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Article

Using Beta Function and Factorization to Find New π-series

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Abstract: Using the method of the beta function, Sun has recently evaluated some series of the type $\sum_{k=0}^{\infty}(ak+b)x^k/\binom{mk}{nk}$. By factorization of the polynomial $t^{m-n}(1-t)^n - x$, we will give a general method to find new π-series.

Keywords: beta function; polynomial factorization; convergence; π-series
AMS Classifications: 40G99; 05A20; 41A58

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1. Introduction

The classical rational Ramanujan-type series for π introduced by Cooper[6] have the form

$$\sum_{k=0}^{\infty} \frac{bk+c}{m^k} a(k) = \frac{\lambda\sqrt{d}}{\pi}.$$

In 1905, Glaisher[7] proved that

$$\sum_{k=0}^{\infty} \frac{(4k-1)\binom{2k}{k}^4}{(2k-1)^4 256^k} = -\frac{8}{\pi^2}.$$

Chan, Chan and Liu[4] proved that

$$\sum_{n=0}^{\infty} \frac{5n+1}{64^n} D_n = \frac{8}{\sqrt{3}\pi},$$

where D_n denotes the Domb number $\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}$.

Let $b, c \in \mathbb{Z}$, for each $n \in \mathbb{N}$, the coefficient of x^n in the expansion of $(x^2+bx+c)^n$ is denoted by $T_n(b,c)$, Sun[11] gave many conjectural series for $\frac{1}{\pi}$ containing the $T_n(b,c)$ such as

$$\sum_{k=0}^{\infty} \frac{3990k+1147}{(-288)^{3k}} T_k(62, 95^2)^3 = \frac{432}{95\pi}(94\sqrt{2}+195\sqrt{14}).$$

In[10], Sun derived several identities involving π by the telescoping method. For example, from Bauer's series[2]

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi}$$

and the telescoping sum

$$\sum_{k=0}^{n} \frac{(16k^3-4k^2-2k+1)\binom{2k}{k}^2}{(2k-1)^2(-64)^k} = \frac{8(2n+1)}{(-64)^n}\binom{2n}{n}^3,$$

he deduced

$$\sum_{k=0}^{\infty} \frac{k(4k-1)\binom{2k}{k}^3}{(2k-1)^2(-64)^k} = -\frac{1}{\pi}. \tag{1}$$

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By Gosper's algorithm[8], Hou and Li[9] give a systematic method to construct π-series of the form (1). For more results on the π-series, we refer to [3][5] .

In 1974 Gosper announced the new identity

$$\sum_{k=0}^{\infty} \frac{25k-3}{2^k\binom{3k}{k}} = \frac{\pi}{2}.$$

Motivated by Gosper's identity, Almkvist, Krattenthaler and Petersson[1] found some new identities of the type

$$\sum_{k=0}^{\infty} \frac{P(k)}{\binom{mk}{nk}x^k} = \pi,$$

where $P(k)$ is a polynomial in k. If $P(k)$ has degree d, they also get that

$$\sum_{k=0}^{\infty} \frac{P(k)}{\binom{mk}{nk}x^k} = \int_0^1 \frac{S(t,x)}{(t^n(1-t)^{m-n}-x)^{d+2}}dt, \tag{2}$$

where $S(t,x)$ is a polynomial in t of degree $m(d+1)$. They find a good way to get π is to have $arctan(t)$ or $arctan(t-1)$ after integration of (2). Then they get $t^n(1-t)^{m-n}-x$ must have the factor t^2+1 or $(t-1)^2+1$. This restricts m and n and gives the value of x.

Using the method of the beta function, Sun[12] has recently evaluated some series of the type $\sum_{k=0}^{\infty}(ak+b)x^k/\binom{mk}{nk}$. For example, he completely determined the values of

$$\sum_{k=1}^{\infty} \frac{k^r x^k}{\binom{3k}{k}} \left(-\frac{27}{4} < x < \frac{27}{4}\right)$$

for $r = 0, \pm 1$.

We first consider the convergence of the left of the equality (2), by Stirling's formula

$$k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k, \quad k \to +\infty.$$

If $m > n > 0$ are integers, then $k \to +\infty$, we have

$$1/\binom{mk}{nk} \sim \frac{\sqrt{2\pi n(m-n)k}}{\sqrt{m}}\left(\frac{n^n(m-n)^{m-n}}{m^m}\right).$$

So we get that if

$$|x| > \frac{n^n(m-n)^{m-n}}{m^m}, \tag{3}$$

the left of the equality (2) is convergent.

The Gamma function is pointed out by Euler as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad (x>0).$$

The beta function is defined as

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \quad a>0, \quad b>0.$$

The connection between Gamma function and beta function is given by Euler as

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

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Now we present an auxiliary proposition.

Proposition 1. *Let $m > n > 0$ be integers, $0 \le t \le 1$ and $|x| > n^n(m-n)^{m-n}/m^m$, then we have*

$$|\frac{t^n(1-t)^{m-n}}{x}| < 1.$$

Proof. Note that for $0 \le t \le 1$, we have

$$\sqrt[m]{\left(\frac{t}{n}\right)^n \left(\frac{1-t}{m-n}\right)^{m-n}} \le \frac{n \cdot \frac{t}{n} + (m-n) \cdot \frac{1-t}{m-n}}{m} = \frac{1}{m},$$

Hence we have

$$|\frac{t^n(1-t)^{m-n}}{x}| \le \frac{n^n(m-n)^{m-n}}{m^m}|\frac{1}{x}| < 1.$$

By Proposition (1) and beta function we get the following lemma we will use later. □

Lemma 1. *Let $m > n > 0$ be integers, and let x be real numbers with $|x| > (n^n(m-n)^{m-n})/m^m$, then we have*

$$\sum_{k=0}^{\infty} \frac{1}{x^k \binom{mk}{nk}} = \int_0^1 \frac{x[(m-1)t^{m-n}(1-t)^n + x)]}{(t^{m-n}(1-t)^n - x)^2} dt.$$

Proof. Clearly we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{x^k \binom{mk}{nk}} &= \sum_{k=0}^{\infty} \frac{(nk)!((m-n)k)!}{x^k (mk)!} \\
&= \sum_{k=0}^{\infty} \frac{(mk+1)\Gamma(nk+1)\Gamma((m-n)k+1)}{x^k \Gamma(mk+2)} \\
&= \sum_{k=0}^{\infty} \frac{mk+1}{x^k} B(nk+1, (m-n)k+1) \\
&= \sum_{k=0}^{\infty} \frac{mk+1}{x^k} \int_0^1 t^{(m-n)k}(1-t)^{nk} dt \\
&= \int_0^1 \sum_{k=0}^{\infty} (mk+1) \left(\frac{t^{m-n}(1-t)^n}{x}\right)^k dt.
\end{aligned}$$

Let

$$y = \frac{t^{m-n}(1-t)^n}{x}.$$

Then by Proposition 1 we have $|y| < 1$, then we get

$$\begin{aligned}
&\int_0^1 \sum_{k=0}^{\infty} (mk+1) y^k dt \\
&= \int_0^1 \frac{(m-1)y+1}{(1-y)^2} dt \\
&= \int_0^1 \frac{x[(m-1)t^{m-n}(1-t)^n + x)]}{(t^{m-n}(1-t)^n - x)^2} dt.
\end{aligned}$$

□

Using the same method of Lemma 1, we get more equalities we will use later.

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Lemma 2. *Let $m > n > 0$ be integers, and let x be real numbers with $|x| > (n^n(m-n)^{m-n})/m^m$, then we have*

$$\sum_{k=0}^{\infty} \frac{1}{(mk+1)x^k\binom{mk}{nk}} = \int_0^1 \frac{-x}{t^{m-n}(1-t)^n - x} dt. \tag{4}$$

$$\sum_{k=0}^{\infty} \frac{k}{x^k\binom{mk}{nk}} = \int_0^1 -\frac{t^{m-n}(1-t)^n[(m-1)t^{m-n}(1-t)^n + (m+1)x]x}{(t^{m-n}(1-t)^n - x)^3} dt. \tag{5}$$

$$\sum_{k=1}^{\infty} \frac{1}{kx^k\binom{mk}{nk}} = \int_0^1 -\frac{nt^{m-n}(1-t)^{n-1}}{[t^{m-n}(1-t)^n - x]} dt. \tag{6}$$

Proof. The proof of equality (4) and (5) is similar as the Lemma 1. To prove the equality (6), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{kx^k\binom{mk}{nk}} &= \sum_{k=1}^{\infty} \frac{1}{kx^k} \frac{(nk)!((m-n)k)!}{(mk)!} \\
&= \sum_{k=1}^{\infty} \frac{nk}{kx^k} \frac{\Gamma(nk)\Gamma((m-n)k+1)}{\Gamma(mk+1)} \\
&= \sum_{k=1}^{\infty} \frac{n}{x^k} B(nk, (m-n)k+1) \\
&= \sum_{k=1}^{\infty} n \int_0^1 \frac{t^{(m-n)k}(1-t)^{nk-1}}{x^k} dt \\
&= \int_0^1 \sum_{k=1}^{\infty} \frac{n}{1-t} \left(\frac{t^{m-n}(1-t)^n}{x} \right)^k dt.
\end{aligned}$$

Let

$$y = \frac{t^{m-n}(1-t)^n}{x}.$$

Then by Proposition 1 we have $|y| < 1$, then we get

$$\begin{aligned}
&\int_0^1 \sum_{k=1}^{\infty} \frac{n}{1-t} y^k dt \\
&= \int_0^1 \frac{ny}{(1-t)(1-y)} dt \\
&= \int_0^1 -\frac{nt^{m-n}(1-t)^{n-1}}{[t^{m-n}(1-t)^n - x]} dt.
\end{aligned}$$

By equality (2) and Lemma 1 and 2, we find the denominator of the integration contains the form $t^{m-n}(1-t)^n - x$, if we integrate directly, as x is a symbol we just don't know, in most times we will fail. □

In this paper, we replace x with a suitable rational function $x(b)$, then by Mathematica we can integrate the rational function $f(t,b)$ from 0 to 1 which the denominator contains the form $t^{m-n}(1-t)^n - x(b)$. Since after integration we get equalities with b and the variable t no longer exists. Then we combine these equalities and let b be a suitable number, this gives lights to find new π-series. we will show how to get π-series containing the type $1/\binom{2k}{k}$, $1/\binom{3k}{k}$, $1/\binom{4k}{k}$, $1/\binom{4k}{2k}$. For example, we get new π-series as follows

Example 1.

$$\sum_{k=0}^{\infty} \frac{(-3+2k)2^k}{\binom{2k}{k}} = \frac{\pi}{2}.$$

$$\sum_{k=1}^{\infty}\frac{(9+35k)8^{k-1}}{k(1+3k)3^k\binom{3k}{k}}=\frac{\sqrt{3}}{2}\pi-1.$$

$$\sum_{k=1}^{\infty}\frac{(-64+211k+1805k^2)9^k}{k(1+4k)8^k\binom{4k}{k}}=45+16\sqrt{3}\pi.$$

$$\sum_{k=1}^{\infty}\frac{3+14k}{2k(1+4k)(-3)^k\binom{4k}{2k}}=\frac{1}{6}(-6+\sqrt{3}\pi).$$

$$\sum_{k=1}^{\infty}\frac{(1+6k)2^{2k-1}}{k(1+4k)\binom{4k}{2k}}=\frac{\pi}{2}-1.$$

2. π-series containing

In this section, we will give methods to find π-series containing the type $1/\binom{2k}{k}$.

Theorem 1. *If $b\neq 0$, we have*

$$\sum_{k=0}^{\infty}\frac{1}{(b^2+1/4)^k\binom{2k}{k}}=\frac{1+4b^2}{8b^3}\left(2b+arctan\left(\frac{1}{2b}\right)\right). \tag{7}$$

If $|b|>\frac{\sqrt{2}}{2}$, we have

$$\sum_{k=0}^{\infty}\frac{1}{(1/4-b^2)^k\binom{2k}{k}}=\frac{4b^2-1}{8b^3}\left(2b-arctanh\left(\frac{1}{2b}\right)\right),$$

Proof. By Lemma 1 we have

$$\sum_{k=0}^{\infty}\frac{1}{x^k\binom{2k}{k}}=\int_0^1\frac{-x(t^2-t-x)}{(t^2-t+x)^2}dt.$$

By inequality (3) we have $|x|>\frac{1}{4}$. □

Since $t^2-t+x=(t-\frac{1}{2})^2+x-\frac{1}{4}$. If $x>\frac{1}{4}$ we let $x-\frac{1}{4}=b^2$, then we $b\neq 0$ and by Mathematica we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{x^k\binom{2k}{k}}&=\int_0^1\frac{-x(t^2-t-x)}{(t^2-t+x)^2}dt\\&=\int_0^1\frac{-(b^2+\frac{1}{4})(t^2-t-b^2-\frac{1}{4})}{((t-\frac{1}{2})^2+b^2)^2}dt\\&=\frac{1+4b^2}{8b^3}\left(2b+arctan\left(\frac{1}{2b}\right)\right).\end{aligned}$$

If $x<-\frac{1}{4}$ we let $x-\frac{1}{4}=-b^2$, then $|b|>\frac{\sqrt{2}}{2}$ and by Mathematica we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{x^k\binom{2k}{k}}&=\int_0^1\frac{-x(t^2-t-x)}{(t^2-t+x)^2}dt\\&=\int_0^1\frac{-(b^2-\frac{1}{4})(t^2-t+b^2+\frac{1}{4})}{((t-\frac{1}{2})^2-b^2)^2}dt\\&=\frac{4b^2-1}{8b^3}\left(2b-arctanh\left(\frac{1}{2b}\right)\right).\end{aligned}$$

Using the same method of Theorem 1 and by Lemma 2 we have

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Theorem 2. *If $b \neq 0$, we have*

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(b^2+1/4)^k \binom{2k}{k}} = \frac{1+4b^2}{2b} arctan\left(\frac{1}{2b}\right). \tag{8}$$

$$\sum_{k=0}^{\infty} \frac{k}{(b^2+1/4)^k \binom{2k}{k}} = \frac{(1+4b^2)[6b+(3+4b^2)arctan\left(\frac{1}{2b}\right)]}{64b^5}. \tag{9}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(b^2+1/4)^k \binom{2k}{k}} = \frac{arctan\left(\frac{1}{2b}\right)}{b}. \tag{10}$$

If $|b| > \frac{\sqrt{2}}{2}$, we have

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(1/4-b^2)^k \binom{2k}{k}} = \frac{4b^2-1}{2b} arctanh\left(\frac{1}{2b}\right).$$

$$\sum_{k=0}^{\infty} \frac{k}{(1/4-b^2)^k \binom{2k}{k}} = -\frac{(4b^2-1)[6b+(4b^2-3)arctanh\left(\frac{1}{2b}\right)]}{64b^5}.$$

$$\sum_{k=1}^{\infty} \frac{1}{k(1/4-b^2)^k \binom{2k}{k}} = -\frac{arctanh\left(\frac{1}{2b}\right)}{b}.$$

By Theorem 1 and 2 we can get propositions of equality.

Proposition 2. *If $b \neq 0$, Via $(-3) \times (7) + 8b^2 \times (9)$ we have*

$$\sum_{k=0}^{\infty} \frac{24b^2-k}{(1/4+b^2)^k \binom{2k}{k}} = \frac{1+4b^2}{2b} arctan\left(\frac{1}{2b}\right) \tag{11}$$

By equality (8) and (11) we let $b = \frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2}$, we can get series involves π, here are the examples.

Example 2.

$$\sum_{k=0}^{\infty} \frac{2^k}{(1+2k)\binom{2k}{k}} = \frac{\pi}{2}.$$

$$\sum_{k=0}^{\infty} \frac{3^k}{(1+2k)\binom{2k}{k}} = \frac{4\pi}{3\sqrt{3}}.$$

$$\sum_{k=0}^{\infty} \frac{1}{(1+2k)\binom{2k}{k}} = \frac{2\pi}{3\sqrt{3}}.$$

$$\sum_{k=0}^{\infty} \frac{(-3+2k)2^k}{\binom{2k}{k}} = \frac{\pi}{2}.$$

$$\sum_{k=0}^{\infty} \frac{(-9+2k)3^k}{\binom{2k}{k}} = \frac{4\pi}{\sqrt{3}}.$$

$$\sum_{k=0}^{\infty} \frac{(-1+2k)}{\binom{2k}{k}} = \frac{2\pi}{9\sqrt{3}}.$$

3. π-series containing $1/\binom{3k}{k}$

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In this section, we will give methods to find π-series containing the type $1/\binom{3k}{k}$. We first give an auxiliary lemma we will use.

Lemma 3. *Let $f(t) = t^3 - t^2 + x$, if $|x| > \frac{4}{27}$, let $x = -2b(1+2b)^2$, then we can factor $f(t)$ as*

$$f(t) = (t-1-2b)((t+b)^2 + b(2+3b)),$$

where $b < -\frac{2}{3}$ or $b > \frac{1}{6}\left((1+\sqrt{2})^{2/3} + \frac{1}{(1+\sqrt{2})^{2/3}}\right) - \frac{1}{3} \approx 0.059$.

Proof. Consider $f(t) = 0$, since $|x| > \frac{4}{27}$, it is easy to calculate that

$$\Delta = \frac{x(27x-4)}{108} > 0$$

Then $f(t) = 0$ has one real roots and one pair of unequal conjugate complex roots. Suppose we factor $f(t)$ as

$$f(t) = (t+a)((t+b)^2 + c),$$

then we have

$$\begin{cases} a + 2b = -1 \\ 2ab + b^2 + c = 0 \\ a(b^2 + c) = x \end{cases}$$

That is to say $a = -(1+2b)$ and $c = b(3b+2)$ and $x = -2b(1+2b)^2$. Since $|x| > \frac{4}{27}$, we have $|-2b(1+2b)^2| > \frac{4}{27}$, then we get $b < -\frac{2}{3}$ or $b > \frac{1}{6}\left((1+\sqrt{2})^{2/3} + \frac{1}{(1+\sqrt{2})^{2/3}}\right) - \frac{1}{3}$. □

Theorem 3. *Let μ be defined by*

$$\mu = \frac{1}{6}\left((1+\sqrt{2})^{2/3} + \frac{1}{(1+\sqrt{2})^{2/3}}\right) - \frac{1}{3}.$$

If we define $q(b)$ as

$$q(b) = \begin{cases} \arctan\left(\sqrt{\frac{2+3b}{b}}\frac{1}{4b+3}\right), & b < -\frac{3}{4}, \quad or \quad b > \mu \\ -\frac{\pi}{2}, & b = -\frac{3}{4} \\ \arctan\left(\sqrt{\frac{2+3b}{b}}\frac{1}{4b+3}\right) - \pi, & -\frac{3}{4} < b < -\frac{3}{2} \end{cases}$$

If $b > \mu$ or $b < -\frac{2}{3}$, we have

$$\sum_{k=0}^{\infty} \frac{1}{\binom{3k}{k}(-2b(1+2b)^2)^k} = \frac{27b(1+2b)^2}{(2+3b)(1+6b)^2} + \frac{6b(1+2b)}{(1+6b)^3}\log\left(\frac{2b}{1+2b}\right) + \frac{2(1+2b)(18b^2+6b-1)}{(2+3b)(1+6b)^3}\sqrt{\frac{b}{2+3b}}q(b), \tag{12}$$

Proof. By Lemma 1 we have

$$\sum_{k=0}^{\infty} \frac{1}{x^k\binom{3k}{k}} = \int_0^1 \frac{-x(2t^3-2t^2-x)}{(t^3-t^2+x)^2}dt.$$

By equation (3) we have $|x| > \frac{4}{27}$. □

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Let $x = -2b(1+2b)^2$, then we have $t^3 - t^2 + x = (t-(2b+1))((t+b)^2 + b(3b+2))$. If $|x| > \frac{4}{27}$, we have $b < -\frac{2}{3}$ or $b > \mu$. If $b \neq -\frac{3}{4}$, we have

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{1}{x^k \binom{3k}{k}}\\ &= \int_0^1 \frac{-x(2t^3 - 2t^2 - x)}{(t^3 - t^2 + x)^2} dt\\ &= \int_0^1 \frac{2b(1+2b)^2(2t^3 - 2t^2 + 2b(1+2b)^2)}{(t-(2b+1))^2((t+b)^2 + b(3b+2))^2} dt\end{aligned}$$

We will calculate the integral by Mathematica. Then we get

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{1}{(-2b(1+2b)^2)^k \binom{3k}{k}}\\ &= \frac{\sqrt{b}}{(2+3b)^{3/2}} \frac{1+2b}{(1+6b)^3} [27\sqrt{b}\sqrt{2+3b}(2b+1)(6b+1)\\ &+6\sqrt{b}\sqrt{2+3b}(2+3b)\log\left(\frac{2b}{1+b}\right)\\ &+2(18b^2+6b-1)\left(arctan\frac{1+b}{\sqrt{b}\sqrt{2+3b}} - arctan\left(\frac{\sqrt{b}}{\sqrt{2+3b}}\right)\right)]\end{aligned}$$

Let

$$q(b) = arctan\frac{1+b}{\sqrt{b}\sqrt{2+3b}} - arctan\left(\frac{\sqrt{b}}{\sqrt{2+3b}}\right),$$

It's easy to deduce that

$$q(b) = \begin{cases} arctan\left(\sqrt{\frac{2+3b}{b}}\frac{1}{4b+3}\right), & b < -\frac{3}{4}, \quad or \quad b > 0 \\ arctan\left(\sqrt{\frac{2+3b}{b}}\frac{1}{4b+3}\right) - \pi, & -\frac{3}{4} < b < -\frac{2}{3} \end{cases}$$

In the domain of b we have

$$\frac{\sqrt{b}}{(2+3b)^{3/2}}\sqrt{b}\sqrt{2+3b} = \frac{b}{2+3b}$$

and

$$\frac{\sqrt{b}}{\sqrt{2+3b}} = \sqrt{\frac{b}{2+3b}}.$$

If $b = -\frac{3}{4}$, the theorem also holds, so we complete the proof.

Using the same method of Theorem 3 and by Lemma 2 we have

Theorem 4. *Let μ be defined by*

$$\mu = \frac{1}{6}\left((1+\sqrt{2})^{2/3} + \frac{1}{(1+\sqrt{2})^{2/3}}\right) - \frac{1}{3}.$$

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If we define $q(b)$ as

$$q(b)=\begin{cases} arctan\left(\sqrt{\frac{2+3b}{b}}\frac{1}{4b+3}\right), & b<-\frac{3}{4}, \quad or \quad b>\mu \\ -\frac{\pi}{2}, & b=-\frac{3}{4} \\ arctan\left(\sqrt{\frac{2+3b}{b}}\frac{1}{4b+3}\right)-\pi, & -\frac{3}{4}<b<-\frac{3}{2} \end{cases}$$

If $b>\mu$ or $b<-\frac{2}{3}$, we have

$$\sum_{k=0}^{\infty}\frac{1}{(1+3k)\binom{3k}{k}(-2b(1+2b)^2)^k} = -\frac{3b(1+2b)}{1+6b}\log\left(\frac{2b}{1+2b}\right)+\frac{2(1+2b)(1+3b)}{(1+6b)}\sqrt{\frac{b}{2+3b}}q(b). \tag{13}$$

$$\sum_{k=0}^{\infty}\frac{k}{(-2b(1+2b)^2)^k\binom{3k}{k}} = -\frac{81b(1+2b)^2}{(2+3b)^2(1+6b)^4}+\frac{6b(1+2b)(-1+8b+12b^2)}{(1+6b)^5}\log\left(\frac{2b}{1+2b}\right) +\frac{2(1+2b)f(b)}{(2+3b)^2(1+6b)^5}\sqrt{\frac{b}{2+3b}}q(b). \tag{14}$$

where $f(b)=1-45b-150b^2+54b^3+648b^4+648b^5$.

$$\sum_{k=1}^{\infty}\frac{1}{k(-2b(1+2b)^2)^k\binom{3k}{k}} = \frac{1}{1+6b}\log\left(\frac{2b}{1+2b}\right)+\frac{2}{1+6b}\sqrt{\frac{b}{2+3b}}q(b). \tag{15}$$

Remark 1. *If $b<0$, $\sqrt{b}$ is defined by $\sqrt{b}=\sqrt{|b|}i$*

By Theorem 3 and 4 we can get propositions of equality.

Proposition 3. *Let μ be defined by*

$$\mu=\frac{1}{6}\left((1+\sqrt{2})^{2/3}+\frac{1}{(1+\sqrt{2})^{2/3}}\right)-\frac{1}{3}.$$

If we define $q(b)$ as

$$q(b)=\begin{cases} arctan\left(\sqrt{\frac{2+3b}{b}}\frac{1}{4b+3}\right), & b<-\frac{3}{4}, \quad or \quad b>\mu \\ -\frac{\pi}{2}, & b=-\frac{3}{4} \\ arctan\left(\sqrt{\frac{2+3b}{b}}\frac{1}{4b+3}\right)-\pi, & -\frac{3}{4}<b<-\frac{3}{2} \end{cases}$$

If $b>\mu$ or $b<-\frac{2}{3}$, we have:

Via (13) $+3b(1+2b)\times$ (15) we get

$$\sum_{k=1}^{\infty}\frac{(18b^2+9b+1)k+3b(1+2b)}{k(1+3k)(-2b(1+2b)^2)^k\binom{3k}{k}}=2(1+2b)\sqrt{\frac{b}{3+2b}}q(b)-1 \tag{16}$$

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Via $(1+6b)^2 \times (12) + 2 \times (13)$ *we get*

$$\sum_{k=0}^{\infty} \frac{(1+6b)^2 k + (1+4b+12b^2)}{(1+3k)(-2b(1+2b)^2)^k \binom{3k}{k}} = \frac{9b(1+2b)^2}{2+3b} + \frac{2(1+2b)^2}{2+3b}\sqrt{\frac{b}{2+3b}} q(b) \tag{17}$$

By Proposition 3, by equality (16) and (17) we let $b=-1$ and $b=-\frac{3}{4}$, we can get four π-series, here are the examples.

Example 3.

$$\sum_{k=1}^{\infty} \frac{3+10k}{k(1+3k)2^k \binom{3k}{k}} = -1 + \frac{\pi}{2} \tag{18}$$

$$\sum_{k=1}^{\infty} \frac{(9+35k)8^{k-1}}{k(1+3k)3^k \binom{3k}{k}} = \frac{\sqrt{3}}{2}\pi - 1 \tag{19}$$

$$\sum_{k=0}^{\infty} \frac{9+25k}{(1+3k)2^k \binom{3k}{k}} = \frac{\pi}{2} + 9 \tag{20}$$

$$\sum_{k=0}^{\infty} \frac{(19+49k)8^k}{4(1+3k)3^k \binom{3k}{k}} = \frac{27}{4} + \sqrt{3}\pi \tag{21}$$

4. π-series containing $1/\binom{4k}{k}$

In this section, we will give methods to find π-series containing the type $1/\binom{4k}{k}$.
We first give an auxiliary lemma we will use.

Lemma 4. *Let* $f(t) = t^4 - t^3 + x$, *let* $x = \frac{8b^3(1+2b)^3}{(1+4b)^2}$, *if* $x > \frac{27}{256}$, *then we can factor* $f(t)$ *as*

$$f(t) = \left(\left(t - b - \frac{1}{2}\right)^2 + \frac{(1+2b)^2(-1+4b)}{4(1+4b)}\right)\left((t+b)^2 + \frac{b^2(3+4b)}{1+4b}\right),$$

where $b > \frac{1}{4}$ *or* $b < -\frac{3}{4}$.
If $x < -\frac{27}{256}$, *then we can factor* $f(t)$ *as*

$$f(t) = \left(t^2 - (1+2b)t + \frac{2b(1+2b)^2}{1+4b}\right)\left((t+b)^2 + \frac{b^2(3+4b)}{1+4b}\right),$$

where $-0.33 \approx -\frac{1}{4} - \mu < b < -\frac{1}{4} + \mu \approx -0.17$ *and* $b \neq -\frac{1}{4}$ *and*

$$\mu = \frac{\sqrt{2}}{8}\sqrt{\frac{-3(1+\sqrt{2})^{2/3} + 2(1+\sqrt{2})^{1/3} + 3}{(1+\sqrt{2})^{1/3}}} \approx 0.08.$$

Proof. Consider $f(t) = 0$, since $|x| > \frac{27}{256}$, we can deduce that

$$\Delta = -12288x^2(256x - 27) \begin{cases} < 0, & x > \frac{27}{256} \\ > 0, & x < -\frac{27}{256} \end{cases}$$

If $x > \frac{27}{256}$, then $f(t) = 0$ has two pairs of unequal conjugate complex roots. Suppose we factor $f(t)$ as

$$f(t) = ((t+a)^2 + c)((t+b)^2 + d),$$

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Then we have

$$\begin{cases} 2(a+b) = -1 \\ a^2 + b^2 + 4ab + c + d = 0 \\ 2(a^2+c)b + 2(b^2+d)a = 0 \\ (a^2+c)(b^2+d) = x \end{cases}$$

That is to say $a = -\frac{1}{2} - b$ and $c = \frac{(1+2b)^2(-1+4b)}{4(1+4b)}$ and $d = \frac{b^2(3+4b)}{1+4b}$ and $x = \frac{8b^3(1+2b)^3}{(1+4b)^2}$. If $x > \frac{4}{27}$, we have $b < -\frac{3}{4}$ or $b > \frac{1}{4}$. $\square$

If $x < -\frac{27}{256}$, then $f(t) = 0$ has two unequal real roots and a pair of conjugate complex roots. Suppose we factor $f(t)$ as

$$f(t) = (t+a)(t+c)((t+b)^2+d) = (t+(a+c)t+ac)((t+b)^2+d),$$

Then we have

$$\begin{cases} a+c+2b = -1 \\ ac + 2(a+c)b + b^2 + d = 0 \\ 2acb + (a+c)(b^2+d) = 0 \\ ac(b^2+d) = x \\ a \neq c \end{cases}$$

That is to say $a + c = -(1+2b)$ and $ac = \frac{2b(1+2b)^2}{1+4b}$ and $d = \frac{b^2(3+4b)}{1+4b}$ and $x = \frac{8b^3(1+2b)^3}{(1+4b)^2}$. Since $x < -\frac{27}{256}$ and $a \neq c$, we have $-\frac{1}{4} - \mu < b < -\frac{1}{4} + \mu$ and $b \neq -\frac{1}{4}$.

Theorem 5. *Let*

$$\mu = \frac{\sqrt{2}}{8}\sqrt{\frac{-3(1+\sqrt{2})^{2/3} + 2(1+\sqrt{2})^{1/3} + 3}{(1+\sqrt{2})^{1/3}}}.$$

If $b > \frac{1}{4}$ or $b < -\frac{3}{4}$ or $-\frac{1}{4} - \mu < b < -\frac{1}{4} + \mu$ and $b \neq -\frac{1}{4}$, then we have

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(1+4b)^{2k}}{(8b^3(1+2b)^3)^k \binom{4k}{k}} \\ =&\frac{2048b^3(1+2b)^3}{(-1+4b)(3+4b)(3+16b+32b^2)^2} \\ &+\frac{6b(1+2b)\sqrt{(-1+4b)(1+4b)}f_1(b)}{(-1+4b)^2(3+16b+32b^2)^3} arctan\left(\frac{\sqrt{(-1+4b)(1+4b)}}{-1+8b^2}\right) \\ &+\frac{6b(1+2b)\sqrt{(1+4b)(3+4b)}f_2(b)}{(3+4b)^2(3+16b+32b^2)^3} arctan\left(\frac{\sqrt{(1+4b)(3+4b)}}{1+8b+8b^2}\right) \\ &+\frac{6b(1+2b)(1+4b)(1+16b+32b^2)}{(3+16b+32b^2)^3}\log\left(\frac{4b^2}{(1+2b)^2}\right) \end{aligned} \tag{22}$$

where

$$f_1(b) = 1 + 12b + 176b^2 + 640b^3 + 512b^4$$

and

$$f_2(b) = -9 - 60b - 16b^2 + 384b^3 + 512b^4.$$

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Proof. By Lemma 1 we have

$$\sum_{k=0}^{\infty}\frac{1}{x^k\binom{4k}{k}}=\int_0^1\frac{-x(3t^4-3t^3-x)}{(t^4-t^3+x)^2}dt.$$

By equation (3) we have $|x|>\frac{27}{256}$. □

Let $x=\frac{8b^3(1+2b)^3}{(1+4b)^2}$. Since $|x|>\frac{27}{256}$, we have $b>\frac{1}{4}$ or $b<-\frac{3}{4}$ or $-\frac{1}{4}-\mu<b<-\frac{1}{4}+\mu$ and $b\neq-\frac{1}{4}$. By Mathematica we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{x^k\binom{4k}{k}}\\ =&\int_0^1\frac{-x(3t^4-3t^3-x)}{(t^4-t^3+x)^2}dt\\ =&\frac{b^3(1+2b)^3}{(1+4b)^2(3+16b+32b^2)^3}\Big[\frac{2048(1+4b)^2(3+16b+32b^2)}{-3+8b+16b^2}\\ &+\frac{6(1+4b)^2f_1(b)}{b^2(1+2b)^2(-1+4b)}\frac{\sqrt{1+4b}}{\sqrt{-1+4b}}\left(arctan\frac{1}{\sqrt{\frac{-1+4b}{1+4b}}}-arctan\frac{-1+2b}{(1+2b)\sqrt{\frac{-1+4b}{1+4b}}}\right)\\ &+\frac{6(1+4b)^2f_2(b)}{b^2(1+2b)^2(3+4b)}\frac{\sqrt{1+4b}}{\sqrt{3+4b}}\left(arctan\frac{(1+b)\sqrt{\frac{1+4b}{3+4b}}}{b}-arctan\sqrt{\frac{1+4b}{3+4b}}\right)\\ &+\frac{6(1+4b)^3(1+16b+32b^2)}{b^2(1+2b)^2}\log\left(\frac{4b^2}{(1+2b)^2}\right)\Big].\end{aligned}$$

where

$$f_1(b)=1+12b+176b^2+640b^3+512b^4$$

and

$$f_2(b)=-9-60b-16b^2+384b^3+512b^4.$$

Since in the domain of b we have

$$\begin{aligned}&\frac{\sqrt{1+4b}}{\sqrt{-1+4b}}\left(arctan\frac{1}{\sqrt{\frac{-1+4b}{1+4b}}}-arctan\frac{-1+2b}{(1+2b)\sqrt{\frac{-1+4b}{1+4b}}}\right)\\ =&\frac{\sqrt{(-1+4b)(1+4b)}}{-1+4b}arctan\left(\frac{\sqrt{(-1+4b)(1+4b)}}{-1+8b^2}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{\sqrt{1+4b}}{\sqrt{3+4b}}\left(arctan\frac{(1+b)\sqrt{\frac{1+4b}{3+4b}}}{b}-arctan\sqrt{\frac{1+4b}{3+4b}}\right)\\ =&\frac{\sqrt{(1+4b)(3+4b)}}{3+4b}arctan\left(\frac{\sqrt{(1+4b)(3+4b)}}{1+8b+8b^2}\right).\end{aligned}$$

Then we get the desired results.

Using the same method of Theorem 5 and by Lemma 2 we have

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Theorem 6. *Let*

$$\mu = \frac{\sqrt{2}}{8}\sqrt{\frac{-3(1+\sqrt{2})^{2/3}+2(1+\sqrt{2})^{1/3}+3}{(1+\sqrt{2})^{1/3}}}.$$

If $b > \frac{1}{4}$ *or* $b < -\frac{3}{4}$ *or* $-\frac{1}{4}-\mu < b < -\frac{1}{4}+\mu$ *and* $b \neq -\frac{1}{4}$, *then we have*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(1+4b)^{2k}}{(8b^3(1+2b)^3)^k(4k+1)\binom{4k}{k}}\\ &=\frac{2b(1+2b)(1+4b+16b^2)\sqrt{(-1+4b)(1+4b)}}{(-1+4b)(1+4b)(3+16b+32b^2)}arctan\left(\frac{\sqrt{(-1+4b)(1+4b)}}{8b^2-1}\right)\\ &+\frac{2b(1+2b)(3+12b+16b^2)\sqrt{(1+4b)(3+4b)}}{(1+4b)(3+4b)(3+16b+32b^2)}arctan\left(\frac{\sqrt{(1+4b)(3+4b)}}{1+8b+8b^2}\right)\\ &-\frac{2b(1+2b)(1+4b)}{3+16b+32b^2}\log\left(\frac{4b^2}{(1+2b)^2}\right).\end{aligned} \tag{23}$$

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{(1+4b)^{2k}}{(8b^3(1+2b)^3)^k k\binom{4k}{k}}\\ &=\frac{3(1+2b)\sqrt{(-1+4b)(1+4b)}}{(-1+4b)(3+16b+32b^2)}arctan\left(\frac{\sqrt{(-1+4b)(1+4b)}}{-1+8b^2}\right)\\ &+\frac{6b\sqrt{(1+4b)(3+4b)}}{(3+4b)(3+16b+32b^2)}arctan\left(\frac{\sqrt{(1+4b)(3+4b)}}{1+8b^+8b^2}\right)\\ &+\frac{3(1+4b)}{2(3+16b+32b^2)}\log\left(\frac{4b^2}{(1+2b)^2}\right).\end{aligned} \tag{24}$$

By Theorem 5 and 6 we can get propositions of equality.

Proposition 4. *Let*

$$\mu = \frac{\sqrt{2}}{8}\sqrt{\frac{-3(1+\sqrt{2})^{2/3}+2(1+\sqrt{2})^{1/3}+3}{(1+\sqrt{2})^{1/3}}}.$$

If $b > \frac{1}{4}$ *or* $b < -\frac{3}{4}$ *or* $-\frac{1}{4}-\mu < b < -\frac{1}{4}+\mu$ *and* $b \neq -\frac{1}{4}$, *then we have:*
Via $(3+4b)(3+16b+32b^2)^2 \times (22) + 3(1+4b)(3+8b)\times(23) - 128b^2(1+2b)^3 \times (24)$ *we have*

$$\begin{aligned}&\frac{[(3+4b)(3+16b+32b^2)^2k^2+f_1(b)k-32b^2(1+2b)^3](1+4b)^{2k}}{k(1+4k)(8b^3(1+2b)^3)^k\binom{4k}{k}}\\ &=\frac{3\left(16b^2(1+2b^2)\sqrt{16b^2-1}arctan\frac{\sqrt{16b^2-1}}{-1+8b^2}-f_2(b)\right)}{(4b-1)^2}\end{aligned} \tag{25}$$

where

$$f_1(b) = 9+96b+328b^2+448b^3+256b^4$$

and

$$f_2(b) = (-1+4b)(1+4b)(-3-8b+8b^2).$$

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Via $(-1+4b)(3+16b+32b^2)^2 \times$ (22) $-3(1+4b)(1+8b) \times$ (23) $-256b^3(1+2b)^2 \times$ (24) *we have*

$$\frac{[(-1+4b)(3+16b+32b^2)^2k^2 - f_3(b)k - 64b^3(1+2b)^2](1+4b)^{2k}}{k(1+4k)(8b^3(1+2b)^3)^k\binom{4k}{k}} = \frac{3\left(f_4(b) - 16b^2(1+2b)^2\sqrt{(1+4b)(3+4b)}\arctan\frac{\sqrt{(1+4b)(3+4b)}}{1+8b+8b^2}\right)}{(3+4b)^2} \tag{26}$$

where

$$f_3(b) = 3+24b+40b^2+64b^3+256b^4$$

and

$$f_2(b) = (1+4b)(3+4b)(3+16b+8b^2).$$

By Proposition 4, in equation (25) we let $b=\frac{1}{2}$ and in (26) we let $b=-1$, we can get one π-series, here is the example.

Example 4.

$$\sum_{k=1}^{\infty}\frac{(-64+211k+1805k^2)9^k}{k(1+4k)8^k\binom{4k}{k}} = 45+16\sqrt{3}\pi \tag{27}$$

5. π-series containing $1/\binom{4k}{2k}$

In this section, we will give methods to find π-series containing the type $1/\binom{4k}{2k}$.
We first give an auxiliary lemma we will use.

Lemma 5. *Let* $f(t) = t^4 - 2t^3 + t^2 - x$*, if* $x < -\frac{1}{16}$*, let* $x = -b(1+b)(1+2b)^2$*, then we can factor* $f(t)$ *as*

$$f(t) = ((t-b-1)^2 + b(1+b))\left((t+b)^2 + b(1+b)\right),$$

where $b > \frac{\sqrt{2+2\sqrt{2}}}{4} - \frac{1}{2} \approx 0.0493$ *or* $b < -\frac{1}{2} - \frac{2+2\sqrt{2}}{4} \approx -1.05$.
If $x > \frac{1}{16}$*, let* $x = b^2$*, then we can factor* $f(t)$ *as*

$$f(t) = \left((t-1/2)^2 + b - 1/4\right)\left((t-1/2)^2 - b - 1/4\right),$$

where $|b| > 1/4$.

Proof. Consider $f(t) = 0$, since $|x| > \frac{1}{16}$, we can deduce that

$$\Delta = 196608x^2(16x-1)\begin{cases} > 0, & x > \frac{1}{16} \\ < 0, & x < -\frac{1}{16} \end{cases}$$

If $x > \frac{1}{16}$, then $f(t) = 0$ has two unequal real roots and a pair of conjugate complex roots. We then get

$$\begin{aligned} f(t) =& t^4 - 2t^3 + t^2 - b^2 \\ =& t^2(t-1)^2 - b^2 \\ =& (t(t-1) - b)(t(t-1) + b) \\ =& ((t-1/2)^2 - b - 1/4)((t-1/2)^2 + b - 1/4) \end{aligned}$$

□

If $x < -\frac{1}{16}$, then $f(t)$ has two pair of conjugate complex roots. Suppose we factor $f(t)$ as

$$f(t) = ((t+a)^2 + c)((t+b)^2 + d),$$

Then we have

$$\begin{cases} 2(a+b) = -2 \\ a^2 + b^2 + 4ab + c + d = 1 \\ 2(a^2+c)b + 2(b^2+d)a = 0 \\ (a^2+c)(b^2+d) = x \end{cases}$$

That is to say $a = -(b+1)$ and $c = b(1+b)$ and $d = b(1+b)$ and $x = -b(1+b)(1+2b)^2$. Since $x < -\frac{1}{16}$, we have $b > \frac{\sqrt{2+2\sqrt{2}}}{4} - \frac{1}{2} \approx 0.0493$ or $b < -\frac{1}{2} - \frac{2+2\sqrt{2}}{4} \approx -1.05$

We first consider the case $x < -\frac{1}{16}$, then we have the following theorem.

Theorem 7. *If $b > \frac{\sqrt{2+2\sqrt{2}}}{4} - \frac{1}{2} \approx 0.0493$ or $b < -\frac{1}{2} - \frac{2+2\sqrt{2}}{4} \approx -1.05$, then we have*

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{(-b(1+b)(1+2b)^2)^k \binom{4k}{2k}} \\ =&\frac{16b(1+b)(1+2b)(1+10b+24b^2+16b^3)}{(1+8b+8b^2)^3} \\ +&\frac{2(1+2b)^2(-1+8b+8b^2)\sqrt{b(1+b)}}{(1+8b+8b^2)^3} arctan \frac{1}{2\sqrt{b(1+b)}} \\ +&\frac{2b(1+b)(1+2b)(3+8b+8b^2)}{(1+8b+8b^2)^3} \log\left(\frac{b}{1+b}\right) \end{aligned} \tag{28}$$

Proof. By Lemma 1 we have

$$\sum_{k=0}^{\infty} \frac{1}{x^k \binom{4k}{2k}} = \int_0^1 \frac{x(3t^2(1-t)^2 + x)}{(t^2(1-t)^2 - x)^2} dt.$$

By inequality (3) we have $|x| > \frac{1}{16}$.

Let $x = -b(1+b)(1+2b)^2$, then we have

$$f(t) = ((t-b-1)^2 + b(1+b)) \left((t+b)^2 + b(1+b)\right),$$

If $x < -\frac{1}{16}$, we have $b > \frac{\sqrt{2+2\sqrt{2}}}{4} - \frac{1}{2} \approx 0.0493$ or $b < -\frac{1}{2} - \frac{2+2\sqrt{2}}{4} \approx -1.05$, then by Mathematica we get

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{x^k \binom{4k}{k}} \\ =&\int_0^1 \frac{x(3t^2(1-t)^2 + x)}{(t^2(1-t)^2 - x)^2} dt \\ =&\frac{2(1+2b)\sqrt{b}\sqrt{1+b}}{(1+8b+8b^2)^3} [8\sqrt{b}\sqrt{1+b}(1+10b+24b^2+16b^3) \\ &+(1+2b)(-1+8b+8b^2)\left(arctan\sqrt{\frac{1+b}{b}} - arctan\sqrt{\frac{b}{1+b}}\right) \\ &+(3+8b+8b^2)\sqrt{b}\sqrt{1+b}\log\left(\frac{b}{1+b}\right)] \end{aligned}$$

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Since in the domain of b we have

$$\sqrt{b}\sqrt{1+b}\sqrt{b}\sqrt{1+b} = b(1+b)$$

and

$$\sqrt{b}\sqrt{1+b}\left(arctan\sqrt{\frac{1+b}{b}} - arctan\sqrt{\frac{b}{1+b}}\right)$$
$$= \sqrt{b(1+b)}arctan\frac{1}{2\sqrt{b(1+b)}}.$$

Then by some simplification we get the desired results. □

Using the same method of Theorem 7 and by Lemma 2 we have

Theorem 8. *If* $b > \frac{\sqrt{2+2\sqrt{2}}}{4} - \frac{1}{2} \approx 0.0493$ *or* $b < -\frac{1}{2} - \frac{2+2\sqrt{2}}{4} \approx -1.05$, *then we have*

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{1}{(-b(1+b)(1+2b)^2)^k(4k+1)\binom{4k}{2k}}\\ &= \frac{2(1+2b)^2\sqrt{b(1+b)}}{1+8b+8b^2}arctan\frac{1}{2\sqrt{b(1+b)}}\\ &- \frac{2b(1+b)(1+2b)}{1+8b+8b^2}\log\left(\frac{b}{1+b}\right)\end{aligned} \tag{29}$$

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{1}{(-b(1+b)(1+2b)^2)^k k\binom{4k}{2k}}\\ &= \frac{4\sqrt{b(1+b)}}{1+8b+8b^2}arctan\frac{1}{2\sqrt{b(1+b)}}\\ &+ \frac{1+2b}{1+8b+8b^2}\log\left(\frac{b}{1+b}\right)\end{aligned} \tag{30}$$

By Theorem 7 and 8 we get some propositions of equality.

Proposition 5. *If* $b > \frac{\sqrt{2+2\sqrt{2}}}{4} - \frac{1}{2}$ *or* $b < -\frac{1}{2} - \frac{2+2\sqrt{2}}{4}$, *then we have:*
Via (29) $+ 2b(1+2b) \times$ (30) *we get*

$$\sum_{k=1}^{\infty} \frac{(1+8b+8b^2)k + 2b(1+b)}{k(4k+1)(-b(1+b)(1+2b)^2)^k\binom{4k}{2k}} = 2\sqrt{b(1+b)}arctan\frac{1}{2\sqrt{b(1+b)}} - 1 \tag{31}$$

Via $(1+8b+8b^2)^2 \times$ (28) $- 2b(1+b)(3+8b+8b^2) \times$ (30) *we get*

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(1+8b+8b^2)^2k - 2b(1+b)(3+8b+8b^2)}{k(-b(1+b)(1+2b)^2)^k\binom{4k}{2k}}\\ &= -\left(1 + 2\sqrt{b(1+b)}arctan\frac{1}{2\sqrt{b(1+b)}}\right)\end{aligned} \tag{32}$$

By Proposition 5 and in equation (31) and (32) we let $b = \frac{1}{2}, \frac{\sqrt{2}-1}{2}, \frac{\sqrt{3}}{3} - \frac{1}{2}$, we can get some π-series, here are the examples.

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Example 5.

$$\sum_{k=1}^{\infty}\frac{3+14k}{2k(1+4k)(-3)^k\binom{4k}{2k}}=\frac{1}{6}(-6+\sqrt{3}\pi) \tag{33}$$

$$\sum_{k=1}^{\infty}\frac{(1+6k)(-2)^k}{2k(1+4k)\binom{4k}{2k}}=\frac{1}{4}(-4+\pi) \tag{34}$$

$$\sum_{k=1}^{\infty}\frac{(1+10k)(-9)^k}{6k(1+4k)\binom{4k}{2k}}=\frac{1}{9}(-9+\sqrt{3}\pi) \tag{35}$$

$$\sum_{k=1}^{\infty}\frac{-27+98k}{2k(-3)^k\binom{4k}{2k}}=-\left(1+\frac{\pi}{2\sqrt{3}}\right) \tag{36}$$

$$\sum_{k=1}^{\infty}\frac{(-5+18)(-2)^k}{2k\binom{4k}{2k}}=-\left(1+\frac{\pi}{4}\right) \tag{37}$$

$$\sum_{k=1}^{\infty}\frac{(-11+50k)(-9)^k}{18k\binom{4k}{2k}}=-\left(1+\frac{\pi}{3\sqrt{3}}\right) \tag{38}$$

Now we consider the case $x > \frac{1}{16}$.

Theorem 9. *If $|b| > \frac{1}{4}$, then we have*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{b^{2k}\binom{4k}{2k}}\\ =&\frac{16b^2}{16b^2-1}+\frac{2b}{\sqrt{-1-4b}(1+4b)}arctan\frac{1}{\sqrt{-1-4b}}\\ +&\frac{2b}{(-1+4b)\sqrt{-1+4b}}arctan\frac{1}{\sqrt{-1+4b}}.\end{aligned} \tag{39}$$

Proof. By Lemma 1 we have

$$\sum_{k=0}^{\infty}\frac{1}{x^k\binom{4k}{2k}}=\int_0^1\frac{x(3t^2(1-t)^2+x)}{(t^2(1-t)^2-x)^2}dt.$$

By equation (3) we have $|x| > \frac{1}{16}$. □

Let $x = b^2$, then we have

$$f(t)=((t-1/2)^2-b-1/4)((t-1/2)^2+b-1/4).$$

If $x > \frac{1}{16}$, we have $|b| > \frac{1}{4}$ then by Mathematica we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{b^{2k}\binom{4k}{k}}\\ =&\int_0^1\frac{b^2(3t^2(1-t)^2+b^2)}{((t-1/2)^2-b-1/4)^2((t-1/2)^2+b-1/4)^2}dt\\ =&\frac{-2b}{(-1-4b)^{3/2}(-1+4b)^{3/2}}[8b\sqrt{-1-4b}\sqrt{-1+4b}\\ +&(-1+4b)^{3/2}arctan\frac{1}{\sqrt{-1-4b}}-(-1-4b)^{3/2}arctan\frac{1}{\sqrt{-1+4b}}].\end{aligned}$$

Then by some simplifications we get the desired results.

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Using the same method of Theorem 9 and by Lemma 2 we have

Theorem 10. *If* $|b| > \frac{1}{4}$, *then we have*

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)b^{2k}\binom{4k}{2k}} =2b\left(\frac{arctan\frac{1}{\sqrt{-1+4b}}}{\sqrt{-1+4b}}-\frac{arctan\frac{1}{\sqrt{-1-4b}}}{\sqrt{-1-4b}}\right). \tag{40}$$

$$\sum_{k=0}^{\infty}\frac{k}{b^{2k}\binom{4k}{2k}}=\frac{24b^2}{(-1+4b)^2(1+4b)^2} +\frac{b(-1+2b)}{(1+4b)^2}\frac{arctan\frac{1}{\sqrt{-1-4b}}}{\sqrt{-1-4b}}+\frac{b(1+2b)}{(-1+4b)^2}\frac{arctan\frac{1}{\sqrt{-1+4b}}}{\sqrt{-1+4b}}. \tag{41}$$

$$\sum_{k=1}^{\infty}\frac{1}{kb^{2k}\binom{4k}{2k}}=2\left(\frac{arctan\frac{1}{\sqrt{-1-4b}}}{\sqrt{-1-4b}}+\frac{arctan\frac{1}{\sqrt{-1+4b}}}{\sqrt{-1+4b}}\right). \tag{42}$$

By Theorem 10 we can get propositions of equality.

Proposition 6. *If* $|b| > \frac{1}{4}$, *via* (40) $+ b \times$ (42) *we have*

$$\sum_{k=1}^{\infty}\frac{(4b+1)k+b}{k(4k+1)b^{2k}\binom{4k}{2k}}=\frac{4b}{\sqrt{4b-1}}arctan\frac{1}{\sqrt{4b-1}}-1. \tag{43}$$

If $|b| > \frac{1}{4}$, *via* $(-1+2b)\times$ (40) $+ 2(1+4b)^2 \times$ (41) *we have*

$$\sum_{k=0}^{\infty}\frac{2(1+4b)^2k(4k+1)+2b-1}{(4k+1)b^{2k}\binom{4k}{2k}}=\frac{48b^2}{(-1+4b)^2}+\frac{8b^2(5+16b^2)}{(-1+4b)^2}\frac{arctan\frac{1}{\sqrt{-1+4b}}}{\sqrt{-1+4b}}. \tag{44}$$

By Proposition 6 and in equation (43) and (44) we let $b=\frac{1}{3},\frac{1}{2},1$, we can get some π-series, here are the examples.

Example 6.

$$\sum_{k=1}^{\infty}\frac{(1+7k)3^{2k-1}}{k(1+4k)\binom{4k}{2k}}=\frac{4\sqrt{3}\pi}{9}-1 \tag{45}$$

$$\sum_{k=1}^{\infty}\frac{1+5k}{k(1+4k)\binom{4k}{2k}}=\frac{2\sqrt{3}\pi}{9}-1 \tag{46}$$

$$\sum_{k=1}^{\infty}\frac{(1+6k)2^{2k-1}}{k(1+4k)\binom{4k}{2k}}=\frac{\pi}{2}-1 \tag{47}$$

$$\sum_{k=1}^{\infty}\frac{(-3+98k+392k^2)9^{k-1}}{(1+4k)\binom{4k}{2k}}=\frac{488\pi}{9\sqrt{3}}+48 \tag{48}$$

$$\sum_{k=1}^{\infty}\frac{1+50k+200k^2}{(1+4k)\binom{4k}{2k}}=\frac{4}{27}(36+7\sqrt{3}\pi) \tag{49}$$

$$\sum_{k=1}^{\infty} \frac{k4^k}{\binom{4k}{2k}} = \frac{\pi}{4} + \frac{2}{3} \quad (50)$$

Data Availability Statement: Data openly available in a public repository.

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