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An Exploration on a Normed Space Called *r*-Normed Space: Some Properties and an Application

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Abstract: In this paper, we explore a new norm on L^p-space called r-norm, and hence producing a new normed space called r-normed space. Measure theoretic and functional analytic properties are covered such as the relation between r-norm and the classical p-norm. An application of r-normed space in probability theory and statsitics is also presented, notably, its role as a rigorous framework in describing the coefficient of determination.

Keywords: r-normed space; r-norm; L^p-space; measure theory; functional analysis; coefficient of determination

1. Introduction

In functional analysis [1,2], a normed-space is a vector space [3] together with a map [4] from the vector space to the set of all nonnegative real numbers which satisfies a set of axioms [4] called normed space axioms [2]. A comprehensive and formal description about normed spaces will be presented later. Mathematicians have studied normed spaces from the late 19th and early 20th centuries as the history is concerned.

In this paper, we explore a new kind of normed space related to L^p functions [1,2] which we refer to as the r-normed space. The formal definition of this normed space will be presented as well its rigorous construction. We also present an application of r-normed space in probability theory and statistics [5].

For the rest of the paper, symbolic first or second order notations [6] will frequently occur in every formal expression. The readers are referred to [4,6] for comprehensive explanations regarding the formal expressions. Foundations in abstract measure theory [7] are also required since we will not present basic concepts in measure theory unless we think necessary.

2. Basics of *Lp* Space and Normed Space

The underlying vector space [3] in the whole discussion is the L^p space, a function space satisfying some Lebesgue integration property [7] which will be described further. The designation of L^p space can be broad since many different vector spaces may be examples of L^p space such as finite dimensional vector space, infinite dimensional vector space in forms of sequences and function space of certain type of Lebesgue integrable functions [2,7]. We will present the general measure theoretic definition of L^p space as follows.

Definition 1 (Equivalence Classes Family L^p)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space [7]. Let $p \in \mathbb{R}$ such that $1 \leq p < \infty$. Let $\mathcal{L}^p(\mu)$ be a family of measurable functions [7] $X \to \mathbb{R}$ such that*

$$\forall f \in \mathcal{L}^p(\mu) : \int_X |f|^p \, d\mu < \infty .$$

Suppose an equivalence relation [4] $\sim$ on $\mathcal{L}^p(\mu)$ defined by

$$\forall f, g \in \mathcal{L}^p(\mu) : \; f \sim g \iff \mu(\{x \in X \mid f(x) \neq g(x)\}) = 0 .$$

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Then we have a family of equivalence classes [4] in $\mathcal{L}^p(\mu)$

$$L^p(\mu) := \{[f] \mid f \in \mathcal{L}^p(\mu)\}$$

in which

$$\forall f, g \in \mathcal{L}^p(\mu): \ g \in [f] \iff f \sim g\,.$$

We will need to show that the family L^p of equivalence classes in definition 1 is a vector space. First, let us observe the axiomatic definition of vector space in the following definition.

Definition 2 (Vector Space)**.** *A set V is a vector space [3] over a field $\mathbb{F}$ together with operations $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ if the following axioms are satisfied:*

V1 $\forall u, v, w \in V: \ (u+v)+w = u+(v+w)$
V2 $\forall u, v \in V: \ u+v = v+u$
V3 *There exists a unique* $0_V \in V$ *such that*

$$\forall u \in V: \ 0_V + u = u + 0_V = u\,.$$

V4 $\forall u, v \in V: \ u+v = v+u = 0_V$
V5 $\forall a, b \in \mathbb{F} \forall u \in V: \ a(bu) = (ab)u$
V6 *There exists a unique* $1_{\mathbb{F}} \in \mathbb{F}$ *such that*

$$\forall u \in V: \ 1_{\mathbb{F}} u = u\,.$$

V7 $\forall a \in \mathbb{F} \forall u, v \in V: \ a(u+v) = au + av$
V8 $\forall a.b \in \mathbb{F} \forall u \in V: \ (a+b)u = au + bu$

The following theorem will show that the family L^p of equivalence classes in definition 1 is in fact a vector space over $\mathbb{R}$ which is referred to as an L^p space.

Theorem 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. The family $L^p(\mu)$ of equivalent classes as described in definition 1 is a vector space over $\mathbb{R}$.*

Proof. To show that $L^p(\mu)$ is a vector space over $\mathbb{R}$ we need to show that all axioms in definition 2 hold.

Let $f, g, h \in L^p(\mu)$. Since f, g, h are real-valued functions, then

$$\forall x \in X: \ (f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$$

and

$$\forall x \in X: \ f(x)+g(x) = g(x)+f(x)$$

holds and show that axioms V1 and V2 are satisfied. Note that in the sense of $L^p(\mu)$, the symbol '=' designates the equivalence relation $\sim$ as described in definition 1. The zero function $e \in L^p(\mu)$ also exists as it satisfies

$$N := \{x \in X \mid e(x) \neq 0\} \ \wedge \ \mu(N) = 0\,.$$

Then we have

$$\forall x \in X \setminus N: \ e(x)+f(x) = f(x)+e(x) = f(x)+0 = f(x)$$

which shows that axiom V3 is satisfied. Then $-f \in L^p(\mu)$ is certain since

$$\int_X |-f|^p \, \mathrm{d}\mu = \int_X |-1|^p |f|^p \, \mathrm{d}\mu = \int_X |f|^p \, \mathrm{d}\mu < \infty\,.$$

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With
$$\forall x \in X: \ f(x) + (-f(x)) = -f(x) + f(x) = 0 = e(x),$$
axiom V4 is satisfied. Note that
$$\forall c \in \mathbb{R} \forall \tilde{f} \in L^p(\mu): \ c\tilde{f} \in L^p(\mu)$$
since
$$\forall c \in \mathbb{R} \forall \tilde{f} \in L^p(\mu): \ \int_X |c\tilde{f}|^p \, \mathrm{d}\mu = \int_X |c|^p |\tilde{f}|^p \, \mathrm{d}\mu = |c|^p \int_X |\tilde{f}|^p \, \mathrm{d}\mu < \infty.$$
Now, let $a, b \in \mathbb{R}$. Then we obtain
$$\forall x \in X: \ a(bf(x)) = (ab)f(x),$$
which shows that axiom V5 is satisfied. Then, certainly $1 \in \mathbb{R}$ and we have
$$\forall x \in X: \ 1f(x) = f(x)$$
which shows that axiom V6 is satisfied. Also
$$\forall x \in X: \ a(f(x) + g(x)) = af(x) + ag(x)$$
and
$$\forall x \in X: \ (a+b)f(x) = af(x) + bf(x)$$
show that axioms V7 and V8 are satisfied. Thus, we conclude that $L^p(\mu)$ is a vector space. □

Another functional analytic [2] notion to necessarily be presented is normed space [2], which is given in the following definition.

Definition 3 (Normed Space)**.** *Let V be a vector space over a field $\mathbb{F}$. A norm on V is a map $\|\cdot\| : V \to [0, \infty)$ satisfying the following axioms:*

- *N1* $\forall u \in V : \|u\| \geq 0$
- *N2* $\forall u \in V : \ \|u\| = 0 \iff u = 0_V$
- *N3* $\forall a \in \mathbb{F} \forall u \in V : \ \|au\| = |a|\|u\|$
- *N4* $\forall u, v \in V : \ \|u + v\| \leq \|u\| + \|v\|$

If $\|\cdot\|$ is a norm on V, then we call $(V, \|\cdot\|)$ a normed space [2].

For a given vector space and a possible norm, the common challenge is proving axiom N4, which is sometimes also referred to as the triangle inequality. An indispensable tools in mathematical analysis to help prove axiom N4 are the so-called “Hölder’s inequality" and “Minkowski inequality" which will be presented. In fact, Minkowski inequality is a consequence of Hölder’s inequality. However, we first need Young’s inequality in order to prove Hölder’s which is presented in the following lemma.

Proposition 1. *Let $p, q \in \mathbb{R}$ such that $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Note that the following expressions are equivalent:*

- *i* $pq - p - q + 1 = 1$
- *ii* $(p-1)(q-1) = 1$
- *iii* $p - 1 = \frac{1}{q-1}, \ q - 1 = \frac{1}{p-1}$
- *iv* $\frac{1}{p} = 1 - \frac{1}{q} = \frac{q-1}{q} \vdash p = \frac{q}{q-1}$
- *v* $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p} \vdash q = \frac{p-1}{p}$

Lemma 1. *Let $a, b \in \mathbb{R}$ such that $a, b \geq 0$. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the inequality*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

known as Young's inequality [2], holds.

Proof. Suppose a function $f : [0, a] \to [0, \alpha]$ such that $b \leq \alpha$, $f(a) = \alpha$ and f is continuous [8] as well as bijective [4]. Note that we will have

$$ab \leq \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y\,.$$

In particular, f can be defined by

$$\forall x \in [0, a] : \; f(x) := x^{p-1}$$

as it is continuous [2] and bijective. And we can show the bijectivity of f if we can show that f is strictly monotone, since strict monotonicity of such a function implies bijectivity. Now let $s, t \in [0, a]$ such that $s < t$. Certainly we have

$$f(s) = s^{p-1} < t^{p-1} = f(t)$$

which shows that f is strictly increasing. Hence f is bijective. It implies that f^{-1} exists, which is now our task in finding it. Follows from proposition 1, we obtain the inverse f^{-1} as given by

$$\forall y \in [0, b] : \; f^{-1}(y) = y^{q-1}\,.$$

And we have

$$ab \leq \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(y)\,\mathrm{d}y = \int_0^a x^{p-1}\,\mathrm{d}x + \int_0^b y^{q-1}\,\mathrm{d}y = \frac{x^p}{p}\bigg|_0^a + \frac{y^q}{q}\bigg|_0^b = \frac{a^p}{p} + \frac{b^q}{q}$$

which proves the lemma. □

Now we present Hölder's inequality in the following theorem.

Theorem 2. *Let $(X, \mathcal{A}, \mu)$ be a measure space [7]. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(\mu)$ and $g \in L^q(\mu)$ [7]. Then*

$$\int_X |fg|\,\mathrm{d}\mu \leq \left(\int_X |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q\,\mathrm{d}\mu\right)^{\frac{1}{q}},$$

holds, which is known as Hölder's inequality.

Proof. Suppose some functions $\tilde{f}, \tilde{g} : X \to \mathbb{R}$ defined by

$$\forall x \in X : \; \tilde{f}(x) := \frac{f(x)}{\left(\int_X |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}}$$

and

$$\forall x \in X : \; \tilde{g}(x) := \frac{g(x)}{\left(\int_X |g|^q\,\mathrm{d}\mu\right)^{\frac{1}{q}}}$$

respectively. Follows from lemma 1, i. e., Young's inequality, we obtain

$$\forall x \in X : \; |\tilde{f}(x)\tilde{g}(x)| \leq \frac{|\tilde{f}(x)|^p}{p} + \frac{|\tilde{g}(x)|^q}{q}\,.$$

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By taking Lebesgue integral [7] to the expression above, we obtain

$$\begin{aligned}
\int_X |\tilde{f}\tilde{g}| \,\mathrm{d}\mu &\leq \int_X \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q} \,\mathrm{d}\mu \\
\int_X \frac{|fg|}{\left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}}} \,\mathrm{d}\mu &\leq \int_X \frac{|f|^p}{p \int_X |f|^p \,\mathrm{d}\mu} + \frac{|g|^q}{q \int_X |g|^q \,\mathrm{d}\mu} \,\mathrm{d}\mu \\
\frac{\int_X |fg| \,\mathrm{d}\mu}{\left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}}} &\leq \frac{\int_X |f|^p \,\mathrm{d}\mu}{p \int_X |f|^p \,\mathrm{d}\mu} + \frac{\int_X |g|^q \,\mathrm{d}\mu}{q \int_X |g|^q \,\mathrm{d}\mu} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1
\end{aligned}$$

which implies

$$\int_X |fg| \,\mathrm{d}\mu \leq \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}},$$

which proves Hölder's inequality. □

Theorem 3. *Let $(X, \mathcal{A}, \mu)$ be a measure space [7]. Let $p \in \mathbb{R}$ such that $1 \leq p < \infty$. Follows from theorem 1, we have a vector space $L^p(\mu)$. Let $f, g \in L^p(\mu)$. Then*

$$\left(\int_X |f+g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}$$

holds, which is known as Minkowski inequality [2].

Proof. Proving for $p = 1$ is trivial, since obviously

$$\forall x \in X: \; |f(x) + g(x)| \leq |f(x)| + |g(x)|,$$

then by the property of Lebesgue integral [7] we have

$$\int_X |f+g| \,\mathrm{d}\mu \leq \int_X |f| \,\mathrm{d}\mu + \int_X |g| \,\mathrm{d}\mu,$$

which proves the inequality for $p = 1$.

Now we prove the inequality for $p > 1$. Let $q \in \mathbb{R}$ such that $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $h : X \to \mathbb{R}$ such that

$$\forall x \in X: \; h(x) := |f(x) + g(x)|^{p-1}.$$

Note that $|f + g| \in L^p(\mu)$ since $L^p(\mu)$ is a vector space, which means that it is closed under addition and scalar multiplication. That is,

$$\forall x \in X: \; |f(x) + g(x)| = \mathrm{sgn}(f(x) + g(x)) \cdot (f(x) + g(x)),$$

where the map $\mathrm{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ is defined by

$$\forall a \in \mathbb{R}: \; \mathrm{sgn}(a) := \begin{cases} 1 & : a > 0 \\ 0 & : a = 0 \\ -1 & : a < 0 \end{cases}.$$

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From the expression above, additive closure shows that $f+g \in L^p(\mu)$, and scalar multiplicative closure shows that $\operatorname{sgn}(f+g)\cdot(f+g) \in L^p(\mu)$, and hence $|f+g| \in L^p(\mu)$. Follows from proposition 1, we obtain

$$\int_X |h|^q \,\mathrm{d}\mu = \int_X |f+g|^{(p-1)q} \,\mathrm{d}\mu = \int_X |f+g|^{(p-1)\frac{p}{p-q}} \,\mathrm{d}\mu = \int_X |f+g|^p \,\mathrm{d}\mu < \infty\,,$$

from the fact that $|f+g| \in L^p(\mu)$. Hence $h \in L^q(\mu)$. Then follows from the triangle inequality and Hölder's inequality we obtain

$$\begin{aligned}
\int_X |f+g|^p \,\mathrm{d}\mu &= \int_X |f+g||f+g|^{p-1} \,\mathrm{d}\mu \\
&\leq \int_X (|f|+|g|)|f+g|^{p-1} \,\mathrm{d}\mu \quad \text{(Triangle inequality)} \\
&= \int_X |f||f+g|^{p-1} \,\mathrm{d}\mu + \int_X |g||f+g|^{p-1} \,\mathrm{d}\mu \\
&= \int_X |fh| \,\mathrm{d}\mu + \int_X |gh| \,\mathrm{d}\mu \\
&\leq \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X |h|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} + \left(\int_X |g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int_X |h|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \quad \text{(Hölder's inequality)} \\
&= \left[\left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}\right] \left(\int_X |h|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \\
&= \left[\left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}\right] \left(\int_X |f+g|^{(p-1)\frac{p}{p-1}} \,\mathrm{d}\mu\right)^{1-\frac{1}{p}} \\
&= \left[\left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}\right] \left(\int_X |f+g|^p \,\mathrm{d}\mu\right)^{1-\frac{1}{p}} \\
&= \left[\left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}\right] \frac{\int_X |f+g|^p \,\mathrm{d}\mu}{\left(\int_X |f+g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}}
\end{aligned}$$

which implies

$$\left(\int_X |f+g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}$$

which proves Minkowski inequality for $p > 1$. Thus, we conclude the proof. □

Now let us observe the very fundamental norm defined on L^p space, which is the p-norm [2]. This norm is a fundamental concept in functional analysis and is the interest of study in Banach spaces [1,2]. This norm will also be essentially related to our formulation of r-norm. We will present this norm on a theorem after we present the following lemma.

Lemma 2. *Let $(X, \mathcal{A}, \mu)$ be a measure space [7]. Let $p \in \mathbb{R}$ such that $1 \leq p < \infty$. Follows from theorem 1, we have a vector space $L^p(\mu)$. The expression*

$$\forall a \in \mathbb{R}\,\forall f \in L^p(\mu):\ af \sim 0 \iff \int_X |af|^p \,\mathrm{d}\mu = 0$$

holds.

Proof. We prove the forward part first. Let $a \in \mathbb{R}$ and $f \in L^p(\mu)$. Suppose $af \sim 0$. Then we have

$$N := \{x \in X \mid af(x) \neq 0\} \ \wedge\ \mu(N) = 0\,,$$

which implies

$$\int_N |af|^p \, \mathrm{d}\mu = 0 \, .$$

And by the property of Lebesgue integral [7], we obtain

$$\int_X |af|^p \, \mathrm{d}\mu = \int_N |af|^p \, \mathrm{d}\mu + \int_{X \setminus N} |af|^p \, \mathrm{d}\mu = 0 + 0 \cdot \mu(X \setminus N) = 0 \, ,$$

which proves the forward part.

Now for the backward part, suppose

$$\int_X |af|^p \, \mathrm{d}\mu = 0 \, .$$

Let $g : X \to \mathbb{R}$ such that $g := |af|^p$. Note that g is measurable and nonnegative. By simple function [7] approximation theorem, for every $n \in \mathbb{N}$ there exists a simple function $s_n : X \to [0, \infty)$ such that

$$\forall x \in X : \; 0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq g(x)$$

and

$$\forall x \in X : \; \lim_{n \to \infty} s_n(x) = g(x)$$

holds. Note that there are only finitely many points in the image of s_n since it is a simple function, for any $n \in \mathbb{N}$. Now let

$$\begin{aligned} m : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto m_n \end{aligned}$$

such that

$$\forall n \in \mathbb{N} : \; |\mathrm{im}(s_n)| = m_n \, .$$

The notation im above refers to the image of a function [4]. Now suppose

$$\forall n \in \mathbb{N} : \; \mathrm{im}(s_n) = \{\alpha_{n,k}\}_{k=1}^{m_n} \, .$$

And suppose

$$\forall n \in \mathbb{N} \forall k \in \{1, \ldots, m_n\} : \; A_{n,k} := \{x \in X \mid s_n(x) = \alpha_{n,k}\} \, .$$

We can now express the simple function as

$$\forall n \in \mathbb{N} \forall x \in X : \; s_n(x) = \sum_{k=1}^{m_n} \alpha_{n,k} \chi_{A_{n,k}}(x)$$

where

$$\forall n \in \mathbb{N} \forall k \in \{1, \ldots, m_n\} \forall x \in X : \; \chi_{A_{n,k}}(x) := \begin{cases} 1 & : x \in A_{n,k} \\ 0 & : x \notin A_{n,k} \, , \end{cases}$$

which is known as an indicator function. And the Lebesgue integral of g over X can be given in accordance with its definition [7] by

$$\int_X g \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \int_X s_n \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \sum_{k=1}^{m_n} \alpha_{n,k} \mu(A_{n,k}) \, .$$

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Recalling the assumption, we now have

$$0 = \int_X |af|^p \, \mathrm{d}\mu = \int_X g \, \mathrm{d}\mu = \sup_{n \in \mathbb{N}} \sum_{k=1}^{m_n} \alpha_{n,k} \mu(A_{n,k}) \, .$$

Note that each term in the summation above is nonnegative. Then we must have

$$\forall j \in \mathbb{N} : \; 0 \leq \sum_{k=1}^{m_j} \alpha_{j,k} \mu(A_{j,k}) \leq \sup_{n \in \mathbb{N}} \sum_{k=1}^{m_n} \alpha_{n,k} \mu(A_{n,k}) = 0 \, ,$$

which implies

$$\forall j \in \mathbb{N} : \; \sum_{k=1}^{m_j} \alpha_{j,k} \mu(A_{j,k}) = 0 \, .$$

The only solution to this circumstance is

$$\bigwedge_{j \in \mathbb{N}} \forall k \in \{1, \ldots, m_j\} : \; \mu(A_{j,k}) \neq 0 \implies \alpha_{j,k} = 0 \, .$$

The statement above implies that

$$\mu(\{x \in X \mid g(x) \neq 0\}) = 0 \, ,$$

which means that $g \sim 0$, or $g = 0$ μ-almost everywhere [7]. It also implies that $af \sim 0$, which proves the backward part. We conclude that

$$af \sim 0 \iff \int_X |af|^p \, \mathrm{d}\mu = 0 \, ,$$

which proves the theorem as a whole. □

Theorem 4. *Let $(X, \mathcal{A}, \mu)$ be a measure space [7]. Let $p \in \mathbb{R}$ such that $1 \leq p < \infty$. Follows from theorem 1, we have a vector space $L^p(\mu)$. A map $\| \cdot \|_p : L^p(\mu) \to [0, \infty)$ defined by*

$$\forall f \in L^p(\mu) : \; \|f\|_p := \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

is a norm on $L^p(\mu)$ known as the p-norm, and hence $(L^p(\mu), \| \cdot \|_p)$ is a normed space.

Proof. To show that $\| \cdot \|_p$ is a norm on $L^p(\mu)$, we need to show that axioms N1, N2, N3 and N4 in definition 3 are satisfied. For the whole discussion in this proof, let $a \in \mathbb{R}$ and $f, g \in L^p(\mu)$.

By definition, $\| \cdot \|_p$ involves the designation of $| \cdot |^p$ within the Lebesgue integral, hence we will always have a nonnegative integrand. By the property of Lebesgue integral [7], since $|f|^p \geq 0$ then

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \geq 0 \, ,$$

which shows that axiom N1 is satisfied.

Now we need to show that $\|f\|_p = 0 \iff f = 0$, which is equivalent to showing $\|f\|_p^p = 0 \iff f = 0$, as shown by

$$\{ \|f\|_p = 0 \iff f = 0 \, , \; \|f\|_p^p = 0 \iff \|f\|_p = 0 \} \vdash \|f\|_p^p \iff f = 0 \, .$$

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Follows from lemma 2, we obtain

$$f \sim 0 \iff \int_X |f|^p \, \mathrm{d}\mu = 0 \,.$$

Note that the equality symbol '=' in terms of $L^p(\mu)$ refers to '$\sim$' in terms of $\mathcal{L}^p(\mu)$. Thus, the statement above shows that $\|f\|_p^p = 0 \iff f = 0$ which implies $\|f\|_p = 0 \iff f = 0$ and hence axiom N2 is satisfied.

By the property of Lebesgue integral [7], we obtain

$$\|af\|_p = \left(\int_X |af|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} = \left(|a|^p \int_X |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} = |a| \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} = |a| \|f\|_p \,,$$

which shows that axiom N3 is satisfied.

Then by the property of Lebesgue integral as well as theorem 3 (Minkowski inequality), we obtain

$$\|f+g\|_p = \left(\int_X |f+g|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \le \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} = \|f\|_p + \|g\|_p \,,$$

which shows that axiom N4 is satisfied.

Thus, we conclude that $\|\cdot\|_p : L^p(\mu) \to \mathbb{R}$ is a norm on $L^p(\mu)$. Hence $(L^p(\mu), \|\cdot\|_p)$ is a normed space. □

3. Definition and Properties of *r*-Normed Space

We formulate r-norm as a norm on L^p space. The formal definition of r-norm and r-normed space follows from the following theorem.

Theorem 5. *Let $(X, \mathcal{A}, \mu)$ be a measure space [7]. Let $p \in \mathbb{R}$ such that $1 \le p < \infty$. Follows from theorem 1, we have a vector space $L^p(\mu)$. Let $r \in L^p(\mu)$ such that*

$$N := \{x \in X \mid r(x) = 0\} \ \wedge \ \mu(N) = 0 \,,$$

i. e., $r \neq 0$ μ-almost everywhere [7]. Let

$$R := \int_X |r|^p \, \mathrm{d}\mu \,.$$

A map $\|\cdot\|_{p:r} : L^p(\mu) \to [0, \infty)$ defined by

$$\forall f \in L^p(\mu) : \ \|f\|_{p:r} := \left(\int_X \frac{|f|^p}{R} \, \mathrm{d}\mu\right)^{\frac{1}{p}}$$

is a norm on $L^p(\mu)$.

Proof. We need to show that axioms N1, N2, N3 and N4 are satisfied. For the whole disucssion in this proof, let $a \in \mathbb{R}$ and $f, g \in L^p(\mu)$.

Note that $r \in L^p(\mu)$, $r \neq 0$ μ-almost everywhere [7] and $|r|^p \ge 0$, which imply $0 < |r|^p < \infty$ μ-almost everywhere. Then

$$0 < R = \int_X |r|^p \, \mathrm{d}\mu < \infty \,.$$

And note that $f \in L^p(\mu)$, which means

$$0 \le \int_X |f|^p \, \mathrm{d}\mu < \infty \,.$$

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Then by the property of Lebesgue integral [7], we obtain

$$\|f\|_{p:r} = \left(\int_X \frac{|f|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} = \frac{1}{\sqrt[p]{R}}\left(\int_X |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \geq 0$$

which shows that axiom N1 is satisfied.

Note that

$$f \sim 0 \iff \frac{f}{\sqrt[p]{R}} \sim 0$$

and

$$\int_X \frac{|f|^p}{R}\,\mathrm{d}\mu = 0 \iff \left(\int_X \frac{|f|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} = 0$$

hold. Follows from lemma 2, we obtain

$$\frac{f}{\sqrt[p]{R}} \sim 0 \iff \int_X \left|\frac{f}{\sqrt[p]{R}}\right|^p \mathrm{d}\mu = \int_X \frac{|f|^p}{R}\,\mathrm{d}\mu = 0\,.$$

Note that the equality symbol '$=$' in terms of $L^p(\mu)$ refers to the equivalence relation [4] symbol '$\sim$' in terms of $\mathcal{L}^p(\mu)$. Then by the transitive property of biconditional statements, we obtain

$$f = 0\ (\mu\text{-almost everywhere}) \iff \|f\|_{p:r} = \left(\int_X \frac{|f|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} = 0$$

which shows that axiom N2 is satisfied.

By the property of Lebesgue integral [7], we obtain

$$\|af\|_{p:r} = \left(\int_X \frac{|af|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} = \left(|a|^p \int_X \frac{|f|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} = |a|\left(\int_X \frac{|f|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} = |a|\|f\|_{p:r}$$

which shows that axiom N3 is satisfied.

Follows from the property of Lebesgue integral [7] and theoerm 3 (Minkowski inequality), we obtain

$$\begin{aligned}
\|f+g\|_{p:r} &= \left(\int_X \frac{|f+g|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} \\
&= \frac{1}{\sqrt[p]{R}}\left(\int_X |f+g|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\sqrt[p]{R}}\left[\left(\int_X |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}\right] \quad \text{(Minkowski inequality)} \\
&= \frac{1}{\sqrt[p]{R}}\left(\int_X |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} + \frac{1}{\sqrt[p]{R}}\left(\int_X |g|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \\
&= \left(\int_X \frac{|f|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X \frac{|g|^p}{R}\,\mathrm{d}\mu\right)^{\frac{1}{p}} \\
&= \|f\|_{p:r} + \|g\|_{p:r}\,,
\end{aligned}$$

which shows that axiom N4 is satisfied. We conclude that the map $\|\cdot\|_{p:r} : L^p(\mu) \to [0,\infty)$ is a norm on $L^p(\mu)$. $\square$

By theorem 5, we can present the formal definition of r-normed and r-normed space in the following definition.

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Definition 4 (r-Normed Space)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space [7]. Let $p \in \mathbb{R}$ such that $1 \leq p < \infty$. Follows from theorem 1, we have a vector space $L^p(\mu)$. Let $r \in L^p(\mu)$ such that $r \neq 0$ μ-almost everywhere [7]. And let*

$$R := \int_X |r|^p \, d\mu \,.$$

Follows from theorem 5, a map $\|\cdot\|_{p:r}$ defined by

$$\forall f \in L^p(\mu): \; \|f\|_{p:r} := \left(\int_X \frac{|f|^p}{R} \, d\mu \right)^{\frac{1}{p}}$$

is a norm on $L^p(\mu)$, which we call r-norm. And the normed space $(L^p(\mu), \|\cdot\|_{p:r})$ is called r-normed space.

As we have mentioned in the earlier section that p-norm and r-norm are closely related. The relationship is described in the following proposition.

Proposition 2. *Suppose an r-normed space $(L^p(\mu), \|\cdot\|_{p:r})$ with respect to a measure space $(X, \mathcal{A}, \mu)$ [7]. Let $f \in L^p(\mu)$. Note that from theorem 4, we have another norm $\|\cdot\|_p : L^p(\mu) \to [0, \infty)$ defined by*

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \,.$$

And follows from definition 4, we obtain

$$\|f\|_{p:r} = \left(\int_X \frac{|f|^p}{R} \, d\mu \right)^{\frac{1}{p}} = \frac{1}{\sqrt[p]{R}} \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} = \frac{\left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}}{\left(\int_X |R|^p \, d\mu \right)^{\frac{1}{p}}} = \frac{\|f\|_p}{\|r\|_p} \,.$$

4. An Application of r-Normed Space

A notable application of r-normed space is in a formal description of the coefficient of determination (CoD) in probability theory and statistics [5]. The CoD is used in statistical models for predictions or hypotheses testing. It provides a numeric representation on the performance of the model [9]. This section will discuss the formal construction of CoD based on the r-norm, begun from defining the underlying measure space [7] and random variables [5].

Suppose we have a real world observation. Note that we will only have finitely many data points in the observation. Let Ω be the possible outcome of the observation, and hence $|\Omega| < \infty$. Let $\mathcal{F}$ be a σ-algebra [7] representing the set of probable events from the observation. Then we have a measurable space $(\Omega, \mathcal{F})$. Then let $P : \mathcal{F} \to [0, 1]$ be a probability measure, and hence we have a probability space $(\Omega, \mathcal{F}, P)$ [5]. Let us consider another measure $\gamma : \mathcal{F} \to \mathbb{N}$ defined as the counting measure [7] on $(\Omega, \mathcal{F})$, i. e.,

$$\forall A \in \mathcal{F}: \; \gamma(A) := |A| \,.$$

Since we only have finitely many data points in the observation, then

$$\gamma(\Omega) = |\Omega| < \infty \,,$$

which means that γ is a finite measure.

Also suppose that the observation produces random variables [5] of which some of them are independent and one of them is dependent. We will focus on the dependent random variable $Y : \Omega \to \mathbb{R}$. Suppose Y is almost surely [5] nonconstant, that is,

$$\forall a \in \mathbb{R}: \; P(\{\omega \in \Omega \mid Y(\omega) = a\}) < 1 \,,$$

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and almost surely [5] bounded, that is,

$$P(\{\omega \in \Omega \mid |Y(\omega)| < \infty\}) = 1\,.$$

On the other hand, suppose $Y \in L^2(\gamma)$. Theorem 5 and definition 4 allow us to define an r-norm on $L^2(\gamma)$. Let $r \in L^2(\gamma)$ such that

$$\forall \omega \in \Omega : \; r(\omega) := Y(\omega) - \mathrm{E}[Y]$$

where $\mathrm{E}[Y]$ denotes the expectation [5] of Y, that is,

$$\mathrm{E}[Y] = \int_\Omega Y \,\mathrm{d}P < \infty\,.$$

The boundedness of the expectation is implied from the designation that Y is almost surely bounded. Also note that $\mathrm{E}[Y]$ is just a constant, which means that it is also an $L^2(\gamma)$ function, since

$$\int_\Omega \mathrm{E}[Y]^2 \,\mathrm{d}\gamma = \mathrm{E}[Y]^2 \int_\Omega \mathrm{d}\gamma = \mathrm{E}[Y]^2 \gamma(\Omega) < \infty\,.$$

Now let

$$R := \int_\Omega r^2 \,\mathrm{d}\gamma = \int_\Omega (Y - \mathrm{E}[Y])^2 \,\mathrm{d}\gamma\,.$$

Then the r-norm $\|\cdot\|_{2:r} : L^2(\gamma) \to [0, \infty)$ is defined by

$$\forall X \in L^2(\gamma) : \; \|X\|_{2:r} := \left(\int_\Omega \frac{X^2}{R} \,\mathrm{d}\gamma \right)^{\frac{1}{2}}\,.$$

Suppose we develop a predictive model to estimate Y, either using a statistical or machine learning (ML) model. Let $\hat{Y} : \Omega \to \mathbb{R}$ be the predictive model. The CoD regarding the predictive model $\hat{Y}$ with respect to Y can be be given by

$$1 - \|Y - \hat{Y}\|_{2:r}^2\,.$$

With that in mind, we can also present a new method of numeric performance evaluation of the model by making use of only the r-norm in exchange for the CoD. We propose the method in the following proposition.

Proposition 3. *Suppose an observation which produces a finite number of data points. Let a measurable space [5] $(\Omega, \mathcal{F})$ be the underlying event space [5]. Note that*

$$|\Omega| < \infty\,.$$

Then let $(\Omega, \mathcal{F}, P)$ be the corresponding probability space. And let $\gamma : \mathcal{F} \to \mathbb{N}$ be a counting measure [7] on $(\Omega, \mathcal{F})$, and hence we have another measure space $(\Omega, \mathcal{F}, \gamma)$. Let $Y : \Omega \to \mathbb{R}$ be a dependent random variable such that Y has a bounded expectation [5], is almost surely nonconstant [5] and $Y \in L^2(\gamma)$. Let $r : \Omega \to \mathbb{R}$ such that

$$\forall \omega \in \Omega : \; r(\omega) := Y(\omega) - \mathrm{E}[Y]\,.$$

And let

$$R := \int_\Omega r^2 \,\mathrm{d}\gamma\,.$$

Follows from theorem 5 and definition 4, there exists a normed space $(L^2(\gamma), \|\cdot\|_{2:r})$ such that

$$\forall X \in L^2(\gamma) : \; \|X\|_{2:r} := \left(\int_\Omega \frac{X^2}{R} \,\mathrm{d}\gamma \right)^{\frac{1}{2}}\,.$$

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Let $\hat{Y} : \Omega \to \mathbb{R}$ be a predictive model on Y such that $\hat{Y} \in L^2(\gamma)$. The model $\hat{Y}$ is ρ-reliable if there exists some $\rho \in \mathbb{R}$ with $0 < \rho < 1$ such that

$$\|Y - \hat{Y}\|_{2:r}^2 < \rho\,.$$

Here is the main difference of the CoD with the performance evaluation presented in proposition 3. In CoD the model can be deemed showing a good performance if the corresponding CoD has a value close to 1. While in our proposed method, the model has a good performance if the corresponding squared r-norm has a value close to 0. On the other hand, an awful model will give a negative CoD, or equivalently a squared r-norm greater than 1.

5. Conclusion and Future Work

We have presented a new norm called r-norm on an L^p space which produces a new normed space called r-normed space. Some property of r-normed space has also been presented regarding its relation with the classical p-norm on the same underlying function space. An application of r-normed space has also been presented in the formal description of coefficient of determination in probability theory and statistics [5]. The r-normed space of square integrable random variable with respect to a measure space space $(\Omega, \mathcal{F}, \gamma)$ and probability space $(\Omega, \mathcal{F}, P)$ of a finite observation Ω, with $\gamma : \mathcal{F} \to \mathbb{N}$ being a counting measure [7], serves as an alternative framework in describing the coefficient of determination. We have also proposed an alternative model performance evaluation using the squared r-norm as presented in proposition 3.

For the future works, we are interested in further exploring the measure theoretic and functional analytic properties of r-normed space. We are also interested in exploring other possible applications of r-normed space in probability theory, statistics, or other fields of science and engineering.

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