials on $\ell_1(\mathbb{Z}_0)$. Note that the set $\{T_n : n \in \mathbb{N}\}$ is an algebraic basis of the algebra $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$. Let us define another important supersymmetric polynomials on $\ell_1(\mathbb{Z}_0)$ which also form an algebraic basis of the algebra $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$. For $n \in \mathbb{N}$ let $W_n : \ell_1(\mathbb{Z}_0) \rightarrow \mathbb{C}$ be defined by

$$W_n((y|x)) = \sum_{k=0}^n G_k(x) B_{n-k}(-y). \quad (17)$$

Note that polynomials W_n can be obtained if we substitute in Newton's formula (4) polynomials T_n instead of F_n [17]. In other words,

$$mW_m((y|x)) = \sum_{k=1}^m (-1)^{k-1} W_{m-k}((y|x)) T_k((y|x)), \quad m \in \mathbb{N}. \quad (18)$$

From (18), in particular, it follows that all polynomials W_n are supersymmetric and form an algebraic basis in $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$.

Let $\mathcal{W}((y|x))(t)$ be the formal series

$$\mathcal{W}((y|x))(t) = \sum_{n=0}^{\infty} t^n W_n((y|x)), \quad W_0 = 1, \quad (19)$$

that is, \mathcal{W} is the generating function for polynomials W_n . By [17, Theorem 2],

$$\mathcal{W}((y|x))(t) = \frac{\mathcal{G}(x)(t)}{\mathcal{G}(y)(t)}, \quad (20)$$

where the equality is true on the common domain of convergence.

Consider a mixed system of bosons and fermions. In [27] it is shown that the partition function for the system, where the total number N of bosons and fermions is fixed, can be represented in the form

$$Z_N(\beta) = \sum_{k=0}^N G_k \left((x_1^{(F)}, x_2^{(F)}, \dots) \right) B_{N-k} \left((x_1^{(B)}, x_2^{(B)}, \dots) \right), \quad (21)$$

where

$$x_i^{(F)} = \exp(-\beta E_i^{(F)}) \quad \text{and} \quad x_i^{(B)} = \exp(-\beta E_i^{(B)}),$$

$E_i^{(F)}$ and $E_i^{(B)}$ are single-particle energies of fermions and bosons resp.

Let $\tilde{W}_n : \ell_1(\mathbb{Z}_0) \rightarrow \mathbb{C}$ be defined by

$$\tilde{W}_n((y|x)) = W_n((-x| -y)), \quad (22)$$

where W_n is defined by (17). By (17) and (22),

$$\tilde{W}_n((y|x)) = \sum_{k=0}^n G_k(-y) B_{n-k}(x). \quad (23)$$

By (21) and (23),

$$Z_N(\beta) = \tilde{W}_N((\tilde{y}|\tilde{x})), \quad (24)$$

where

$$\tilde{y} = (-x_1^{(F)}, -x_2^{(F)}, \dots) \quad (25)$$

and

$$\tilde{x} = (x_1^{(B)}, x_2^{(B)}, \dots). \quad (26)$$

If sequences are finite, we complete them with an infinite number of zeros. Note that the equality (24) makes sense only if \tilde{x} and \tilde{y} belong to ℓ_1 . Otherwise we only can consider (24) as formal equality.

Let us consider the grand canonical partition function. By (13) and (24),

$$Z(z, \beta) = \sum_{N=0}^{\infty} z^N \tilde{W}_N((\tilde{y}|\tilde{x})), \quad \tilde{W}_0 = 1. \quad (27)$$

For $(y|x) \in \ell_1(\mathbb{Z}_0)$ and $t \in \mathbb{C}$ let $\tilde{\mathcal{W}}((y|x))(t)$ be the formal series

$$\tilde{\mathcal{W}}((y|x))(t) = \sum_{n=0}^{\infty} t^n \tilde{W}_n((y|x)). \quad (28)$$

Evidently,

$$Z(z, \beta) = \tilde{\mathcal{W}}((\tilde{y}|\tilde{x}))(z). \quad (29)$$

On the other hand, by (28), (22), (19) and (20),

$$\begin{aligned} \tilde{\mathcal{W}}((y|x))(t) &= \sum_{n=0}^{\infty} t^n \tilde{W}_n((-x| -y)) \\ &= \mathcal{W}((-x| -y))(t) \\ &= \frac{\mathcal{G}(-y)(t)}{\mathcal{G}(-x)(t)} = \frac{\mathcal{B}(x)}{\mathcal{B}(y)}. \end{aligned} \quad (30)$$

So, by (29), (30) and (11)

$$Z(z, \beta) = \frac{\mathcal{G}(-\tilde{y})(t)}{\mathcal{G}(-\tilde{x})(t)} = \frac{\mathcal{B}(\tilde{x})}{\mathcal{B}(\tilde{y})},$$

where \tilde{y} and \tilde{x} are defined by (25) and (26) resp.

Thus, we have represented the grand canonical partition function of the mixed system of bosons and fermions via the generating functions \mathcal{G} and \mathcal{B} for elementary symmetric polynomials.

Let us observe that if we apply the transformation $(y|x) \mapsto (-x|-y)$ to T_n for the case $y = 0$, we will obtain

$$F_n(x) = T_n((0|x)) \mapsto T_n((-x|0) = (-1)^{n-1}T_n((0|x)) = (-1)^{n-1}F_n(x) = (\omega(F_n))(x).$$

In other words, the involution ω on $\mathcal{P}_s(\ell_1)$ can be extended to $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$ setting $(\omega(P))((y|x)) = P((-x|-y))$. In particular, $\omega(W_n) = \tilde{W}_n$. Applying the homomorphism ω to (18), we obtain

$$m\tilde{W}_m((y|x)) = \sum_{k=1}^m \tilde{W}_{m-k}((y|x))T_k((y|x)), \quad m \in \mathbb{N},$$

that is, \tilde{W}_n can be obtained if we substitute T_n instead of F_n to the Newton formula (5) and so, we have another representation for \tilde{W}_n which can be interpreted as another realization of Landsberg's identity. In addition, from (6), (7) we can get

$$\tilde{W}_m = \sum_{k=1}^m (-1)^{k-1} \tilde{W}_{m-k} W_k, \quad m \in \mathbb{N}.$$

4. Semiring structures on the set of variables

4.1. The ring \mathcal{M}_0

First we consider a dense linear subspace c_{00} of ℓ_1 . Let c_{00} be the vector space of all eventually zero sequences of complex numbers. Let $c_{00}(\mathbb{Z}_0)$ be the subspace of $\ell_1(\mathbb{Z}_0)$ consisting of all $(y|x) \in \ell_1(\mathbb{Z}_0)$ such that $x, y \in c_{00}$. To shorten the notation we will write elements of c_{00} as (x_1, \dots, x_n) instead of $(x_1, \dots, x_n, 0, \dots)$. Correspondingly, we will write elements of $c_{00}(\mathbb{Z}_0)$ as $(y_1, \dots, y_m|x_1, \dots, x_n)$.

Let us define the following equivalence relation on $c_{00}(\mathbb{Z}_0)$. For $a, b \in c_{00}(\mathbb{Z}_0)$ let $a \sim b$ if and only if $T_n(a) = T_n(b)$ for every $n \in \mathbb{N}$. Let $\mathcal{M}_0 = c_{00}(\mathbb{Z}_0)/\sim$. Note that we have two types of equivalent elements:

$$(y_1, \dots, y_m|x_1, \dots, x_n) \sim (y_{\tau(1)}, \dots, y_{\tau(m)}|x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

where τ and σ are permutations on sets $\{1, \dots, m\}$ and $\{1, \dots, n\}$ resp., and

$$(y_1, \dots, y_m, c|x_1, \dots, x_n) \sim (y_1, \dots, y_m|x_1, \dots, x_n).$$

Consequently, every element of \mathcal{M}_0 has the representative $(y|x)$, where $x, y \in c_{00}$, such that multisets of nonzero elements of x and y are disjoint. On the other hand, every pair of disjoint finite multisets of nonzero complex numbers define some element of \mathcal{M}_0 . So, we have the bijection between \mathcal{M}_0 and the set of all pairs of disjoint finite multisets of nonzero complex numbers. Let us define ring operations on \mathcal{M}_0 . First we define some auxiliary operations on c_{00} . Let

$$(x_1, \dots, x_n) \bullet (x'_1, \dots, x'_m) = (x_1, \dots, x_n, x'_1, \dots, x'_m)$$

and

$$\begin{aligned} (x_1, \dots, x_n) \diamond (x'_1, \dots, x'_m) \\ = (x_1x'_1, x_1x'_2, \dots, x_1x'_m, x_2x'_1, x_2x'_2, \dots, x_2x'_m, \dots, x_nx'_1, x_nx'_2, \dots, x_nx'_m) \end{aligned}$$

for $(x_1, \dots, x_n), (x'_1, \dots, x'_m) \in c_{00}$. Let

$$[z] + [z'] = [(y \bullet y' | x \bullet x')]$$

and

$$[z][z'] = \left[\left((y \diamond x') \bullet (x \diamond y') \mid (y \diamond y') \bullet (x \diamond x') \right) \right]$$

for $z = (y|x), z' = (y'|x') \in c_{00}(\mathbb{Z}_0)$, where $x, y, x', y' \in c_{00}$. By [17] \mathcal{M}_0 with these operations is a ring, where $-(y|x) = [(x|y)]$. Note that \mathcal{M}_0 is not a linear space, so it is not an algebra [17].

Let $a \sim b$. Since $T_n(a) = T_n(b)$ for every $n \in \mathbb{N}$, it follows that $f(a) = f(b)$ for every supersymmetric function f . That is, the value of a supersymmetric function does not depend on the choice of a representative of a class. So, we can set

$$f([a]) = f(a)$$

for a supersymmetric function f and for $[a] \in \mathcal{M}_0$.

Let us consider how our ring operations interplay with the algebraic basis T_n and the partition function $\widetilde{\mathcal{W}}(z)(t)$. By [17],

$$T_n([z][z']) = T_n([z])T_n([z']) \quad \text{and} \quad T_n([z] + [z']) = T_n([z]) + T_n([z']) \quad (31)$$

for every $n \in \mathbb{N}$ and $[z], [z'] \in \mathcal{M}_0$. In other words, each T_n is a ring homomorphism from \mathcal{M}_0 to \mathbb{C} . Also, it is easy to check (c.f. [17]) that

$$\mathcal{W}([z] + [z'])(t) = \mathcal{W}([z])(t)\mathcal{W}([z'])(t)$$

and

$$\widetilde{\mathcal{W}}([z] + [z'])(t) = \widetilde{\mathcal{W}}([z])(t)\widetilde{\mathcal{W}}([z'])(t).$$

The following example may be interesting for evaluating grand canonical partition functions “at infinity”.

Example 1. Let λ and μ be positive numbers. Set

$$z^{(n)} = \left(\underbrace{-\frac{\mu}{n}, \dots, -\frac{\mu}{n}}_n \mid \underbrace{\frac{\lambda}{n}, \dots, \frac{\lambda}{n}}_n \right).$$

Taking into account [17, pp. 7-8] and relations between \mathcal{W} and $\widetilde{\mathcal{W}}$, we can see that if $n \rightarrow \infty$, then both $\mathcal{W}(z^{(n)})(t)$ and $\widetilde{\mathcal{W}}(z^{(n)})(t)$ approach the function $e^{(\lambda+\mu)t}$. Moreover, at the “limit point” $Z_1(\beta) = \lambda + \mu$, and $Z_N(\beta) = 0$ for every $N > 1$.

Consider the case when sequences \tilde{x} and \tilde{y} , defined by (26) and (25) resp., have only finite number of nonzero elements, i.e., $\tilde{x}, \tilde{y} \in c_{00}$. Then $(\tilde{y}|\tilde{x}) \in c_{00}(\mathbb{Z}_0)$. So, $[(\tilde{y}|\tilde{x})] \in \mathcal{M}_0$. Since functions \widetilde{W}_n , used in the representations (24) and (27) of partition functions are supersymmetric, it follows that values $\widetilde{W}_n(u)$ do not depend on the choice of the representative $u \in [(\tilde{y}|\tilde{x})]$. So, it is natural to consider partition functions as functions on such equivalence classes. Note that all elements of the sequence \tilde{x} are non-negative and all elements of the sequence \tilde{y} are non-positive. So, the equivalence class $[(\tilde{y}|\tilde{x})]$ belongs to the subset \mathcal{M}_0^\pm of \mathcal{M}_0 defined in the following way. Let us denote by \mathcal{M}_0^\pm the set of elements $[u]$, where u is of the form

$$u = (-y_1, \dots, -y_m | x_1, \dots, x_n), \quad x_i \geq 0, \quad y_j \geq 0.$$

Note that \mathcal{M}_0^\pm can be completed with respect to a ring norm on \mathcal{M}_0 (see [15,17]). In Subsection 4.2 we consider such completions more detailed.

For every $[u] \in \mathcal{M}_0^\pm$ and odd number k ,

$$T_k(u) = F_k(x) + F_k(y) \geq 0,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, and it is equal to zero if and only if $u = 0$.

It is known that \mathcal{M}_0 contains divisors of zero. For example,

$$\begin{aligned} [(-1|1)][(0|1, -1)] &= [((-1) \diamond (1, -1)) \bullet ((1) \diamond (0)) | ((1) \diamond (1, -1)) \bullet ((-1) \diamond (0))] \\ &= [(-1, 1) \bullet (0) | (1, -1) \bullet (0)] \\ &= [(-1, 1, 0 | 1, -1, 0)] \\ &= [(0|0)] \\ &= 0. \end{aligned}$$

Proposition 1. *The set \mathcal{M}_0^\pm is a commutative semiring with respect to the ring operations in \mathcal{M}_0 , without divisors of zero.*

Proof. It is easy to check that if $[u]$ and $[v]$ are in \mathcal{M}_0^\pm , then both $[u] + [v]$ and $[u][v]$ are in \mathcal{M}_0^\pm . But for a given $[u] = [(-y_1, \dots, -y_m | x_1, \dots, x_n)] \in \mathcal{M}_0^\pm$, $u \neq 0$, the element $-[u] = [(x_1, \dots, x_n | -y_1, \dots, -y_m)]$ does not belong to \mathcal{M}_0^\pm . Thus, \mathcal{M}_0^\pm is a semiring but not a ring. If $[u][v] = 0$, then, by (31), $T_1(u)T_1(v) = 0$. So either $T_1(u) = 0$ or $T_1(v) = 0$. Thus, either $u = 0$ or $v = 0$. \square

The semiring \mathcal{M}_0^\pm has the following important property that

$$[(-y_1, \dots, -y_m | x_1, \dots, x_n)] = [(-y'_1, \dots, -y'_{m'} | x'_1, \dots, x'_{n'})]$$

if and only if $m = m'$, $n = n'$ and there are permutations σ and τ such that

$$(-y_1, \dots, -y_m | x_1, \dots, x_n) = (-y'_{\tau(1)}, \dots, -y'_{\tau(m)} | x'_{\sigma(1)}, \dots, x'_{\sigma(n)}).$$

Let $\sharp[u]$ be a pair (m, n) such that in the representation $[u] = [(-y_1, \dots, -y_m | x_1, \dots, x_n)]$ in \mathcal{M}_0^\pm the number of nonzero elements y_j is equal to m and the number of nonzero elements x_i is equal to n . From the definition of the ring operations in \mathcal{M}_0 , we have that if $\sharp[u] = (m, n)$ and $\sharp[v] = (k, s)$, then $\sharp([u] + [v]) = (m + k, n + s)$ and $\sharp([u][v]) = (ms + nk, mk + ns)$. In particular, $\sharp[u]^2 = (2mn, m^2 + n^2)$.

Proposition 2. *Every invertible element in \mathcal{M}_0^\pm is of the form $(0|x)$ for some $x > 0$ or $(-y|0)$ for some $y > 0$. Every idempotent $[u]$ in \mathcal{M}_0^\pm is of the form $[u] = [(0|1)]$ or $[u] = [(-1|0)]$.*

Proof. Let $[u][v] = [(0|1)]$, then $\sharp([u][v]) = (0, 1)$ and, so, $\sharp[u] = (0, 1)$ and $\sharp[v] = (0, 1)$ or $\sharp[u] = (1, 0)$ and $\sharp[v] = (1, 0)$. Consequently, $u = (0|x)$ and $v = (0|1/x)$ for some $x > 0$ or $u = (-y|0)$ and $v = (-1/y|0)$ for some $y > 0$.

Let $[u]$ be an idempotent in \mathcal{M}_0^\pm , that is, $[u]^r = [u]$ for some positive integer $r > 1$. Then $\sharp[u]^r = \sharp[u]$ only if $\sharp[u] = (1, 0)$ or $\sharp[u] = (0, 1)$. Elements of the form $[(-a|0)]$ and $[(0|a)]$, $a > 0$, are idempotents only if $a = 1$. \square

Proposition 3. *Elements of the form $[(-x_1, \dots, -x_n | x_1, \dots, x_n)]$, $x_i > 0$ can be represented as*

$$[(-x_1, \dots, -x_n | x_1, \dots, x_n)] = [(-1|1)][(-x_1, \dots, -x_k | x_{k+1}, \dots, x_n)]$$

for every integer k , $0 \leq k \leq n$.

Proof. The straightforward computation. \square

From the proposition it follows that we have no multiplicative cancelation in \mathcal{M}_0^\pm , that is, the equalities $[u][v] = [w]$ and $[u][v'] = [w]$ do not imply $[v] = [v']$.

4.2. A tropical semiring structure

We introduce another semiring structure on \mathcal{M}_0^\pm which is related to Tropical Mathematics. Some applications of tropical semirings to Quantum Mechanics can be found in [34]. Let us recall that the *min tropical semiring* is the semiring $(\mathbb{R} \cup \{+\infty\}, \underline{\oplus}, \odot)$, where the operations $\underline{\oplus}$ and \odot are defined by

$$x \underline{\oplus} y = \min\{x, y\}, \quad x \odot y = x + y, \quad x, y \in \mathbb{R} \cup \{+\infty\}.$$

The operations $\underline{\oplus}$ and \odot are called the *tropical addition* and the *tropical multiplication* respectively. The unit for $\underline{\oplus}$ is $+\infty$, and the unit for \odot is 0.

Similarly, the *max tropical semiring* is the semiring $(\mathbb{R} \cup \{-\infty\}, \overline{\oplus}, \odot)$ such that

$$x \overline{\oplus} y = \max\{x, y\}, \quad x \odot y = x + y, \quad x, y \in \mathbb{R} \cup \{-\infty\}.$$

In this semiring, the unit for $\overline{\oplus}$ is $-\infty$ and the unit for \odot is 0. The semirings are isomorphic with respect to the mapping $x \mapsto -x$. The usual metric $\rho(a, b) = |a - b|$ on \mathbb{R} can be extended to $\mathbb{R} \cup \{-\infty\}$ by setting $\rho(a, -\infty) = 1$ for every $a \in \mathbb{R}$. Similarly, $\rho(a, +\infty) = 1, a \in \mathbb{R}$, for the case $\mathbb{R} \cup \{+\infty\}$.

Let $(-y_m, \dots, -y_1 | x_1, \dots, x_n)$ be a representation of $[u] \in \mathcal{M}_0^\pm$. We say that this representation is *ordered* if $x_1 \geq x_2 \geq \dots \geq x_n$ and $-y_1 \leq -y_2 \leq \dots \leq -y_m$. The ordered representation of $[u]$ is unique and we denote it by $(-y_m, \dots, -y_1 | x_1, \dots, x_n)_o$. Let us denote by ϵ the formal element

$$\epsilon = (\dots, +\infty, \dots, +\infty, +\infty | -\infty, -\infty, \dots, -\infty, \dots).$$

Definition 2. Let us define a tropical semiring \mathcal{M}_0^\oplus as the set $\mathcal{M}_0^\pm \cup \{\epsilon\}$ with operations \oplus and \odot such that

$$\begin{aligned} & (-y_m, \dots, -y_1 | x_1, \dots, x_n)_o \oplus (-d_m, \dots, -d_1 | b_1, \dots, b_n)_o \\ &= ((-y_m) \underline{\oplus} (-d_m), \dots, (-y_1) \underline{\oplus} (-d_1) | x_1 \overline{\oplus} b_1, \dots, x_n \overline{\oplus} b_n)_o \\ &= (\min\{-y_m, -d_m\}, \dots, \min\{-y_1, -d_1\} | \max\{x_1, b_1\}, \dots, \max\{x_n, b_n\})_o \end{aligned}$$

and

$$\begin{aligned} & (-y_m, \dots, -y_1 | x_1, \dots, x_n)_o \odot (-d_m, \dots, -d_1 | b_1, \dots, b_n)_o \\ &= ((-y_m) \odot (-d_m), \dots, (-y_1) \odot (-d_1) | x_1 \odot b_1, \dots, x_n \odot b_n)_o \\ &= (-y_m - d_m, \dots, -y_1 - d_1 | x_1 + b_1, \dots, x_n + b_n)_o. \end{aligned}$$

Proposition 4. $\mathcal{M}_0^\pm \cup \{\epsilon\}$ is a semiring and the unit for \oplus is ϵ and the unit for \odot is 0.

Proof. Let us check the distributive law. From the distributive laws in the min tropical semiring and in the max tropical semiring,

$$\begin{aligned} & (-c_m, \dots, -c_1 | a_1, \dots, a_n)_o \odot ((-y_m, \dots, -y_1 | x_1, \dots, x_n)_o \oplus (-d_m, \dots, -d_1 | b_1, \dots, b_n)_o) \\ &= (-c_m \odot ((-y_m) \underline{\oplus} (-d_m)), \dots, -c_1 \odot ((-y_1) \underline{\oplus} (-d_1)) | a_1 \odot (x_1 \overline{\oplus} b_1), \dots, a_n \odot (x_n \overline{\oplus} b_n))_o \\ &= (-c_m, \dots, -c_1 | a_1, \dots, a_n)_o \odot (-y_m, \dots, -y_1 | x_1, \dots, x_n)_o \\ &\quad \oplus (-c_m, \dots, -c_1 | a_1, \dots, a_n)_o \odot (-d_m, \dots, -d_1 | b_1, \dots, b_n)_o. \end{aligned}$$

\square

Let X be a Banach space with an unconditional Schauder basis (e_n) , $n \in \mathbb{N}$. Then any vector $x \in X$ can be represented as

$$x = \sum_{n=1}^{\infty} x_n e_n = (x_1, \dots, x_n, \dots).$$

Denote by \mathcal{M}_X the ring of elements

$$[u] = [(\dots, y_m, \dots, y_1 | x_1, \dots, x_n, \dots)], \quad x_i, y_j \in \mathbb{C}$$

such that $x = (x_1, \dots, x_n, \dots)$ and $y = (y_1, \dots, y_m, \dots)$ are in X endowed with the following ring norm

$$\|[u]\| = \inf (\|x\|_X + \|y\|_X),$$

where the infimum is taken over all representations $u = (\dots, y_m, \dots, y_1 | x_1, \dots, x_n, \dots)$. It is known that this norm generates a metric $d([u], [v]) = \|[u] - [v]\|$ and \mathcal{M}_X is a complete metric space with respect to the metric. Moreover, the ring operations in \mathcal{M}_X are continuous and \mathcal{M}_0 is a dense subring in \mathcal{M}_X [15,17].

Let us denote by \mathcal{M}_X^{\pm} the closed subset in \mathcal{M}_X consisting of elements

$$[u] = [(\dots, -y_m, \dots, -y_1 | x_1, \dots, x_n, \dots)], \quad x_i \geq 0, \quad y_j \geq 0.$$

Thus \mathcal{M}_X^{\pm} is a complete metric space and a topological semiring.

We can extend the metric to $\mathcal{M}_X^{\oplus} = \mathcal{M}_X^{\pm} \cup \{\epsilon\}$ by setting $d(u, \epsilon) = 1$ for every $u \in \mathcal{M}_X^{\oplus}$. Note that $\mathcal{M}_X^{\oplus} \setminus \{\epsilon\}$ is a commutative group with respect to " \odot " and

$$\|[u]^{\odot k}\| = \left\| \underbrace{[u] \odot \dots \odot [u]}_k \right\| = \|[ku]\| = k\|u\|, \quad k \in \mathbb{N}.$$

Theorem 1. For any Banach space X with an unconditional basis the following statements are true:

1. The tropical operations are continuous in \mathcal{M}_X^{\oplus} ;
2. The mappings

$$\Phi^+[u] = \max_i x_i \quad \text{and} \quad \Phi^-[u] = \min_j (-y_j)$$

are continuous semiring homomorphisms from \mathcal{M}_X^{\oplus} to the max tropical semiring $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$ and to the min tropical semiring $(\mathbb{R} \cup \{-\infty\}, \ominus, \odot)$ respectively.

Proof. 1. If $[u]$ and $[v]$ are not equal to ϵ , then

$$\|[u] \oplus [v]\| \leq \|[u] \odot [v]\| \leq \|[u] \bullet [v]\|$$

and we know that the operation " \bullet " is continuous.

2. Clearly, $\Phi^+([u]) = x_1$ and $\Phi^-([u]) = -y_1$, in particular, $\Phi^+(\epsilon) = +\infty$, and $\Phi^-(\epsilon) = -\infty$. Also,

$$\Phi^+([u] \oplus [u']) = \max(x_1, x'_1) = \Phi^+([u]) \oplus \Phi^+([u'])$$

and

$$\Phi^+([u] \odot [u']) = x_1 + x'_1 = \Phi^+([u]) \odot \Phi^+([u']).$$

Thus Φ^+ is a semiring isomorphism.

To show the continuity, we observe that the function $x = (x_1, x_2, \dots) \mapsto \max_n |x_n|$ is bounded (on bounded subsets) on every Banach space X with a Schauder basis (e_n) . Indeed, if (π_n) , $n \in \mathbb{N}$ be the sequence of projections,

$$\pi_n(x) = \sum_{k=1}^n x_k e_k,$$

then

$$\sup_n \|\pi_n\| = K < \infty$$

(see [35], pp. 1-2) and so $|x_n| = |\pi_{n+1}(x) - \pi_n(x)| \leq 2K\|x\|_X$. Hence $|\Phi^+([u])| \leq 2K\|u\|$. The continuity of Φ^+ follows from ([36], Theorem 11.22) taking into account that Φ^+ is a bounded homomorphism of the multiplicative normed group $\mathcal{M}_X^\oplus \setminus \epsilon$ such that $\|[u]^{\odot k}\| = k\|u\|$, $k \in \mathbb{N}$. The same works for Φ^- . \square

5. Discussions and Conclusions

In this paper we continue to develop the ideas proposed by Schmidt and Schnack in [2,3] about involving symmetric polynomials for investigations of the partition functions of ideal quantum gases. The first goal of the paper was to find a correspondence between algebraic bases of supersymmetric polynomials and partition functions of ideal gases consisting of both bosons and fermions. We can see that combinatorial relations in the algebra of supersymmetric polynomials have corresponding physical interpretations. Taking into account that two elements (vectors) z and z' , in the set of possible energy levels are equivalent if and only if $P(z) = P(z')$ for every supersymmetric polynomial P , it is naturally to consider the quotient set with respect to the equivalence as a natural domain. For such a quotient set the usual vector operations are not valid and we introduced new ring operations (addition and multiplication) on the quotient set \mathcal{M}_0 . It seems to be that the new addition can be obtained using the direct sum of operators $\exp(-\beta H_1)$ and $\exp(-\beta H_2)$, while the new product leads to the tensor product of operators. Note that the elements of \mathcal{M}_0 have the physical interpretation if $x_i \geq 0$ and $y_j \leq 0$. Otherwise, we can get a system where the cancelation rule $[(y, a|a, x)] = [(x|y)]$ plays a non-trivial role, and where we can get a negative energy. It leads us to tachyonic particles that cannot exist because they are inconsistent with the known laws of physics. But such an approach can be interesting for tachyon condensation (for details on tachyon condensation see [37]).

The fact that the energy on a level can not be negative suggests using elements in \mathcal{M}_0 which have very specific form, $[z] = [(-y_1, \dots, -y_m|x_1, \dots, x_n)]$, where all x_i and y_j are non-negative. The subset of such elements forms a semiring without divisions of zero, denoted by \mathcal{M}_0^\pm . We considered algebraic properties of this semiring and its completions \mathcal{M}_X^\pm with respect to various metrics associated with different Banach spaces X . Also, we introduced new operations on $\mathcal{M}_X^\oplus = \mathcal{M}_X^\pm \cup \epsilon$ that lead to an infinite-dimensional analog of the so-called tropical semirings. We proved the continuity of the operations on \mathcal{M}_X^\oplus and constructed some real-valued homomorphisms of \mathcal{M}_X^\oplus .

For further investigation we are going to use block-symmetric (or MacMahon) and block-supersymmetric polynomials on $\ell_1(\mathbb{C}^s)$ and their applications to partition functions of quantum gases. The space $\ell_1(\mathbb{C}^s)$ can be defined as a vector space of sequences

$$\mathbf{x} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}, \dots)$$

such that every element $\mathbf{x}^{(n)} = (x_1^{(n)}, \dots, x_s^{(n)})$ is a vector in \mathbb{C}^s , and

$$\|\mathbf{x}\| = \sum_{n=1}^{\infty} \|\mathbf{x}^{(n)}\|.$$

A polynomial is block-symmetric on $\ell_1(\mathbb{C}^s)$ if it is symmetric with respect to all permutations of the vectors (blocks) $\mathbf{x}^{(n)}$. We can expect that models based on block-symmetric (and maybe

block-supersymmetric) polynomials can be useful for describing quantum gases with entanglement particles.

Combinatorial properties of block-symmetric polynomials were considered in [38]. Algebras of block-symmetric polynomials and analytic functions and corresponding bases of polynomials on $\ell_1(\mathbb{C}^s)$ were studied in [39–44]. Applications of block-symmetric polynomials for the quantum product of symmetric functions were proposed in [45].

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References

1. Macdonald, I. G. *Symmetric Functions and Orthogonal Polynomials*; AMS: University Lecture Serie 12, Providence, RI, 1997.
2. Schmidt, H. J.; Schnack, J. Symmetric polynomials in physics. In: Gazeau, J.-P., Kerner, R., Antoine, J.-P., Métens, S., Thibon, J.-Y. *Inst. Phys. Conf. Ser.* **2003**, *173*, IOP: Bristol, Philadelphia, 147–152.
3. Schmidt, H.-J.; Schnack, J. Partition functions and symmetric polynomials. *American Journal of Physics* **2002**, *70(1)*, 53–57. doi:10.1119/1.1412643
4. Giraud, O.; Grabsch, A.; Texier, C. Correlations of occupation numbers in the canonical ensemble and application to a Bose-Einstein condensate in a one-dimensional harmonic trap. *Phys. Rev. A* **2018**, *97*, 053615. <https://link.aps.org/doi/10.1103/PhysRevA.97.053615>.
5. Mullin, W. J.; Fernández, J. P. Bose-Einstein condensation, fluctuations, and recurrence relations in statistical mechanics. *Am. J. Phys.* **2002**, *71*, 661–669. <https://doi.org/10.1119/1.1544520>.
6. Pain, J.-C.; Gilleron, F.; Porcherot, Q. Generating functions for canonical systems of fermions. *Phys. Rev. E* **2011**, *83*, 067701. <https://doi.org/10.1103/PhysRevE.83.067701>.
7. Peña, J. J.; Ponce, A. R.; Morales, J. On the generalization of statistical thermodynamic functions by a Riccati differential equation. *J. Phys.: Conf. Ser.* **2016**, *738*, 012095. <https://doi.org/10.1088/1742-6596/738/1/012095>.
8. Zhou, C.-C.; Dai, W.-S. A statistical mechanical approach to restricted integer partition functions. *Journal of Statistical Mechanics: Theory and Experiment* **2018**, *2018*, 5053111. <https://doi.org/10.1088/1742-5468/aabfc9>.
9. Alencar, R.; Aron, R.; Galindo, P.; Zagorodnyuk, A. Algebra of symmetric holomorphic functions on ℓ_p . *Bull. Lond. Math. Soc.* **2003**, *35*, 55–64. <https://doi.org/10.1112/S0024609302001431>.
10. Aron, R.; Galindo, P.; Pinasco, D.; Zalduendo, I. Group-symmetric holomorphic functions on a Banach space. *Bull. Lond. Math. Soc.* **2016**, *48*, 779–796. <https://doi.org/10.1112/blms/bdw043>.
11. Chernega, I.; Galindo, P.; Zagorodnyuk, A. Some algebras of symmetric analytic functions and their spectra. *Proc. Edinb. Math. Soc.* **2012**, *55*, 125–142. <https://doi.org/10.1017/S0013091509001655>.
12. García, D.; Maestre, M.; Zalduendo, I. The spectra of algebras of group-symmetric functions. *Proc. Edinb. Math. Soc.* **2019**, *62(3)*, 609–623. <https://doi.org/10.1017/S0013091518000603>.
13. González, M.; Gonzalo, R.; Jaramillo, J.A. Symmetric polynomials on rearrangement-invariant function spaces. *J. Lond. Math. Soc.* **1999**, *59*, 681–697. <https://doi.org/10.1112/S0024610799007164>.
14. Falcó, J.; García, D.; Jung, M.; Maestre, M. Group-invariant separating polynomials on a Banach space. *Publ. Mat.* **2022**, *66*, 207–233. <https://doi.org/10.5565/PUBLMAT6612209>.
15. Chernega, I.; Zagorodnyuk, A. Supersymmetric Polynomials and a Ring of Multisets of a Banach Algebra. *Axioms* **2022**, *11*, 511. <https://doi.org/10.3390/axioms11100511>.

16. Chopyuk, Y.; Vasylyshyn, T.; Zagorodnyuk, A. Rings of Multisets and Integer Multinumerals. *Mathematics* **2022**, *10*, 778. <https://doi.org/10.3390/math10050778>
17. Jawad, F.; Zagorodnyuk, A. Supersymmetric polynomials on the space of absolutely convergent series. *Symmetry* **2019**, *11*(9), 1111. <https://doi.org/10.3390/sym11091111>
18. Olshanski, G.; Regev, A.; Vershik, A.; Ivanov, V. Frobenius-Schur Functions. In *Studies in Memory of Issai Schur. Progress in Mathematics*; Joseph, A., Melnikov, A., Rentschler, R., Eds.; Birkhauser: Boston, MA, USA, 2003; Volume 210, pp. 251–299.
19. Sergeev, A.N. On rings of supersymmetric polynomials. *J. Algebra* **2019**, *517*, 336–364. <https://doi.org/10.1016/j.jalgebra.2018.10.003>.
20. Stembridge, J.R. A characterization of supersymmetric polynomials. *J. Algebra* **1985**, *95*, 439–444. [https://doi.org/10.1016/0021-8693\(85\)90115-2](https://doi.org/10.1016/0021-8693(85)90115-2).
21. Martsinkiv, M.; Vasylyshyn, S.; Vasylyshyn, T.; Zagorodnyuk, A. Lipschitz symmetric functions on Banach spaces with symmetric bases. *Carpathian Math. Publ.* **2021**, *13*, 727–733. <https://doi.org/10.15330/cmp.13.3.727-733>.
22. Dineen, S. *Complex Analysis on Infinite Dimensional Spaces*; Springer: Berlin/Heidelberg, Germany, 1999. <https://doi.org/10.1007/978-1-4471-0869-6>.
23. Mujica, J. *Complex Analysis in Banach Spaces*; North-Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK, 1986.
24. Golan, J. S. *Semirings and their Applications*; Springer Science and Business Media, 2013. <https://doi.org/10.1007/978-94-015-9333-5>.
25. Speyer, D.E.; Sturmfels, B. Tropical mathematics. *Math. Mag.* **2009**, *82*, 163–173. <https://doi.org/10.1080/0025570X.2009.11953615>.
26. Chernega, I.; Galindo, P.; Zagorodnyuk, A. The convolution operation on the spectra of algebras of symmetric analytic functions. *J. Math. Anal. Appl.* **2012**, *395*, 569–577. <https://doi.org/10.1016/j.jmaa.2012.04.087>.
27. Balantekin, A.B. Partition functions in statistical mechanics, symmetric functions, and group representation. *Physical Review E* **2001**, *64*(6 II), 066105/1-066105/8.
28. Landsberg, P. T. *Thermodynamics with quantum statistical illustrations*, Interscience Publishers, New York, London (1961).
29. Nemirovskii, A.; Semenov, S. On polynomial approximation of functions on Hilbert space. *Mat. USSR-Sb.* **1973**, *21*, 255–277. <https://doi.org/10.1070/SM1973v021n02ABEH002016>.
30. Burtnyak, I.V.; Chopyuk, Yu.Yu.; Vasylyshyn, S.I.; Vasylyshyn T.V. Algebras of weakly symmetric functions on spaces of Lebesgue measurable functions. *Carpathian Math. Publ.* **2023**, *15*, 411–419. <https://doi.org/10.15330/cmp.15.2.411-419>.
31. Galindo, P.; Vasylyshyn, T.; Zagorodnyuk, A. Analytic structure on the spectrum of the algebra of symmetric analytic functions on L_∞ . *RACSAM* **2020**, *114*, 56. <https://doi.org/10.1007/s13398-020-00791-w>.
32. Burtnyak, I.; Chernega, I.; Hladkyi, V.; Labachuk, O.; Novosad, Z. Application of symmetric analytic functions to spectra of linear operators. *Carpathian Math. Publ.* **2021**, *13*(3), 701–710. <https://doi.org/10.15330/cmp.13.3.701-710>.
33. Li, H.-D.; Li, S.-L.; Chen, Y.-J. Li, W.-D.; Dai, W.-S. Energy spectrum of interacting gas: cluster expansion method. *Chem. Phys.* **2022**, *559*, 111537. <https://doi.org/10.1016/j.chemphys.2022.111537>.
34. Litvinov, G. L. Maslov dequantization, idempotent and tropical mathematics: A brief introduction. *J. Math. Sci.* **2007**, *140*, 426–444. <https://doi.org/10.1007/s10958-007-0450-5>.
35. Lindenstrauss, J.; Tzafriri L. *Classical Banach Spaces, vol. I, Sequence Spaces*, Springer-Verlag, 1977.
36. Bingham, N.H.; Ostaszewski, A.J. Normed versus topological groups: dichotomy and duality. *Diss. Math.* **2010**, *472*, 138. <https://doi.org/10.4064/dm472-0-1>.
37. Erler, T.; Schnabl, M. A simple analytic solution for tachyon condensation. *Journal of High Energy Physics* **2009**, *10*, 066. <https://doi.org/10.1088/1126-6708/2009/10/066>
38. Rosas, M. MacMahon symmetric functions, the partition lattice, and Young subgroups. *J. Combin. Theory Ser. A* **2001**, *96* 326–340. <https://doi.org/10.1006/jcta.2001.3186>.

39. Bandura, A.; Kravtsiv, V.; Vasylyshyn, T. Algebraic basis of the algebra of all symmetric continuous polynomials on the Cartesian product of ℓ_p -Spaces. *Axioms* **2022**, *11*, 41. <https://doi.org/10.3390/axioms11020041>.
40. Kravtsiv, V. Algebraic basis of the algebra of block-symmetric polynomials on $\ell_1 \oplus \ell_\infty$. *Carpathian Math. Publ.* **2019**, *11*, 89–95. <https://doi.org/10.15330/cmp.11.1.89-95>.
41. Kravtsiv, V.V. Analogues of the Newton formulas for the block-symmetric polynomials. *Carpathian Math. Publ.* **2020**, *12*, 17–22. <https://doi.org/10.15330/cmp.12.1.17-22>.
42. Kravtsiv, V. Zeros of block-symmetric polynomials on Banach spaces. *Mat. Stud.* **2020**, *53*, 2016–2011. <https://doi.org/10.30970/ms.53.2.206-211>.
43. Kravtsiv, V.; Vitrykus, D. Generating elements of the algebra of block-symmetric polynomials on the product of Banach spaces \mathbb{C}^s . *AIP Conference Proceedings* **2022**, *2483*, 030010. <https://doi.org/10.1063/5.0115680>.
44. Kravtsiv, V. The analogue of Newton's formula for block-symmetric polynomials. *Int. J. Math. Anal.* **2016**, *10*, 323–327. <http://dx.doi.org/10.12988/ijma.2016.617>
45. Diaz, R.; Pariguan, E. Quantum product of symmetric functions. *Int. J. Math. Math. Sci.* **2015**, 476926 (2015). <https://doi.org/10.1155/2015/476926>.

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