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About Unitary States and Commutative Gates in a Two-Qubit Quantum System

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† Dedicated to our friend, the founder of four-dimensional mathematics Abenov Maksut Mnaidarovych

Abstract: This work is devoted to the construction and study of commutative gates for a two-qubit quantum system. Using four-dimensional algebra developed by the Kazakh mathematician Abenov M.M. all groups of commutative gates have been constructed, and among all states of a two-qubit quantum system, unitary states with which a specific gate is connected have been identified. An explicit type of gate is described that transfers a quantum system from one unitary state to another unitary state. The proposed approach opens up new possibilities for the design of quantum algorithms not only for two-qubit quantum systems, but also for n-qubit quantum systems.

Keywords: quantum computing; quantum algorithm; gate; unitary operator; four-dimensional mathematics; abelian group

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1. Introduction

Quantum computation, in a more strict fundamental understanding, is the movement of a point in a 2n-dimensional complex space, where the initial and final positions of the point correspond to the input and output of the computation.

Therefore, the process of quantum computation is nothing more than a linear transformation of a 2^n - dimensional vector from a state $\mid \psi >$ to $\mid \phi >$.

Thus, the coordinates of vector $\mid \psi >$ are the initial conditions, and the coordinates of vector $\mid \phi >$ are the result of the computation - the output.

An ideal quantum computation is then one that performs a direct linear transformation from $\mid \psi >$ to $\mid \phi >$ in a single step - a singular transform, a single computation execution.

Ideal here means that there is no more efficient, in terms of computational power, computational expense of the quantum computer's work.

To make ideal quantum computation technically possible, it is necessary to find a group of unitary matrices, the elements of which form a universal set of quantum gates for the direct transformation from $\mid \psi >$ to $\mid \phi >$. As is known, quantum computations use quantum bits or qubits instead of classical bits, which have two basis states $\mid 0 >$ и $\mid 1 >$. All other states of a qubit are defined as a linear combination of basis states with complex coefficients, that is

$$\mid \psi >= \lambda_1 \mid 0 > +\lambda_2 \mid 1 >, \tag{1}$$

where $\lambda_1 \in C, \lambda_2 \in C$ and $\mid \lambda_1 \mid^2 + \mid \lambda_2 \mid^2 = 1, C$ - being the space of complex numbers.

The basis states of a qubit $\mid 0 >$ and $\mid 1 >$ are also denoted using vectors $(1,0)^T$ and $(0,1)^T$ respectively, where the index T denotes the transposition sign.

A two-qubit quantum system consists of two qubits and has four basis states $\mid 00 >= (1,0,0,0)^T, \mid 01 >= (0,1,0,0)^T, \mid 10 >= (0,0,1,0)^T$ and $\mid 11 >= (0,0,0,1)^T$. Then, an arbitrary state of a two-qubit quantum system can be written as [7,8]

$$\mid \psi >= \lambda_1 \mid 00 > +\lambda_2 \mid 01 > +\lambda_3 \mid 10 > +\lambda_4 \mid 11 >, \tag{2}$$

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where $\lambda_i \in C, i = 1,2,3,4$, and $\mid \lambda_1 \mid^2 + \mid \lambda_2 \mid^2 + \mid \lambda_3 \mid^2 + \mid \lambda_4 \mid^2 = 1$. A similar record for an arbitrary state of a two-qubit system looks like [7,8]

$$\mid \psi >= \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}, \quad (3)$$

i.e., any quantum state of a two-qubit quantum system is uniquely determined by the complex amplitudes $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Similarly, n-qubit quantum systems, consisting of n qubits, are defined. Such a system would have 2^n basis states and any of its states are determined as a linear combination of basis states with complex amplitudes. In this work, we only consider two-qubit quantum systems, although all results can in principle be generalized for an n-qubit quantum system. However, this would require the development of the basics of 2^n-dimensional mathematics with commutative multiplication. Here, we rely on four-dimensional mathematics, the foundations of which were laid by the great Kazakh mathematician Abenov M.M. [1].

Quantum computations consist in the sequential application of unitary U operators to the quantum state of the quantum system, which are called gates. Unitary operators or gates applied to an n-qubit quantum system are represented in the form of a matrix of size $2^n \times 2^n$. For example, unitary operators for a two-qubit system have the form of a 4×4 matrix U:

$$U = \begin{pmatrix} u_{11} + v_{11}i & u_{12} + v_{12}i & u_{13} + v_{13}i & u_{14} + v_{14}i \\ u_{21} + v_{21}i & u_{22} + v_{22}i & u_{23} + v_{23}i & u_{24} + v_{24}i \\ u_{31} + v_{31}i & u_{32} + v_{32}i & u_{33} + v_{33}i & u_{34} + v_{34}i \\ u_{41} + v_{41}i & u_{42} + v_{42}i & u_{43} + v_{43}i & u_{44} + v_{44}i \end{pmatrix}, \quad (4)$$

$$u_{ij}, v_{ij} \in R, i, j = 1,2,3,4.$$

Note that if a certain matrix (4) is a two-qubit gate, then the Hermitian conjugate matrix to it

$$U^* = \begin{pmatrix} u_{11} - v_{11}i & u_{21} - v_{21}i & u_{31} - v_{31}i & u_{41} - v_{41}i \\ u_{12} - v_{12}i & u_{22} - v_{22}i & u_{32} - v_{32}i & u_{42} - v_{42}i \\ u_{13} - v_{13}i & u_{23} - v_{23}i & u_{33} - v_{33}i & u_{43} - v_{43}i \\ u_{14} - v_{14}i & u_{24} - v_{24}i & u_{34} - v_{34}i & u_{44} - v_{44}i \end{pmatrix} \quad (5)$$

is also a gate.

The main two-qubit gates, or binary operators, are SWAP, CNOT, CZ, represented by matrices [4,8]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (6)$$

Note that two-qubit gates can be applied to any n-qubit quantum system ($n \geq 2$), with any two qubits from the n-qubit system selected to which the two-qubit gate is applied, and the identity operator, which is defined by the unit matrix of the required dimension, is applied to the other qubits.

In addition to the gates indicated in (6), one can construct many different two-qubit gates, but a general description of all possible two-qubit gates is absent. In the work [3], with the participation of one of the authors, an Abelian group of commutating two-qubit gates was constructed. However, the constructed group of gates is not complete, as none of the gates (6) belongs to the mentioned group. This work is a continuation of [3], and here we have constructed all other commutative groups of two-qubit gates, as well as defined the correspondence between quantum states and gates of two-qubit quantum systems, introduced the concept of a unitary state and shown the possibility of transitioning in one step from any unitary state to any basis state, and also from any unitary state to any other unitary state. The obtained results open up new possibilities for constructing quantum algorithms not only for two-qubit quantum systems.

2. Four-Dimensional Number Spaces with Commutative Multiplication

In the study [2], all spaces of four-dimensional numbers R^4 with commutative multiplication are examined. There are, in total, six such spaces, denoted as $M_2, M_3, M_4, M_5, M_6, M_7$. To each four-dimensional number $Z = (z_1, z_2, z_3, z_4) \in R^4$ from any of these spaces, a certain 4×4 matrix M, is associated, with its elements being the components of the four-dimensional number Z, and this mapping is bijective. Moreover, this bijection is a homomorphism with respect to the multiplication operation of four-dimensional numbers, meaning the group of matrices forms a commutative group with identity. The results obtained in work [2] can be transferred to the case of four-dimensional numbers $Z \in C^4$. Retaining the same designations for the spaces $M_j (j = 2, 3, ..., 7)$ describe the necessary properties of these spaces for the case of complex-valued four-dimensional numbers. As we will see below, in each of the spaces $M_2, ..., M_7$, there exist two groups of matrices corresponding to one operation of commutative multiplication of four-dimensional numbers [2]. The operations of addition $X + Y$ and subtraction $X - Y$ of four-dimensional numbers $X \in C^4$ and $Y \in C^4$ are defined as component-wise addition and subtraction. The multiplication operation of four-dimensional numbers can be defined in various ways, among which we are only interested in commutative multiplication. All ways of defining commutative multiplication are given in [2], where for each method, the corres-ponding space of four-dimensional numbers $M_j (j = 2, 3, ..., 7)$ is defined. Without going into details, we will go through these spaces and generalize the results needed for our purposes to the case of complex-valued four-dimensional numbers.

Let's consider the space M_2, in which the multiplication of numbers $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ is defined as follows:

$$\begin{aligned} z_1 &= x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 - x_4y_3 - x_3y_4 \\ z_3 &= x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 &= x_4y_1 + x_3y_2 + x_2y_3 + x_1y_4 \end{aligned}$$

where $Z(z_1, z_2, z_3, z_4) = X \cdot Y$. If we set $x_j = a_j + b_j i$, $y_j = c_j + d_j i$, $j = 1, 2, 3, 4$, where i is the imaginary unit, then this multiplication can be rewritten as

$$\begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ a_3 + b_3 i \\ a_4 + b_4 i \end{pmatrix} \cdot \begin{pmatrix} c_1 + d_1 i \\ c_2 + d_2 i \\ c_3 + d_3 i \\ c_4 + d_4 i \end{pmatrix} = \begin{pmatrix} e_1 + g_1 i \\ e_2 + g_2 i \\ e_3 + g_3 i \\ e_4 + g_4 i \end{pmatrix} \tag{7}$$

where

$$\begin{aligned} e_1 &= a_1c_1 + a_2c_2 - a_3c_3 - a_4c_4 - b_1d_1 - b_2d_2 + b_3d_3 + b_4d_4, \\ g_1 &= a_1d_1 + a_2d_2 - a_3d_3 - a_4d_4 + b_1c_1 + b_2c_2 - b_3c_3 - b_4c_4, \\ e_2 &= a_1c_2 + a_2c_1 - a_3c_4 - a_4c_3 - b_1d_2 - b_2d_1 + b_3d_4 + b_4d_3, \\ g_2 &= a_1d_2 + a_2d_1 - a_3d_4 - a_4d_3 + b_1c_2 + b_2c_1 - b_3c_4 - b_4c_3, \\ e_3 &= a_3c_1 + a_4c_2 - a_1c_3 - a_2c_4 - b_3d_1 - b_4d_2 - b_1d_3 - b_2d_4, \\ g_3 &= a_3d_1 + a_4d_2 + a_1d_3 + a_2d_4 + b_3c_1 + b_4c_2 + b_1c_3 + b_2c_4, \\ e_4 &= a_4c_1 + a_3c_2 + a_2c_3 + a_1c_4 - b_4d_1 - b_3d_2 - b_2d_3 - b_1d_4, \\ g_4 &= a_4d_1 + a_3d_2 + a_2d_3 + a_1d_4 + b_4c_1 + b_3c_2 + b_2c_3 + b_1c_4. \end{aligned}$$

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One can readily verify that the multiplication of four-dimensional numbers defined in this way is commutative. To the four-dimensional number $X = (x_1, x_2, x_3, x_4) \in C^4$ we associate the matrix

$$M_{20}(X) = \begin{pmatrix} x_1 & x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}. \tag{8}$$

The mapping $F : X \to M_{20}(X)$ is bijective and onto. Indeed, two different numbers X and Y correspond to different matrices, and for any matrix in the form of (8), a corresponding four-dimensional number from C^4 can be found.

Theorem 1. The set of all matrices in the form of (8) is closed with respect to the operations of matrix addition, subtraction, multiplication, and multiplication by scalar. For the mapping $F : X \to M_{20}(X)$ the relationships $F(X \pm Y) = F(X) \pm F(Y)$, $F(XY) = F(X)F(Y)$ hold for any $X \in C^4, Y \in C^4$.

The proof is conducted by direct verification.

Thus, there is a bijection between the space of four-dimensional numbers and the space of matrices of the form (8), which preserves arithmetic operations, meaning the existing bijection is a homomorphism. From Theorem 1, it also follows that the operation of matrix multiplication of the form (8) is commutative.

It is further noted that if we multiply the j-th row and j-th column of matrix (8) by -1, we obtain another matrix with the same properties as the matrix M_{20}, that is, the statements of Theorem 1 remain valid. Moreover, if we multiply the j-th row and three columns of matrix M_20, with indexes not equal to j, by -1, we also get a matrix corresponding to the multiplication of four-dimensional numbers (7) and possessing the properties of matrix M_20. The matrix transposed to M_20 also possesses all the properties of matrix M_20. To describe such operations, let us denote by $M_{20}^{(j,k)}$, where j and k are one, two, or three indices with values from 1 to 4, the matrix obtained by multiplying by -1 the rows with numbers from index j and columns with numbers from index k. For example, $M_{20}^{(24,134)}$ is a matrix obtained from matrix M_20 by multiplying the second and fourth rows by -1, and also by multiplying the first, third, and fourth columns by -1. Let's describe all possible operations that lead to matrices for which the statements of Theorem 1 are valid. It is easy to verify that such operations are operations of the following types: $M_{20}^{(j,j)}$, $M_{20}^{(j,klm)}$, $M_{20}^{(klm,j)}$, $M_{20}^{(klm,klm)}$, $M_{20}^{(jk,jk)}$, $M_{20}^{(jk,lm)}$, where j,k,l,m are pairwise distinct indices with values from 1 to 4. In addition, the operation of transposing a matrix also does not change its properties.

The number of different operations of the form $M_{20}^{(j,j)}$ is four when $j = 1,2,3,4$, respectively, we get four new matrices:

$$M_{20}^{(1,1)} = \begin{pmatrix} x_1 & -x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & -x_3 \\ -x_3 & x_4 & x_1 & x_2 \\ -x_4 & x_3 & x_2 & x_1 \end{pmatrix} M_{20}^{(2,2)} = \begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ -x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix},$$

$$M_{20}^{(3,3)} = \begin{pmatrix} x_1 & x_2 & x_3 & -x_4 \\ x_2 & x_1 & x_4 & -x_3 \\ -x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & -x_2 & x_1 \end{pmatrix} M_{20}^{(4,4)} = \begin{pmatrix} x_1 & x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}.$$

But as we can easily notice,

$$\begin{aligned} M_{20}^{(1,1)}(x_1, x_2, x_3, x_4) &= M_{20}^{(2,2)}(x_1, x_2, -x_3, -x_4), \\ M_{20}^{(1,1)}(x_1, x_2, x_3, x_4) &= M_{20}^{(3,3)}(x_1, -x_2, x_3, -x_4), \\ M_{20}^{(1,1)}(x_1, x_2, x_3, x_4) &= M_{20}^{(4,4)}(x_1, -x_2, -x_3, x_4), \end{aligned}$$

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that is, these matrices lie in one group. Similarly, it can be shown that the matrices from the groups $M_{20}^{(j,klm)}$, $M_{20}^{(klm,j)}$, $M_{20}^{(klm,klm)}$, totaling 12 (4 in each group), lie in the same group. In addition, we include in this group all transposed matrices of this group, as it is easy to check that they also lie in this group. And the matrices from the groups $M_{20}^{(jk,jk)}$, $M_{20}^{(jk,lm)}$, totaling 12 (6 in each group) and the transposed matrices to them, also lie in one group, but different from the first group. For example, the matrices $M_{20}^{(12,12)}$ and $M_{20}^{(12,34)}$ respectively have the form:

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ -x_3 & -x_4 & x_1 & x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{pmatrix}, \begin{pmatrix} -x_1 & -x_2 & -x_3 & -x_4 \\ -x_2 & -x_1 & -x_4 & -x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & x_3 & -x_2 & -x_1 \end{pmatrix},$$

from which it follows that $M_{20}^{(12,12)}(x_1, x_2, x_3, x_4) = -M_{20}^{(12,34)}(x_1, x_2, x_3, x_4)$, but from $M_{20}^{(12,12)}$ it is impossible to obtain $M_{20}^{(1,1)}$ or another matrix from the first group. Thus, there are two groups of matrices that are closed with respect to the operations of addition, multiplication, and these operations are commutative. Any matrix from the corresponding group can be taken as a representative of these groups. As a representative of the second group, we take the matrix M_{20}, and as a representative of the first group, we take, for example, the matrix $M_{20}^{(1,1)}$, which we denote by M_{21}:

$$M_{21}(X) = \begin{pmatrix} x_1 & -x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & -x_3 \\ -x_3 & x_4 & x_1 & x_2 \\ -x_4 & x_3 & x_2 & x_1 \end{pmatrix} \tag{9}$$

Thus, to each four-dimensional number $X \in C^4$ two matrices M_{20} and M_{21} can be associated, that is, to define two mappings, $F_{20} : X \to M_{20}(X)$ and $F_{21} : X \to M_{21}(X)$, which are bijective and onto. The products of matrices from one class are closed with respect to the operations of addition and multiplication, and the multiplication operation corresponds to the multiplication of four-dimensional numbers (7).

Note. Other transformations $M_{20}^{(j,k)}$ of the matrix M_{20} can be considered and used to build unitary operators.

Now, let us consider the space M_3, where the multiplication operation of four-dimensional numbers $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ is defined as follows [2]:

$$\begin{aligned} z_1 &= x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 &= x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4 \\ z_4 &= x_4y_1 + x_3y_2 + x_2y_3 + x_1y_4 \end{aligned}$$

where $Z = (z_1, z_2, z_3, z_4) = X \cdot Y$. A detailed exposition of the algebra and analysis over the four-dimensional space of real numbers M_3 is presented in monograph [1]. To the four-dimensional number $X = (x_1, x_2, x_3, x_4) \in C^4$ we associate the matrix

$$M_{30}(X) = \begin{pmatrix} x_1 & -x_2 & x_3 & -x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}. \tag{10}$$

The mapping $F_{30} : X \to M_{30}(X)$ is bijective and onto. Indeed, two different numbers X and Y correspond to different matrices, and for any matrix of form (10), one can find the corresponding four-dimensional number from C^4. For the matrix M_{30} and the mapping F_{30} , the statements of

Theorem 1 hold true. Similarly to the previous case, by considering transformations of the form $M_{30}^{(j,j)}$, $M_{30}^{(j,klm)}$, $M_{30}^{(klm,j)}$, $M_{30}^{(klm,klm)}$, $M_{30}^{(jk,jk)}$, $M_{30}^{(jk,lm)}$, where j, k, l, m are pairwise distinct indices with values from 1 to 4, we find that there exist two groups of matrices corresponding to the commutative multipli-cation in the space M_3 and satisfying the conditions of Theorem 1. One group is represented by the matrix (10), and the other group by the following matrix $M_{31}(X)$:

$$M_{31}(X) = \begin{pmatrix} x_1 & x_2 & -x_3 & x_4 \\ -x_2 & x_1 & x_4 & x_3 \\ -x_3 & -x_4 & x_1 & -x_2 \\ -x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

Similarly, for the space M_4, with commutative multiplication [2]

$$\begin{aligned} z_1 &= x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 &= x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 &= x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{aligned}$$

we obtain two groups of matrices $M_{40}(X)$ and $M_{41}(X)$:

$$M_{40}(X) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}, \tag{11}$$

$$M_{41}(X) = \begin{pmatrix} x_1 & x_2 & x_3 & -x_4 \\ -x_2 & x_1 & x_4 & x_3 \\ -x_3 & x_4 & x_1 & x_2 \\ -x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}.$$

Proceeding with similar reasoning and corresponding calculations for spaces $M_5, M_6, M_7,$, the multiplication operations of which are defined respectively as [2]

$$\begin{aligned} z_1 &= x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 - x_4y_3 - x_3y_4 \\ z_3 &= x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4 \\ z_4 &= x_4y_1 + x_3y_2 + x_2y_3 + x_1y_4 \end{aligned},$$

$$\begin{aligned} z_1 &= x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 - x_4y_3 - x_3y_4 \\ z_3 &= x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 &= x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{aligned},$$

$$\begin{aligned} z_1 &= x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4 \\ z_2 &= x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 &= x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4 \\ z_4 &= x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{aligned},$$

we determine the corresponding groups of matrices satisfying the conditions of Theorem 1 for the indicated commutative multiplication operations:

$$M_{50}(X) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}, \tag{12}$$

$$M_{51}(X) = \begin{pmatrix} x_1 & x_2 & x_3 & -x_4 \\ -x_2 & x_1 & -x_4 & -x_3 \\ -x_3 & -x_4 & x_1 & -x_2 \\ -x_4 & x_3 & x_2 & x_1 \end{pmatrix},$$

for the space M_5,

$$M_{60}(X) = \begin{pmatrix} x_1 & -x_2 & x_3 & -x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}, \tag{13}$$

$$M_{61}(X) = \begin{pmatrix} x_1 & x_2 & -x_3 & x_4 \\ -x_2 & x_1 & -x_4 & -x_3 \\ -x_3 & x_4 & x_1 & x_2 \\ -x_4 & -x_3 & -x_2 & x_1 \end{pmatrix},$$

for the space M_6,

$$M_{70}(X) = \begin{pmatrix} x_1 & x_2 & -x_3 & -x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}, \tag{14}$$

$$M_{71}(X) = \begin{pmatrix} x_1 & -x_2 & x_3 & x_4 \\ -x_2 & x_1 & x_4 & x_3 \\ -x_3 & -x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & -x_2 & x_1 \end{pmatrix},$$

for the space M_7. Note that the foundations of the four-dimensional space of real numbers M5 can be found in works [5,6].

A careful examination of the obtained matrices reveals that

$$\begin{aligned} M_{20}(x_1,x_2,x_3,x_4) &= M_{71}(x_1,-x_2,-x_3,-x_4), \\ M_{30}(x_1,x_2,x_3,x_4) &= M_{61}(x_1,-x_2,-x_3,-x_4), \\ M_{40}(x_1,x_2,x_3,x_4) &= M_{51}(x_1,-x_2,-x_3,-x_4), \\ M_{50}(x_1,x_2,x_3,x_4) &= M_{41}(x_1,-x_2,-x_3,-x_4), \\ M_{60}(x_1,x_2,x_3,x_4) &= M_{31}(x_1,-x_2,-x_3,-x_4), \\ M_{70}(x_1,x_2,x_3,x_4) &= M_{21}(x_1,-x_2,-x_3,-x_4). \end{aligned}$$

This implies that there are in fact six independent groups of matrices corresponding to the six spaces of four-dimensional numbers M_j, $(j = 2, 3, ..., 7)$. As such matrices, we shall take M_{20}, M_{30}, M_{40}, M_{50}, M_{60} and M_{70}, defined by equations (8), (10) - (14). Each of these matrices is bijective to the space of four-dimensional complex-valued numbers, closed with respect to the multiplication operation, and forms an abelian group with respect to the matrix multiplication operation. Moreover, as evident from the construction, no other abelian groups of matrices exist.

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3. Commutative groups of two-qubit gates

In the previous section, we constructed six abelian groups of matrices with elements formed from the components of a four-dimensional number. Based on these matrices, it is possible to construct gates for two-qubit quantum systems; in other words, under certain additional conditions, the constructed matrices transform into unitary operators. Let us formulate the corresponding conditions.

Theorem 2. Let $X = (x_1, x_2, x_3, x_4) \in C^4$. Let the components of the complex numbers $x_j = u_j + v_j i (j = 1, 2, 3, 4)$ satisfy the conditions:

$$\begin{array}{c} \rho_0 \equiv \sum_{j=1}^{4}(u_j^2 + v_j^2) = 1, \\ \rho_{21} \equiv u_1u_2 + u_3u_4 + v_1v_2 + v_3v_4 = 0, \\ \rho_{22} \equiv u_1v_3 + u_2v_4 - u_3v_1 - u_4v_2 = 0 \\ \rho_{23} \equiv u_1v_4 + u_2v_3 - u_3v_2 - u_4v_1 = 0. \end{array} \tag{15}$$

Then the matrix $M_{20}(X)$ is a two-qubit gate.

Proof. The system (15) is consistent and has an infinite number of solutions. Consider the Hermitian conjugate matrix M_{20}^* to the matrix M_{20}:

$$M_{20}^* = \begin{pmatrix} u_1 - v_1 i & u_2 - v_2 i & u_3 - v_3 i & u_4 - v_4 i \\ u_2 - v_2 i & u_1 - v_1 i & u_4 - v_4 i & u_3 - v_3 i \\ -u_3 + v_3 i & -u_4 + v_4 i & u_1 - v_1 i & u_2 - v_2 i \\ -u_4 + v_4 i & -u_3 + v_3 i & u_2 - v_2 i & u_1 - v_1 i \end{pmatrix},$$

and multiply it by the matrix M_{20}:

$$M_{20} \cdot M_{20}^* = \begin{pmatrix} \rho_0 & 2\rho_{21} & -2\rho_{22} i & -2\rho_{23} i \\ 2\rho_{21} & \rho_0 & -2\rho_{23} i & -2\rho_{22} i \\ 2\rho_{22} i & 2\rho_{23} i & \rho_0 & 2\rho_{21} \\ 2\rho_{23} i & 2\rho_{22} i & 2\rho_{21} & \rho_0 \end{pmatrix}.$$

Using relations (15) we obtain that $M_{20} \cdot M_{20}^* = E$, where E is the identity matrix of size 4×4, hence M_{20} is a unitary matrix.

Thus, when conditions (15) are met, the group of matrices $M_{20}(X)$ forms a commutative group of gates for a two-qubit quantum system.

Corollary 1. The group of commutative gates with real elements is of the form

$$RM_{20} = \begin{pmatrix} u_1 & u_2 & -u_3 & -u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & u_3 & u_2 & u_1 \end{pmatrix},$$

where

$$\begin{array}{c} \sum_{j=1}^{4} u_j^2 = 1, \\ u_1u_2 + u_3u_4 = 0. \end{array} \tag{16}$$

Corollary 2. The group of commutative gates with purely imaginary elements is of the form

$$IM_{20} = \begin{pmatrix} v_1 i & v_2 i & -v_3 i & -v_4 i \\ v_2 i & v_1 i & -v_4 i & -v_3 i \\ v_3 i & v_4 i & v_1 i & v_2 i \\ v_4 i & v_3 i & v_2 i & v_1 i \end{pmatrix},$$

where

$$\sum_{j=1}^{4} v_j^2 = 1,$$
$$v_1 v_2 + v_3 v_4 = 0.$$

Theorem 3. Let $X = (x_1, x_2, x_3, x_4) \in C^4$. Let the components of the complex numbers $x_j = u_j + v_j i (j = 1, 2, 3, 4)$ satisfy the conditions:

$$\begin{aligned} \rho_0 &\equiv \sum_{j=1}^{4}(u_j^2 + v_j^2) = 1, \\ \rho_{31} &\equiv u_1 u_3 + u_2 u_4 + v_1 v_3 + v_2 v_4 = 0, \\ \rho_{32} &\equiv u_1 v_2 - u_2 v_1 + u_3 v_4 - u_4 v_3 = 0 \\ \rho_{33} &\equiv u_1 v_4 - u_2 v_3 + u_3 v_2 - u_4 v_1 = 0. \end{aligned} \tag{17}$$

Then the matrix $M_{30}(X)$ is a two-qubit gate.

The proof is analogous to the proof of Theorem 2.

This theorem coincides with Theorem 1 from work [3]. Analogous to Corollary 1 and Corollary 2 to Theorem 2, we can write the forms of gates with real and imaginary elements for the matrix M_{30}.

Similarly, we consider matrices M_{40}, M_{50}, M_{60} and M_{70} and list the corres-ponding conditions for their unitarity.

Theorem 4. Let $X = (x_1, x_2, x_3, x_4) \in C^4$. Let the components of the complex numbers $x_j = u_j + v_j i (j = 1, 2, 3, 4)$ satisfy the conditions:

$$\begin{aligned} \rho_0 &\equiv \sum_{j=1}^{4}(u_j^2 + v_j^2) = 1, \\ \rho_{41} &\equiv u_1 u_4 + u_2 u_3 + v_1 v_4 + v_2 v_3 = 0, \\ \rho_{42} &\equiv u_1 v_2 - u_2 v_1 - u_3 v_4 + u_4 v_3 = 0 \\ \rho_{43} &\equiv u_1 v_3 - u_2 v_4 - u_3 v_1 + u_4 v_2 = 0. \end{aligned} \tag{18}$$

Then the matrix $M_{40}(X)$ is a two-qubit gate.

Theorem 5. Let $X = (x_1, x_2, x_3, x_4) \in C^4$. Let the components of the complex numbers $x_j = u_j + v_j i (j = 1, 2, 3, 4)$ satisfy the conditions:

$$\begin{aligned} \rho_0 &\equiv \sum_{j=1}^{4}(u_j^2 + v_j^2) = 1, \\ \rho_{51} &\equiv u_1 u_4 - u_2 u_3 + v_1 v_4 - v_2 v_3 = 0, \\ \rho_{52} &\equiv u_1 v_2 - u_2 v_1 + u_3 v_4 - u_4 v_3 = 0 \\ \rho_{53} &\equiv u_1 v_3 + u_2 v_4 - u_3 v_1 - u_4 v_2 = 0. \end{aligned} \tag{19}$$

Then the matrix $M_{50}(X)$ is a two-qubit gate.

Theorem 6. Let $X = (x_1, x_2, x_3, x_4) \in C^4$. Let the components of the complex numbers $x_j = u_j + v_j i (j = 1, 2, 3, 4)$ satisfy the conditions:

$$\begin{aligned} \rho_0 &\equiv \sum_{j=1}^{4}(u_j^2 + v_j^2) = 1, \\ \rho_{61} &\equiv u_1 u_3 - u_2 u_4 + v_1 v_3 - v_2 v_4 = 0, \\ \rho_{62} &\equiv u_1 v_2 - u_2 v_1 - u_3 v_4 + u_4 v_3 = 0 \\ \rho_{63} &\equiv u_1 v_4 + u_2 v_3 - u_3 v_2 - u_4 v_1 = 0. \end{aligned} \tag{20}$$

Then the matrix $M_{60}(X)$ is a two-qubit gate.

Theorem 7. Let $X = (x_1, x_2, x_3, x_4) \in C^4$. Let the components of the complex numbers $x_j = u_j + v_j i (j = 1, 2, 3, 4)$ satisfy the conditions:

$$\begin{aligned} \rho_0 &\equiv \sum_{j=1}^{4}(u_j^2 + v_j^2) = 1, \\ \rho_{71} &\equiv u_1 u_2 - u_3 u_4 + v_1 v_2 - v_3 v_4 = 0, \\ \rho_{72} &\equiv u_1 v_3 - u_2 v_4 - u_3 v_1 + u_4 v_2 = 0 \\ \rho_{73} &\equiv u_1 v_4 - u_2 v_3 + u_3 v_2 - u_4 v_1 = 0. \end{aligned} \tag{21}$$

Then the matrix $M_{70}(X)$ is a two-qubit gate.

Thus, we have defined 6 groups of two-qubit gates that are closed with respect to the operation of matrix multiplication and within each group, the gates commute with each other. It is clear that each group contains a continuum of gates. It is easy to see that gates from different groups are not commutative, although their product is also a gate. That is, if we apply gates from one of these groups to the state of a (two-qubit) quantum system in sequence, then we ultimately obtain a gate from the same group, in other words, any number of sequentially applied gates from one group $M_j, (j = 20, 30, 40, 50, 60, 70)$ can always be replaced by a single gate from the same group, and the order of application of these gates is not important. Since any quantum algorithm is essentially the product of a sequence of gates, it can be reduced to a sequence of applications of two-qubit gates, each taken from the different specified 6 groups (provided that all the gates of the algorithm belong to the specified groups).

Observation. We have constructed all commutative groups of two-qubit gates. If we consider other transformations of matrices M_n, with indices $(n = 20, ..., 70)$, in the form $M_n^{(j,k)}$, where j and k are one, two, or three indices with values from 1 to 4 that do not satisfy the assertions of Theorem 1, then we can also obtain groups of unitary operators that fulfill the corresponding conditions (15), (17) - (21). These gates are not commutative and are not closed with respect to multiplication. Nonetheless, investigating their properties and relationships with the commutative groups that have been constructed is a relevant task. These are the directions for future research.

In the case of an $n-$qubit quantum system, a quantum algorithm typically consists of a sequence of single-qubit (unary), two-qubit (binary), three-qubit (ternary), and other multi-qubit gates. In this context, sequentially applied two-qubit gates can be simplified or reduced due to the commutativity and closure of these operators within the described groups.

4. Unitary States of Quantum Systems

An arbitrary state of a two-qubit quantum system is specified as in expressions (2) or (3). The ultimate purpose of any quantum algorithm is to transform the state of the quantum system into a state suitable for solving the given problem. Therefore, the task of finding a gate that transitions the quantum system from one specified state to another is of significant importance. Let us consider this task for a two-qubit quantum system, which can be generalized to an n-qubit system.

Definition. The state of a two-qubit system $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ is called a unitary state if the components of complex numbers $\lambda_j = u_j + v_j i$ $(j = 1, 2, 3, 4)$ satisfy at least one of the conditions (15), (17) - (21).

Note that all basis states of quantum systems are unitary since they fulfill all the listed conditions. Furthermore, only those states, called quasi-basis states, satisfy all conditions (15), (17) - (21) simultaneously.

Definition. Quantum states of the following forms $\Lambda = (\lambda, 0, 0, 0)^T$, $\Lambda = (0, \lambda, 0, 0)^T$, $\Lambda = (0, 0, \lambda, 0)^T$, $\Lambda = (0, 0, 0, \lambda)^T$, where $\lambda \in C, \mid \lambda \mid= 1$ are called quasi-basis states.

Evidently, all basis states are quasi-basis states.

Theorem 8. The state of a two-qubit system $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ satisfies all conditions (15), (17) - (21) simultaneously if and only if Λ is a quasi-basis state.

The proof is straightforwardly achieved by concurrently solving system (15), (17) - (21).

From the previously proven Theorems 1 - 7, it follows that each unitary state of a two-qubit system corresponds to at least one unitary matrix. Moreover, if the state is a quasi-basis state, then it corresponds to no fewer than six gates. Unitary states play a crucial role in the construction of quantum algorithms since for them we can explicitly specify a gate that transitions the quantum system from one specified state to another.

Remark. When we say that each unitary state corresponds to at least one unitary matrix, not exactly one matrix, we imply that in addition to commutative unitary matrices, there exist non-commutative unitary matrices whose elements satisfy the same conditions (15), (17) - (21). Since we are only considering commutative gates here, the matrices corresponding to unitary states are only matrices $M_{20}, ..., M_{70}$ and the corresponding unitary states are denoted by $S_2(X) - S_7(X)$. That is, $S_2(X)$ - unitary states satisfying condition (15), $S_3(X)$ - unitary states satisfying condition (17), and so on, $U_7(X)$ - unitary states satisfying condition (21),

Theorem 9. Let $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ — be a unitary state of the two-qubit quantum system from S_j, where $j = 2, ...7$. Then, for any quasi-basis state B, among the unitary matrices $M_{10j}(\Lambda)$, there exists a matrix G, such that $G\Lambda = B$.

Proof. Let us prove the theorem for $j = 2$ as an example. The matrix M_{20}, corresponding to the unitary state Λ is of the form (8)

$$M_{20}(\Lambda) = \begin{pmatrix} u_1 + v_1 i & u_2 + v_2 i & -u_3 - v_3 i & -u_4 - v_4 i \\ u_2 + v_2 i & u_1 + v_1 i & -u_4 - v_4 i & -u_3 - v_3 i \\ u_3 + v_3 i & u_4 + v_4 i & u_1 + v_1 i & u_2 + v_2 i \\ u_4 + v_4 i & u_3 + v_3 i & u_2 + v_2 i & u_1 + v_1 i \end{pmatrix},$$

where $u_j + v_j i = \lambda_j$ $(j = 1, 2, 3, 4)$ satisfy conditions (15). Consequently,

$$M_{20}^{-1}(\Lambda) = \begin{pmatrix} u_1 - v_1 i & u_2 - v_2 i & u_3 - v_3 i & u_4 - v_4 i \\ u_2 - v_2 i & u_1 - v_1 i & u_4 - v_4 i & u_3 - v_3 i \\ -u_3 + v_3 i & -u_4 + v_4 i & u_1 - v_1 i & u_2 - v_2 i \\ -u_4 + v_4 i & -u_3 + v_3 i & u_2 - v_2 i & u_1 - v_1 i \end{pmatrix}.$$

Then

$$M_{20}^{-1}(\Lambda)\Lambda = \begin{pmatrix} u_1^2 + u_2^2 + u_3^2 + u_4^2 + v_1^2 + v_2^2 + v_3^2 + v_4^2 \\ 2(u_1 u_2 + v_1 v_2 + u_3 u_4 + v_3 v_4) \\ 2(u_1 v_3 - u_3 v_1 + u_2 v_4 - u_4 v_2) i \\ 2(u_1 v_4 - u_4 v_1 + u_2 v_3 - u_3 v_2) i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

by virtue of conditions (15). Similarly, it can be shown that $M_{20}^{-1}(B)B = (1, 0, 0, 0)^T$, whence $M_{20}(B)(1, 0, 0, 0)^T = B$. Now, by applying gate $M_{20}^{-1}(\Lambda)$ to state Λ first, and then gate $M_{20}(B)$ we obtain $M_{20}(B)M_{20}^{-1}(\Lambda)\Lambda = B$. That is, the gate $G = M_{20}(B)M_{20}^{-1}(\Lambda)$ transitions the state Λ into the quasi-basis state B. The proof for other values of j is conducted analogously.

The theorem is proven.

Remark. There are several gates G, the existence of which is asserted in Theorem 9. We could have transitioned to the quasi-basis state B not through the basis state $(1, 0, 0, 0)^T$ but, for example, through the basis state $(0, 1, 0, 0)^T$, and so forth.

Corollary. Let Λ_1 and Λ_2 be two unitary states from S_j. Then there exists a gate from the Abelian group M_{10j}, that transitions the state of the quantum system from Λ_1 to Λ_2.

The proof is conducted absolutely analogously to the proof of Theorem 9.

Thus, we have explicitly described a gate that translates any unitary state into any quasi-basis state, including any basis state within a specific group of unitary states $S_j(j = 2, ..., 7$ and vice versa. From Theorems 8 and 9, the following important result easily follows.

Theorem 10. Let Λ_1 and Λ_2 be two unitary states of a two-qubit quantum system. Then there exists a gate G, which translates the quantum system from state Λ_1 to state Λ_2 in one step.

Proof. The proof follows from Theorems 9 and 8. According to Theorem 9, if the state Λ_1 lies in the group S_j for some j, then there is a unitary matrix G_1 from the group M_{10j}, which translates the system into some quasi-basis state Λ. According to Theorem 8, the state Λ belongs to all groups S_j for all $j = 2, ..., 7$. Now, by Theorem 9, we can translate the system from state Λ into state Λ_2 using some matrix G_2 from the group M_{10k}, where S_k is the group in which the state Λ_2 is located. Then the gate $G_2 G_1$ translates the system from state Λ_1 into state Λ_2 in one step. Moreover, we can explicitly write the matrices G_1 and G_2.

The theorem is proven.

In this way, we have divided all possible states of a two-qubit quantum system into two classes: unitary and non-unitary. Unitary states include all quasi-basis states (and thus all basis states) and play an important role in the construction of quantum algorithms. Theorem 10 allows transitioning from any unitary state to any other unitary state in one step, and the gate of this transition can be explicitly described, and there are infinitely many such gates (since there are infinitely many quasi-basis states). If the solution to a certain quantum problem is represented as a non-unitary state, there is always a unitary state that lies as close as desired to this non-unitary state [1]. This gives us the possibility of searching for an approximate solution to the problem of finding a gate that translates the quantum system from any state to any other state.

5. Conclusion

This work has considered an important class of quantum states called unitary states and has solved the problem of finding a gate that translates a two-qubit quantum system from one unitary state to any other unitary state. If the initial and final states are not unitary, then the proposed approach can generally be used to approximate the solution to the problem of finding a gate that translates the system from one state to any other arbitrary state. However, that is the subject of future research.

The paper focuses on commutative two-qubit gates and leaves without consideration non-commutative two-qubit gates, which have the form of matrices similar to matrices M_j, where $j = 20, 30, ..., 70$. Investigating the properties of such gates and their interrelationships with commutative gates is of great practical interest for the development of quantum algorithms. Constructing all non-commutative groups of two-qubit gates makes the picture more complete.

In this context, the question of the completeness of the constructed groups (commutative and others) of two-qubit gates is interesting, namely, whether it is possible to describe the action of any unitary operator with any degree of accuracy using the constructed gates.

The results obtained can be extended to the case of n-qubit systems, but for this, it is necessary to develop an algebra of 2n-dimensional mathematics, similar to four-dimensional mathematics. However, for the practical application of such developments, it is important to have ready-made libraries, for example, in Python, that allow performing computations of multidimensional algebra.

All these issues, in our opinion, can be resolved using the apparatus of multidimensional mathematics. In general, the apparatus of multidimensional commutative algebra not only opens new possibilities for the existing mathe-matical model of quantum computing but may also give a new impetus to the development of a differential mathematical model of quantum computing, because in multidimensional mathematics with commutative multiplication, it is possible to effectively develop integral and differential calculus, as has been done for the four-dimensional case [1,2,6]. Four-dimensional mathematical analysis has allowed for new approaches to solving complex three-dimensional non-stationary problems for systems of partial differential equations [9,10].

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