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Article

Some New Fractional Inequalities For Coordinated Convexity Over Convex Set Pertaining To Fuzzy-Number Valued Settings Governed By Fractional Integrals

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Abstract: In this study, we first discover some new concept coordinated UD-convex mappings with fuzzy-number values. After that, we look into Hermite-Hadamard type inequalities via fuzzy-number-valued coordinated UD-convex fuzzy-number-valued mapping (coordinated UD-convex $FNVM$). In the case of coordinated UD-convex $FNVM$, novel conclusions are derived by making particular decisions in recently proven inequalities. Additionally, it is demonstrated that the recently discovered inequalities are expansions of comparable findings in the literature. It is important to note that the main outcomes are validated by nontrivial examples.

Keywords: fuzzy-number valued mappings; generalized fractional integral; coordinated convex mappings; coordinated UD-convexity; Hermite-Hadamard's inequalities

1. Introduction

One of the most well-known concepts in the field of function theory is the Hermite-Hadamard inequality, which was found by C. Hermite and J. Hadamard (and described in sources such as [1], [2] p.137). This inequality has several real-world applications in addition to its geometric interpretation.

The Hermite-Hadamard inequalities have been established by numerous mathematicians. It's important to note that the Hermite-Hadamard inequality, which naturally follows from Jensen's inequality, can be seen as a development of the idea of convexity. Recently, there has been renewed interest in the Hermite-Hadamard inequality for convex functions, leading to a wide range of improvements and expansions that have been thoroughly investigated (see, for example, publications like [3–8]).

Interval analysis is a crucial topic since it is used in math and computer models as one method of addressing interval uncertainty. Even though this theory has a long history going back to Archimedes' calculation of a circle's circumference, significant research on the subject was not published until the 1950s. The first book [9] on interval analysis was published in 1966 by Ramon E. Moore, who is credited with developing interval calculus. After that, other academics studied the theory and uses of interval analysis.

Furthermore, by taking into account interval-valued functions in [10–13], well-known inequality types as Ostrowski, Minkowski, and Beckenbach, as well as some of their applications, were supplied. Additionally, Budak et al. in [14] developed a few inequalities utilizing interval-valued Riemann-Liouville fractional integrals. The definition of interval-valued harmonically convex

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functions was provided by Liu et al. in [15], and as a result, they are able to derive several Hermite-Hadamard type inequalities, including interval fractional integrals. The authors provided a fuzzy integral-based variation of Jensen's inequality for interval-valued functions in [16] and [17] and demonstrated several integral inequalities, [18–29]. In their proofs of Hermite-Hadamard type inequalities for set-valued functions in [30] and [31,32], Mitroi et al. made use of general forms of interval-valued convex functions. Rom'an Flores et al. found a few Gronwal type inequalities for interval-valued functions in [33]. Zhao et al. showed many kinds of integral inequalities for interval-valued functions in [34,35].

In [36], Jleli and Samet discovered brand-new Hermite-Hadamard type inequality involving fractional integrals with regard to a different function. Fractional integrals of a function with respect to another function were first introduced by Tunc in [37]. The Riemann-Liouville and Hadamard fractional integrals were generalized into a single form by Katugompala's novel fractional integration. Budak and Agarwal used generalized fractional integrals, which generalize some significant fractional integrals like the Riemann-Liouville fractional integrals, the Hadamard fractional integrals, and the Katugampola fractional integrals in [38], to establish the Hermite-Hadamard-type inequalities for co-ordinated convex function. Interval-valued left- and right-sided generalized fractional double integrals were defined by Kara et al. [39]. Numerous authors have concentrated on interval-valued functions in recent years. The authors of [40] introduced the idea of interval-valued general convex functions and used it to demonstrate a number of novel Hermite-Hadamard type inequalities. A fractional version of Hermite-Hadamard type inequalities for interval-valued harmonically convex functions was also provided by the authors in [41]. Researchers recently expanded the idea of interval-valued convexity and described various types of UD-convexity for interval-valued functions in [42–46]. For UD-fuzzy-number-valued convex functions, they also discovered a large number of Hermite-Hadmard type inequalities.

To express the collection of all positive fuzzy numbers over the real numbers, we introduce the notation $\mathbb{F}_0$ in the context of this article. The terms $\mathcal{A}_{[\upsilon,ȡ]}$, $\mathcal{IA}_{[\upsilon,ȡ]}$, and $\mathcal{FA}_{([\upsilon,ȡ])}$ refer to the set of all $FNVM$ that are Riemann integrable real valued functions, Aumann's integrable *IV-Fs* and fuzzy Aumann's integrable on the interval $[\upsilon, ȡ]$. The following theorem draws a link between functions that are integrable in the sense of Riemann ($\mathcal{A}$-integrable) and functions that are integrable in the sense of $\mathcal{FA}$. Additionally, the sign "$\supseteq_{\mathbb{F}}$" is used to denote the up and down (UD) fuzzy inclusion relationship for $\widetilde{Ð}$ and $\widetilde{ሸ}$ belonging to $\mathbb{F}_0$, where $\widetilde{ሸ}$ is thought of as a fuzzy subset of $\widetilde{Ð}$. If and only if for $ȥ$-levels, the conditions $[\widetilde{Ð}]^{ȥ} \supseteq_I [\widetilde{ሸ}]^{ȥ}$ is met, this UD-inclusion is true. Integral fuzzy inequalities generated from $FNVM$s have recently attracted the attention of several academics:

Theorem 1 [35]: Assume that the UD-convex FNVM $\widetilde{Ɣ}: [\upsilon, ȡ] \to \mathbb{F}_0$ is IVM with $Ɣ_{ȥ}(\vartheta) = [Ɣ_*(\vartheta, ȥ), Ɣ^*(\vartheta, ȥ)]$ for all $\vartheta \in [\upsilon, ȡ]$ and for all $ȥ \in [0,1]$. Then there are the disparities:

$$\widetilde{Ɣ}\left(\frac{\upsilon+ȡ}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{ȡ-\upsilon} \odot (FA)\int_{\upsilon}^{ȡ} \widetilde{Ɣ}(\vartheta)d\vartheta \supseteq_{\mathbb{F}} \frac{\widetilde{Ɣ}(ȡ)\oplus\widetilde{Ɣ}(\upsilon)}{2}. \quad (1)$$

We provide the ideas of generalized fractional integrals for two-variable $FNVMs$ in order to demonstrate Hermite-Hadmard type inequalities for convex and coordinated convex functions, which are inspired by ongoing investigations. The main benefit of the newly established inequalities is that they can be converted into classical Hermite-Hadamard integral inequalities for coordinated UD-convex $FNVM$s as well as fuzzy Riemann-Liouville fractional Hermite-Hadamard, Hadamard, and Katugampola fractional Hermite-Hadamard inequalities without having to prove each one separately.

The format of this essay is as follows: A brief summary of the foundations of fuzzy-number-valued calculus and other relevant works in this area are presented in Section 2. In Section 3, we provide some generalized fractional integrals for UD-convex $FNVM$ with two variables. For UD-convex $FNVM$, we create a novel Hermite-Hadamard type inequality. Several Hermite-Hadamard type inequalities for coordinated UD-convex $FNVM$ are parented in Section 3. It is also taken into consideration how these findings compare to findings of a similar nature in the literature. Finally, Section 4 makes some suggestions for additional study.

2. Preliminaries

We will go through the fundamental terminologies and findings in this section, which aid in comprehending the ideas behind our fresh findings.

Definition 1 ([47,48]). Given $\tilde{Đ} \in \mathbb{F}_0$, the level sets or cut sets are given by $[\tilde{Đ}]^{\mathfrak{z}} = \{\theta \in \Re |\ \tilde{Đ}(\theta) > \mathfrak{z}\}$ $\forall\ \mathfrak{z} \in [0,1]$ and by

$$[\tilde{Đ}]^{0} = \{\theta \in \Re |\ \tilde{Đ}(\theta) > 0\}. \tag{2}$$

These sets are known as $\mathfrak{z}$-level sets or $\mathfrak{z}$-cut sets of $\tilde{Đ}$.

Proposition 1 ([49]). Let $\tilde{Đ}, \tilde{m} \in \mathbb{F}_0$. Then, relation " $\leq_{\mathbb{F}}$ " is given on $\mathbb{F}_0$ by $\tilde{Đ} \leq_{\mathbb{F}} \tilde{m}$ when and only when $[\tilde{Đ}]^{\mathfrak{z}} \leq_I [\tilde{m}]^{\mathfrak{z}}$, for every $\mathfrak{z} \in [0,1]$, which are left- and right-order relations.

Proposition 2 ([46]). Let $\tilde{Đ}, \tilde{m} \in \mathbb{F}_0$. Then, relation " $\supseteq_{\mathbb{F}}$ " is given on $\mathbb{F}_0$ by $\tilde{Đ} \supseteq_{\mathbb{F}} \tilde{m}$ when and only when $[\tilde{Đ}]^{\mathfrak{z}} \supseteq_I [\tilde{m}]^{\mathfrak{z}}$ for every $\mathfrak{z} \in [0,1]$, which is *the UD*-order relation on $\mathbb{F}_0$.

Remember the approaching notions, which are offered in the literature. If $\tilde{Đ}, \tilde{m} \in \mathbb{F}_0$ and $t \in \Re$, then, for every $\mathfrak{z} \in [0,1]$, the arithmetic operations addition "$\oplus$", multiplication "$\otimes$", and scaler multiplication "$\odot$" are defined by

$$[\tilde{Đ} \oplus \tilde{m}]^{\mathfrak{z}} = [\tilde{Đ}]^{\mathfrak{z}} + [\tilde{m}]^{\mathfrak{z}}, \tag{3}$$
$$[\tilde{Đ} \otimes \tilde{m}]^{\mathfrak{z}} = [\tilde{Đ}]^{\mathfrak{z}} \times [\tilde{m}]^{\mathfrak{z}}, \tag{4}$$
$$[t \odot \tilde{Đ}]^{\mathfrak{z}} = t[\tilde{Đ}]^{\mathfrak{z}}, \tag{5}$$

Equations (4) through (6) have immediate consequences for these outcomes.

Theorem 2 ([47]). The space $\mathbb{F}_0$ dealing with a supremum metric, i.e., for $\tilde{Đ}, \tilde{m} \in \mathbb{F}_0$

$$d_{\infty}(\tilde{Đ}, \tilde{m}) = \sup_{0 \leq \mathfrak{z} \leq 1} d_H([\tilde{Đ}]^{\mathfrak{z}}, [\tilde{m}]^{\mathfrak{z}}), \tag{6}$$

is a complete metric space, where H indicates the well-known Hausdorff metric on the space of intervals.

Theorem 3. Let $\tilde{Y}: [\upsilon, ȡ] \subset \Re \to \mathbb{F}_0$ be a *FNVM*, its *IVM*s are classified according to their $\mathfrak{z}$-levels $Y_{\mathfrak{z}}: [\upsilon, ȡ] \subset \Re \to \mathbb{R}_I$ are given by $Y_{\mathfrak{z}}(\theta) = [Y_*(\theta, \mathfrak{z}), Y^*(\theta, \mathfrak{z})]$ $\forall\ \theta \in [\upsilon, ȡ]$ and $\forall\ \mathfrak{z} \in (0,1]$. Then, $\tilde{Y}$ is *FA*-integrable over $[\upsilon, ȡ]$ if and only if, $Y_*(\theta, \mathfrak{z})$ and $Y^*(\theta, \mathfrak{z})$ are both *A*-integrable over $[\upsilon, ȡ]$. Moreover, if $\tilde{Y}$ is *FA*-integrable over $[\upsilon, ȡ]$, then

$$\left[(FA)\int_{\upsilon}^{ȡ} \tilde{Y}(\theta)d\theta\right]^{\mathfrak{z}} = \left[(A)\int_{\upsilon}^{ȡ} Y_*(\theta,\mathfrak{z})d\theta, (A)\int_{\upsilon}^{ȡ} Y^*(\theta,\mathfrak{z})d\theta\right] = (IA)\int_{\upsilon}^{ȡ} Y_{\mathfrak{z}}(\theta)d\theta, \tag{7}$$

$\forall\ \mathfrak{z} \in (0,1]$. $\forall\ \mathfrak{z} \in (0,1]$, $\mathcal{FA}_{([\upsilon,ȡ],\mathfrak{z})}$ denotes the collection of all *FA*-integrable *FNVM*s over $[\upsilon, ȡ]$.

Fuzzy Aumann's and fractional calculus on coordinates

Definition 2. [16,48] Let $Y: [\delta, n] \to \mathbb{R}_I^+$ be *IVM* and $Y \in \mathcal{IR}_{[\delta,n]}$. Then interval Riemann-Liouville-type integrals of Y are defined as

$$\mathcal{I}_{\delta^+}^{\gamma} Y(\psi) = \frac{1}{\Gamma(\gamma)}\int_{\delta}^{\psi} (\psi - t)^{\gamma-1} Y(t)\,dt \quad (\psi > \delta), \tag{8}$$
$$\mathcal{I}_{n^-}^{\gamma} Y(\psi) = \frac{1}{\Gamma(\gamma)}\int_{\psi}^{n} (t - \psi)^{\gamma-1} Y(t)\,dt \quad (\psi < n), \tag{9}$$

where $\gamma > 0$ and Γ is the gamma function.

Recently, Allahviranloo et al. [49] introduced the fuzzy version of defined the fractional integral integrals such that:

Definition 3. Let $\gamma > 0$ and $L([\eth, n], \mathbb{F}_0)$ be the collection of all Lebesgue measurable *FNVM*s on $[\eth, n]$. Then, the fuzzy left and right Riemann-Liouville fractional integral of $\tilde{Y} \in L([\eth, n], \mathbb{F}_0)$ with order $\gamma > 0$ are defined by

$$\mathcal{J}_{\eth^+}^{\gamma}\,\tilde{Y}(\psi) = \frac{1}{\Gamma(\gamma)}\int_{\eth}^{\psi}(\psi - t)^{\gamma-1}\tilde{Y}(t)\,dt, \quad (\psi > \eth), \tag{10}$$

and

$$\mathcal{J}_{n^-}^{\gamma}\,\tilde{Y}(\psi) = \frac{1}{\Gamma(\gamma)}\int_{\psi}^{n}(t - \psi)^{\gamma-1}\tilde{Y}(t)\,dt, \quad (\psi < n), \tag{11}$$

respectively, where $\Gamma(\psi) = \int_0^{\infty} t^{\psi-1}e^{-t}\,dt$ is the Euler gamma function. The fuzzy left and right Riemann-Liouville fractional integral ψ based on left and right end point functions can be defined, that is

$$\left[\mathcal{J}_{\eth^+}^{\gamma}\tilde{Y}(\psi)\right]^{\mathfrak{z}} = \frac{1}{\Gamma(\gamma)}\int_{\eth}^{\psi}(\psi - t)^{\gamma-1}Y_{\mathfrak{z}}(t)\,dt$$

$$= \frac{1}{\Gamma(\gamma)}\int_{\eth}^{\psi}(\psi - t)^{\gamma-1}[Y_*(t,\mathfrak{z}), Y^*(t,\mathfrak{z})]\,dt, (\psi > \eth), \tag{12}$$

where

$$\mathcal{J}_{\eth^+}^{\gamma}\,Y_*(\psi,\mathfrak{z}) = \frac{1}{\Gamma(\gamma)}\int_{\eth}^{\psi}(\psi - t)^{\gamma-1}Y_*(t,\mathfrak{z})\,dt, \quad (\psi > \eth), \tag{13}$$

and

$$\mathcal{J}_{\eth^+}^{\gamma}\,Y^*(\psi,\mathfrak{z}) = \frac{1}{\Gamma(\gamma)}\int_{\eth}^{\psi}(\psi - t)^{\gamma-1}Y^*(t,\mathfrak{z})\,dt, \quad (\psi > \eth), \tag{14}$$

The right Riemann-Liouville fractional integral, denoted by $\left[\mathcal{J}_{n^-}^{\gamma}\tilde{Y}(\psi)\right]^{\mathfrak{z}}$, can also be defined using the left and right end point functions.

Theorem 4. [27] Let $\tilde{Y}: [\upsilon, \mathrm{d}] \to \mathbb{F}_0$ be a *UD*-convex *FNVM* on $[\upsilon, \mathrm{d}]$, whose $\mathfrak{z}$-cuts set up the sequence of *IVM*s $Y_{\mathfrak{z}}: [\upsilon, \mathrm{d}] \subset \mathbb{R} \to \mathbb{R}_C^{\,+}$ are given by $Y_{\mathfrak{z}}(\psi) = [Y_*(\psi,\mathfrak{z}), Y^*(\psi,\mathfrak{z})]$ for all $\psi \in [\upsilon, \mathrm{d}]$ and for all $\mathfrak{z} \in [0,1]$. If $\tilde{Y} \in L([\upsilon, \mathrm{d}], \mathbb{F}_0)$, then

$$\tilde{Y}\left(\frac{\upsilon+\mathrm{d}}{2}\right) \supseteq_{\mathbb{F}} \frac{\Gamma(\gamma+1)}{2(\mathrm{d}-\upsilon)^{\gamma}}\left[\mathcal{J}_{\upsilon^+}^{\gamma}\tilde{Y}(\mathrm{d}) \oplus \mathcal{J}_{\mathrm{d}^-}^{\gamma}\tilde{Y}(\upsilon)\right] \supseteq_{\mathbb{F}} \frac{\tilde{Y}(\upsilon)\oplus\tilde{Y}(\mathrm{d})}{2}. \tag{15}$$

Theorem 5. [27] Let $\tilde{Y}, \tilde{\mathcal{J}} : [\upsilon, \mathrm{d}] \to \mathbb{F}_0$ be two *UD*-convex *FNVM*s. Then, from $\mathfrak{z}$-cuts, we set up the sequence of *IVM*s $Y_{\mathfrak{z}}$, $\mathcal{J}_{\mathfrak{z}}: [\upsilon, \mathrm{d}] \subset \mathbb{R} \to \mathbb{R}_I^+$ are given by $Y_{\mathfrak{z}}(\theta) = [Y_*(\theta,\mathfrak{z}), Y^*(\theta,\mathfrak{z})]$ and $\mathcal{J}_{\mathfrak{z}}(\theta) = [\mathcal{J}_*(\theta,\mathfrak{z}), \mathcal{J}^*(\theta,\mathfrak{z})]$ for all $\theta \in [\upsilon, \mathrm{d}]$ and for all $\mathfrak{z} \in [0,1]$. If $\tilde{Y} \otimes \tilde{\mathcal{J}} \in L([\upsilon, \mathrm{d}], \mathbb{F}_0)$ is fuzzy Riemann integrable, then

$$\frac{\Gamma(\gamma+1)}{2(\mathrm{d}-\upsilon)^{\gamma}}\left[\mathcal{J}_{\upsilon^+}^{\gamma}\tilde{Y}(\mathrm{d}) \otimes \tilde{\mathcal{J}}(\mathrm{d}) \oplus \mathcal{J}_{\mathrm{d}^-}^{\gamma}\tilde{Y}(\upsilon) \otimes \tilde{\mathcal{J}}(\upsilon)\right]$$
$$\supseteq_{\mathbb{F}} \left(\frac{1}{2} - \frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\widetilde{\mathcal{M}}(\upsilon,\mathrm{d}) \oplus \left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\widetilde{\mathcal{N}}(\upsilon,\mathrm{d}), \tag{16}$$

and

$$\tilde{Y}\left(\frac{\upsilon+\mathrm{d}}{2}\right) \otimes \tilde{\mathcal{J}}\left(\frac{\upsilon+\mathrm{d}}{2}\right) \supseteq_{\mathbb{F}} \frac{\Gamma(\gamma+1)}{4(\mathrm{d}-\upsilon)^{\gamma}}\left[\mathcal{J}_{\upsilon^+}^{\gamma}\tilde{Y}(\mathrm{d}) \otimes \tilde{\mathcal{J}}(\mathrm{d}) \oplus \mathcal{J}_{\mathrm{d}^-}^{\gamma}\tilde{Y}(\upsilon) \otimes \tilde{\mathcal{J}}(\upsilon)\right]$$

$$+\frac{1}{2}\left(\frac{1}{2} - \frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\widetilde{\mathcal{M}}(\upsilon,\mathrm{d}) \oplus \frac{1}{2}\left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\widetilde{\mathcal{N}}(\upsilon,\mathrm{d}), \tag{17}$$

where $\widetilde{\mathcal{M}}(\upsilon,\mathrm{d}) = \tilde{Y}(\upsilon) \otimes \tilde{\mathcal{J}}(\upsilon) \oplus \tilde{Y}(\mathrm{d}) \otimes \tilde{\mathcal{J}}(\mathrm{d})$, $\widetilde{\mathcal{N}}(\upsilon,\mathrm{d}) = \tilde{Y}(\upsilon) \otimes \tilde{\mathcal{J}}(\mathrm{d}) \oplus \tilde{Y}(\mathrm{d}) \otimes \tilde{\mathcal{J}}(\upsilon)$,
$\mathcal{M}_{\mathfrak{z}}(\upsilon,\mathrm{d}) = \left[\mathcal{M}_*((\upsilon,\mathrm{d}),\mathfrak{z}), \mathcal{M}^*((\upsilon,\mathrm{d}),\mathfrak{z})\right]$, and $\mathcal{N}_{\mathfrak{z}}(\upsilon,\mathrm{d}) = \left[\mathcal{N}_*((\upsilon,\mathrm{d}),\mathfrak{z}), \mathcal{N}^*((\upsilon,\mathrm{d}),\mathfrak{z})\right]$.

Interval and fuzzy Aumann's type integrals are defined as follows for coordinated *IVM* $Y(\theta,\psi)$ and coordinated *FNVM* $\tilde{Y}(\theta,\psi)$:

Theorem 6. [34] Let $\tilde{Y}: \Delta[\eth, n] \times [\upsilon, \mathrm{d}] \subset \mathbb{R}^2 \to \mathbb{F}_0$ be a *FNVM* on coordinates, whose $\mathfrak{z}$-cuts set up the sequence of *IVM*s $Y_{\mathfrak{z}}: \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I$ are given by $Y_{\mathfrak{z}}(\theta,\psi) = \left[Y_*((\theta,\psi),\mathfrak{z}), Y^*((\theta,\psi),\mathfrak{z})\right]$ for all $(\theta,\psi) \in \Delta = [\eth, n] \times [\upsilon, \mathrm{d}]$ and for all $\mathfrak{z} \in [0,1]$. Then $\tilde{Y}$ is fuzzy double integrable (*FD*-integrable) over Δ if and only if $Y_*(\theta,\mathfrak{z})$ and $Y^*(\theta,\mathfrak{z})$ both are *D*-integrable over Δ. Moreover, if $\tilde{Y}$ is *FD*-integrable over Δ, then

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$$\left[(FD)\int_{\eth}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}(\vartheta,\psi)d\psi d\vartheta\right]^{\mathfrak{z}}=\left[(D)\int_{\eth}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z})d\psi d\vartheta,(D)\int_{\eth}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}\mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})d\psi d\vartheta\right]$$
$$=(ID)\int_{\eth}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}\mathrm{Y}_{\mathfrak{z}}(\vartheta,\psi)d\psi d\vartheta, \quad (18)$$

for all $\mathfrak{z}\in[0,1]$.

The family of all FD-integrable of $FNVM$s over coordinates and D-integrable functions over coordinates are denoted by $\mathcal{FD}_\Delta$ and $\mathfrak{D}_{(\Delta,\mathfrak{z})}$, for all $\mathfrak{z}\in[0,1]$.

Here is the main definition of fuzzy Riemann-Liouville fractional integral on the coordinates of the function $\widetilde{\mathrm{Y}}(\vartheta,\psi)$ by:

Definition 4. [29] Let $\widetilde{\mathrm{Y}}:\Delta\to\mathbb{F}_0$ and $\widetilde{\mathrm{Y}}\in\mathcal{FD}_\Delta$. The double fuzzy interval Riemann-Liouville-type integrals $\mathcal{J}_{\eth^+,\upsilon^+}^{\mathfrak{r},\beta},\mathcal{J}_{\eth^+,\mathrm{d}^-}^{\mathfrak{r},\beta},\mathcal{J}_{\mathrm{n}^-,\upsilon^+}^{\mathfrak{r},\beta},\mathcal{J}_{\mathrm{n}^-,\mathrm{d}^-}^{\mathfrak{r},\beta}$ of Y order $\mathfrak{r},\beta>0$ are defined by:

$$\mathcal{J}_{\eth^+,\upsilon^+}^{\mathfrak{r},\beta}\widetilde{\mathrm{Y}}(\vartheta,\psi)=\frac{1}{\Gamma(\mathfrak{r})\Gamma(\beta)}\int_{\eth}^{\vartheta}\int_{\upsilon}^{\psi}(\vartheta-\mathrm{t})^{\mathfrak{r}-1}(\psi-\mathrm{s})^{\beta-1}\widetilde{\mathrm{Y}}(\mathrm{t},\mathrm{s})\,d\mathrm{s}d\mathrm{t},\ (\vartheta>\eth,\psi>\upsilon), \quad (19)$$
$$\mathcal{J}_{\eth^+,\mathrm{d}^-}^{\mathfrak{r},\beta}\widetilde{\mathrm{Y}}(\vartheta,\psi)=\frac{1}{\Gamma(\mathfrak{r})\Gamma(\beta)}\int_{\eth}^{\vartheta}\int_{\psi}^{\mathrm{d}}(\vartheta-\mathrm{t})^{\mathfrak{r}-1}(\mathrm{s}-\psi)^{\beta-1}\widetilde{\mathrm{Y}}(\mathrm{t},\mathrm{s})\,d\mathrm{s}d\mathrm{t},\ (\vartheta>\eth,\psi<\mathrm{d}), \quad (20)$$
$$\mathcal{J}_{\mathrm{n}^-,\upsilon^+}^{\mathfrak{r},\beta}\widetilde{\mathrm{Y}}(\vartheta,\psi)=\frac{1}{\Gamma(\mathfrak{r})\Gamma(\beta)}\int_{\vartheta}^{\mathrm{n}}\int_{\upsilon}^{\psi}(\mathrm{t}-\vartheta)^{\mathfrak{r}-1}(\psi-\mathrm{s})^{\beta-1}\widetilde{\mathrm{Y}}(\mathrm{t},\mathrm{s})\,d\mathrm{s}d\mathrm{t},\ (\vartheta<\mathrm{n},\psi>\upsilon), \quad (21)$$
$$\mathcal{J}_{\mathrm{n}^-,\mathrm{d}^-}^{\mathfrak{r},\beta}\widetilde{\mathrm{Y}}(\vartheta,\psi)=\frac{1}{\Gamma(\mathfrak{r})\Gamma(\beta)}\int_{\vartheta}^{\mathrm{n}}\int_{\psi}^{\mathrm{d}}(\mathrm{t}-\vartheta)^{\mathfrak{r}-1}(\mathrm{s}-\psi)^{\beta-1}\widetilde{\mathrm{Y}}(\mathrm{t},\mathrm{s})\,d\mathrm{s}d\mathrm{t},\ (\vartheta<\mathrm{n},\psi<\mathrm{d}). \quad (22)$$

Here is the newly defined concept of coordinated convexity over fuzzy number space in the codomain via UD-relation given by:

Definition 5. [34] The $FNVM$ $\widetilde{\mathrm{Y}}:\Delta\to\mathbb{F}_0$ is referred to be coordinated UD-convex $FNVM$ on Δ if

$$\widetilde{\mathrm{Y}}(\varepsilon\eth+(1-\varepsilon)\mathrm{n},s\upsilon+(1-s)\mathrm{d})$$
$$\supseteq_{\mathbb{F}}\varepsilon s\widetilde{\mathrm{Y}}(\eth,\upsilon)\oplus\varepsilon(1-s)\widetilde{\mathrm{Y}}(\eth,\mathrm{d})\oplus(1-\varepsilon)s\widetilde{\mathrm{Y}}(\mathrm{n},\upsilon)\oplus(1-\varepsilon)(1-s)\widetilde{\mathrm{Y}}(\mathrm{n},\mathrm{d}), \quad (23)$$

for all $(\eth,\mathrm{n}),(\upsilon,\mathrm{d})\in\Delta$, and $\varepsilon,s\in[0,1]$, where $\widetilde{\mathrm{Y}}(\vartheta)\geq_{\mathbb{F}}\tilde{0}$. If inequality (23) is reversed, then $\widetilde{\mathrm{Y}}$ is referred to be coordinate concave $FNVM$ on Δ.

Lemma 1. [34] Let $\widetilde{\mathrm{Y}}:\Delta\to\mathbb{F}_0$ be a coordinated $FNVM$ on Δ. Then, $\widetilde{\mathrm{Y}}$ is coordinated UD-convex $FNVM$ on Δ if and only if there exist two coordinated UD-convex $FNVM$s $\widetilde{\mathrm{Y}}_\vartheta:[\upsilon,\mathrm{d}]\to\mathbb{F}_0$, $\widetilde{\mathrm{Y}}_\vartheta(w)=\widetilde{\mathrm{Y}}(\vartheta,w)$ and $\widetilde{\mathrm{Y}}_\psi:[\eth,\mathrm{n}]\to\mathbb{F}_0$, $\widetilde{\mathrm{Y}}_\psi(z)=\widetilde{\mathrm{Y}}(z,\psi)$.

Theorem 7. [34] Let $\widetilde{\mathrm{Y}}:\Delta\to\mathbb{F}_0$ be a $FNVM$ on Δ. Then, from $\mathfrak{z}$-levels, we get the collection of IVMs $\mathrm{Y}_{\mathfrak{z}}:\Delta\to\mathbb{R}_I^+\subset\mathbb{R}_I$ are given by

$$\mathrm{Y}_{\mathfrak{z}}(\vartheta,\psi)=[\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z}),\mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})],$$

for all $(\vartheta,\psi)\in\Delta$ and for all $\mathfrak{z}\in[0,1]$. Then, $\widetilde{\mathrm{Y}}$ is coordinated UD-convex $FNVM$ on Δ, if and only if, for all $\mathfrak{z}\in[0,1]$, $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z})$ and $\mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ are coordinated convex and concave functions, respectively.

Example 1. We consider the $FNVM$ $\widetilde{\mathrm{Y}}:[0,1]\times[0,1]\to\mathbb{F}_0$ defined by,

$$\mathrm{Y}(\vartheta)(\sigma)=\begin{cases}\dfrac{\sigma-\vartheta\psi}{5-\vartheta\psi}, & \sigma\in[\vartheta\psi,5]\\ \dfrac{(6+e^{\vartheta})(6+e^{\psi})-\sigma}{(6+e^{\vartheta})(6+e^{\psi})-5}, & \sigma\in(5,(6+e^{\vartheta})(6+e^{\psi})]\\ 0, & \text{otherwise,}\end{cases}$$

Then, for each $\mathfrak{z}\in[0,1]$, we have $\mathrm{Y}_{\mathfrak{z}}(\vartheta)=[(1-\mathfrak{z})\vartheta\psi+5\mathfrak{z},(1-\mathfrak{z})(6+e^{\vartheta})(6+e^{\psi})+5\mathfrak{z}]$. Since endpoint functions $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z})$, $\mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ are coordinate concave functions for each $\mathfrak{z}\in[0,1]$. Hence, $\widetilde{\mathrm{Y}}(\vartheta,\psi)$ is coordinate UD-convex $FNVM$.

From Lemma 1 and Example 1, we can easily note that each UD-convex $FNVM$ is coordinated UD-convex $FNVM$. But the converse is not true.

Remark 1. If one assumes that $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z})=\mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z}=1$, then Y is referred to be as a coordinated convex function if Y meets the stated inequality here, see [41]:

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$$\Upsilon(\varepsilon\eth + (1-\varepsilon)\mathfrak{n}, s\upsilon + (1-s)\mathfrak{d})$$
$$\leq \varepsilon s\Upsilon(\eth,\upsilon) + \varepsilon(1-s)\Upsilon(\eth,\mathfrak{d}) + (1-\varepsilon)s\Upsilon(\mathfrak{n},\upsilon) + (1-\varepsilon)(1-s)\Upsilon(\mathfrak{n},\mathfrak{d}),$$

Let one assumes that $\Upsilon_*((\vartheta,\psi),\mathfrak{z}) \neq \Upsilon^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z}=1$ and $\Upsilon_*((\vartheta,\psi),\mathfrak{z})$ is affine function and $\Upsilon^*((\vartheta,\psi),\mathfrak{z})$ is a concave function. If the stated inequality here, see [32:]

$$\Upsilon(\varepsilon\eth + (1-\varepsilon)\mathfrak{n}, s\upsilon + (1-s)\mathfrak{d})$$
$$\supseteq \varepsilon s\Upsilon(\eth,\upsilon) + \varepsilon(1-s)\Upsilon(\eth,\mathfrak{d}) + (1-\varepsilon)s\Upsilon(\mathfrak{n},\upsilon) + (1-\varepsilon)(1-s)\Upsilon(\mathfrak{n},\mathfrak{d}),$$

is true.

Definition 6. Let $\tilde{\Upsilon}: \Delta \to \mathbb{F}_0$ be a $FNVM$ on Δ. Then, from $\mathfrak{z}$-levels, we get the collection of IVMs $\Upsilon_{\mathfrak{z}}: \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ are given by

$$\Upsilon_{\mathfrak{z}}(\vartheta,\psi) = [\Upsilon_*((\vartheta,\psi),\mathfrak{z}), \Upsilon^*((\vartheta,\psi),\mathfrak{z})], \tag{24}$$

for all $(\vartheta,\psi) \in \Delta$ and for all $\mathfrak{z} \in [0,1]$. Then, $\tilde{\Upsilon}$ is coordinated left-UD-convex (concave) $FNVM$ on Δ, if and only if, for all $\mathfrak{z} \in [0,1]$, $\Upsilon_*((\vartheta,\psi),\mathfrak{z})$ and $\Upsilon^*((\vartheta,\psi),\mathfrak{z})$ are coordinated convex (concave) and affine functions on Δ, respectively.

Definition 7. Let $\tilde{\Upsilon}: \Delta \to \mathbb{F}_0$ be a $FNVM$ on Δ. Then, from $\mathfrak{z}$-levels, we get the collection of IVMs $\Upsilon_{\mathfrak{z}}: \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ are given by

$$\Upsilon_{\mathfrak{z}}(\vartheta,\psi) = [\Upsilon_*((\vartheta,\psi),\mathfrak{z}), \Upsilon^*((\vartheta,\psi),\mathfrak{z})], \tag{25}$$

for all $(\vartheta,\psi) \in \Delta$ and for all $\mathfrak{z} \in [0,1]$. Then, $\tilde{\Upsilon}$ is coordinated right-UD-convex (concave) $FNVM$ on Δ, if and only if, for all $\mathfrak{z} \in [0,1]$, $\Upsilon_*((\vartheta,\psi),\mathfrak{z})$ and $\Upsilon^*((\vartheta,\psi),\mathfrak{z})$ are coordinated affine and convex (concave) functions on Δ, respectively.

Theorem 8. Let Δ be a coordinated convex set, and let $\tilde{\Upsilon}: \Delta \to \mathbb{F}_0$ be a $FNVM$. Then, from $\mathfrak{z}$-levels, we obtain the collection of IVMs $\Upsilon_{\mathfrak{z}}: \Delta \to \mathbb{R}_I^+ \subset \mathbb{R}_I$ are given by

$$\Upsilon_{\mathfrak{z}}(\vartheta,\psi) = [\Upsilon_*((\vartheta,\psi),\mathfrak{z}), \Upsilon^*((\vartheta,\psi),\mathfrak{z})], \tag{26}$$

for all $(\vartheta,\psi) \in \Delta$ and for all $\mathfrak{z} \in [0,1]$. Then, $\tilde{\Upsilon}$ is coordinated UD-concave $FNVM$ on Δ, if and only if, for all $\mathfrak{z} \in [0,1]$, $\Upsilon_*((\vartheta,\psi),\mathfrak{z})$ and $\Upsilon^*((\vartheta,\psi),\mathfrak{z})$ are coordinated concave and convex functions, respectively.

Proof. The demonstration of proof of Theorem 8 is similar to the demonstration proof of Theorem 7.

Example 2. We consider the $FNVM$s $\tilde{\Upsilon}: [0,1] \times [0,1] \to \mathbb{F}_0$ defined by,

$$\tilde{\Upsilon}(\vartheta)(\sigma) = \begin{cases} \frac{\sigma-(6-e^{\vartheta})(6-e^{\psi})}{(6-e^{\vartheta})(6-e^{\psi})-25}, & \sigma \in [(6-e^{\vartheta})(6-e^{\psi}), 25] \\ \frac{35\vartheta\psi-\sigma}{35\vartheta\psi-25}, & \sigma \in (25,\ 35\vartheta\psi] \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Then, for each $\mathfrak{z} \in [0,1]$, we have $\Upsilon_{\mathfrak{z}}(\vartheta,\psi) = [(1-\mathfrak{z})(6-e^{\vartheta})(6-e^{\psi}) + 25\mathfrak{z}, 35(1-\mathfrak{z})\vartheta\psi + 25\mathfrak{z}]$. Since endpoint functions $\Upsilon_*((\vartheta,\psi),\mathfrak{z})$, $\Upsilon^*((\vartheta,\psi),\mathfrak{z})$ are coordinate concave and convex functions for each $\mathfrak{z} \in [0,1]$. Hence $\tilde{\Upsilon}(\vartheta,\psi)$ is coordinated UD-concave $FNVM$.

In the next results, to avoid confusion, we will not include the symbols (R), (IR), (FR), (ID), and (FD) before the integral sign.

The main goal of this article is to develop a number of original fractional coordinated integral inequalities for the Hermite-Hadamard types using an coordinated UD-concave $FNVM$. We acquired the most recent estimates for mappings whose products are coordinated UD-concave $FNVM$s using the fuzzy fractional operators.

3. Main Results

Here is first result of coordinated integral inequalities for the Hermite-Hadamard type using the fuzzy fractional operators via coordinated UD-concave $FNVM$s.

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Theorem 9. Let $\widetilde{Y}:\Delta\to \mathbb{F}_0$ be a coordinate *UD*-convex *FNVM* on Δ. Then, from $\mathfrak{z}$-cuts, we set up the sequence of *IVMs* $Y_{\mathfrak{z}}:\Delta\to \mathbb{R}_I^+$ are given by $Y_{\mathfrak{z}}(\theta,\psi)=[Y_*((\theta,\psi),\mathfrak{z}),Y^*((\theta,\psi),\mathfrak{z})]$ for all $(\theta,\psi)\in\Delta$ and for all $\mathfrak{z}\in[0,1]$. If $\widetilde{Y}\in\mathcal{FO}_\Delta$, then following inequalities holds:

$$\begin{aligned}\widetilde{Y}\left(\frac{\delta+n}{2},\frac{\upsilon+d}{2}\right)&\supseteq_{\mathbb{F}}\frac{\Gamma(\gamma+1)}{4(n-\delta)^{\gamma}}\left[\mathcal{J}_{\delta^+}^{\gamma}\widetilde{Y}\left(n,\frac{\upsilon+d}{2}\right)\oplus\mathcal{J}_{n^-}^{\gamma}\widetilde{Y}\left(\delta,\frac{\upsilon+d}{2}\right)\right]\oplus\frac{\Gamma(\beta+1)}{4(d-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\widetilde{Y}\left(\frac{\delta+n}{2},d\right)\oplus\mathcal{J}_{d^-}^{\beta}\widetilde{Y}\left(\frac{\delta+n}{2},\upsilon\right)\right]\\&\supseteq_{\mathbb{F}}\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\delta)^{\gamma}(d-\upsilon)^{\beta}}\left[\mathcal{J}_{\delta^+,\upsilon^+}^{\gamma,\beta}\widetilde{Y}(n,d)\oplus\mathcal{J}_{\delta^+,d^-}^{\gamma,\beta}\widetilde{Y}(n,\upsilon)\oplus\mathcal{J}_{n^-,\upsilon^+}^{\gamma,\beta}\widetilde{Y}(\delta,d)\oplus\mathcal{J}_{n^-,d^-}^{\gamma,\beta}\widetilde{Y}(\delta,\upsilon)\right]\\&\supseteq_{\mathbb{F}}\frac{\Gamma(\gamma+1)}{8(n-\delta)^{\gamma}}\left[\mathcal{J}_{\delta^+}^{\gamma}\widetilde{Y}(n,\upsilon)\oplus\mathcal{J}_{\delta^+}^{\gamma}\widetilde{Y}(n,d)\oplus\mathcal{J}_{n^-}^{\gamma}\widetilde{Y}(\delta,\upsilon)\oplus\mathcal{J}_{n^-}^{\gamma}\widetilde{Y}(\delta,d)\right]\\&\oplus\frac{\Gamma(\beta+1)}{8(d-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\widetilde{Y}(\delta,d)\oplus\mathcal{J}_{d^-}^{\beta}\widetilde{Y}(n,\upsilon)\oplus\mathcal{J}_{\upsilon^+}^{\beta}\widetilde{Y}(n,d)\oplus\mathcal{J}_{d^-}^{\beta}\widetilde{Y}(n,\upsilon)\right]\\&\supseteq_{\mathbb{F}}\frac{\widetilde{Y}(\delta,\upsilon)\oplus\widetilde{Y}(n,\upsilon)\oplus\widetilde{Y}(\delta,d)\oplus\widetilde{Y}(n,d)}{4}.\end{aligned}\tag{28}$$

If $Y(\theta)$ coordinated concave *FNVM* then,

$$\begin{aligned}\widetilde{Y}\left(\frac{\delta+n}{2},\frac{\upsilon+d}{2}\right)&\subseteq_{\mathbb{F}}\frac{\Gamma(\gamma+1)}{4(n-\delta)^{\gamma}}\left[\mathcal{J}_{\delta^+}^{\gamma}\widetilde{Y}\left(n,\frac{\upsilon+d}{2}\right)\oplus\mathcal{J}_{n^-}^{\gamma}\widetilde{Y}\left(\delta,\frac{\upsilon+d}{2}\right)\right]\oplus\frac{\Gamma(\beta+1)}{4(d-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\widetilde{Y}\left(\frac{\delta+n}{2},d\right)\oplus\mathcal{J}_{d^-}^{\beta}\widetilde{Y}\left(\frac{\delta+n}{2},\upsilon\right)\right]\\&\subseteq_{\mathbb{F}}\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\delta)^{\gamma}(d-\upsilon)^{\beta}}\left[\mathcal{J}_{\delta^+,\upsilon^+}^{\gamma,\beta}\widetilde{Y}(n,d)\oplus\mathcal{J}_{\delta^+,d^-}^{\gamma,\beta}\widetilde{Y}(n,\upsilon)\oplus\mathcal{J}_{n^-,\upsilon^+}^{\gamma,\beta}\widetilde{Y}(\delta,d)\oplus\mathcal{J}_{n^-,d^-}^{\gamma,\beta}\widetilde{Y}(\delta,\upsilon)\right]\\&\subseteq_{\mathbb{F}}\frac{\Gamma(\gamma+1)}{8(n-\delta)^{\gamma}}\left[\mathcal{J}_{\delta^+}^{\gamma}\widetilde{Y}(n,\upsilon)\oplus\mathcal{J}_{\delta^+}^{\gamma}\widetilde{Y}(n,d)\oplus\mathcal{J}_{n^-}^{\gamma}\widetilde{Y}(\delta,\upsilon)\oplus\mathcal{J}_{n^-}^{\gamma}\widetilde{Y}(\delta,d)\right]\\&\oplus\frac{\Gamma(\beta+1)}{8(d-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\widetilde{Y}(\delta,d)\oplus\mathcal{J}_{d^-}^{\beta}\widetilde{Y}(n,\upsilon)\oplus\mathcal{J}_{\upsilon^+}^{\beta}\widetilde{Y}(n,d)\oplus\mathcal{J}_{d^-}^{\beta}\widetilde{Y}(\delta,\upsilon)\right]\\&\subseteq_{\mathbb{F}}\frac{\widetilde{Y}(\delta,\upsilon)\oplus\widetilde{Y}(n,\upsilon)\oplus\widetilde{Y}(\delta,d)\oplus\widetilde{Y}(n,d)}{4}.\end{aligned}\tag{29}$$

Proof. Let $\widetilde{Y}:[\delta,n]\to\mathbb{F}_0$ be a coordinated *UD*-convex *FNVM*. Then, by hypothesis, we have

$$4\widetilde{Y}\left(\frac{\delta+n}{2},\frac{\upsilon+d}{2}\right)\supseteq_{\mathbb{F}}\widetilde{Y}(\varepsilon\delta+(1-\varepsilon)n,\varepsilon\upsilon+(1-\varepsilon)d)\oplus\widetilde{Y}((1-\varepsilon)\delta+\varepsilon n,(1-\varepsilon)\upsilon+\varepsilon d).$$

By using Theorem 7, for every $\mathfrak{z}\in[0,1]$, we have

$$\begin{aligned}&4Y_*\left(\left(\frac{\delta+n}{2},\frac{\upsilon+d}{2}\right),\mathfrak{z}\right)\\&\leq Y_*((\varepsilon\delta+(1-\varepsilon)n,\varepsilon\upsilon+(1-\varepsilon)d),\mathfrak{z})+Y_*\left(((1-\varepsilon)\delta+\varepsilon n,(1-\varepsilon)\upsilon+\varepsilon d),\mathfrak{z}\right),\\&4Y^*\left(\left(\frac{\delta+n}{2},\frac{\upsilon+d}{2}\right),\mathfrak{z}\right)\\&\geq Y^*((\varepsilon\delta+(1-\varepsilon)n,\varepsilon\upsilon+(1-\varepsilon)d),\mathfrak{z})+Y^*\left(((1-\varepsilon)\delta+\varepsilon n,(1-\varepsilon)\upsilon+\varepsilon d),\mathfrak{z}\right).\end{aligned}$$

By using Lemma 1, we have

$$\begin{aligned}2Y_*\left(\left(\theta,\frac{\upsilon+d}{2}\right),\mathfrak{z}\right)&\leq Y_*((\theta,\varepsilon\upsilon+(1-\varepsilon)d),\mathfrak{z})+Y_*((\theta,(1-\varepsilon)\upsilon+\varepsilon d),\mathfrak{z}),\\2Y^*\left(\left(\theta,\frac{\upsilon+d}{2}\right),\mathfrak{z}\right)&\geq Y^*((\theta,\varepsilon\upsilon+(1-\varepsilon)d),\mathfrak{z})+Y^*((\theta,(1-\varepsilon)\upsilon+\varepsilon d),\mathfrak{z}),\end{aligned}\tag{30}$$

and

$$\begin{aligned}2Y_*\left(\left(\frac{\delta+n}{2},\psi\right),\mathfrak{z}\right)&\leq Y_*((\varepsilon\delta+(1-\varepsilon)n,\psi),\mathfrak{z})+Y_*\left(((1-\varepsilon)\delta+tn,\psi),\mathfrak{z}\right),\\2Y^*\left(\left(\frac{\delta+n}{2},\psi\right),\mathfrak{z}\right)&\geq Y^*((\varepsilon\delta+(1-\varepsilon)n,\psi),\mathfrak{z})+Y^*\left(((1-\varepsilon)\delta+tn,\psi),\mathfrak{z}\right).\end{aligned}\tag{31}$$

From (30) and (31), we have

$$\begin{aligned}&2\left[Y_*\left(\left(\theta,\frac{\upsilon+d}{2}\right),\mathfrak{z}\right),Y^*\left(\left(\theta,\frac{\upsilon+d}{2}\right),\mathfrak{z}\right)\right]\\&\supseteq_I[Y_*((\theta,\varepsilon\upsilon+(1-\varepsilon)d),\mathfrak{z}),Y^*((\theta,\varepsilon\upsilon+(1-\varepsilon)d),\mathfrak{z})]\\&+[Y_*((\theta,(1-\varepsilon)\upsilon+\varepsilon d),\mathfrak{z}),Y^*((\theta,(1-\varepsilon)\upsilon+\varepsilon d),\mathfrak{z})],\end{aligned}$$

and

$$\begin{aligned}&2\left[Y_*\left(\left(\frac{\delta+n}{2},\psi\right),\mathfrak{z}\right),Y^*\left(\left(\frac{\delta+n}{2},\psi\right),\mathfrak{z}\right)\right]\\&\supseteq_I[Y_*((\varepsilon\delta+(1-\varepsilon)n,\psi),\mathfrak{z}),Y^*((\varepsilon\delta+(1-\varepsilon)n,\psi),\mathfrak{z})]\end{aligned}$$

$$+[\Upsilon_*((\varepsilon\mathfrak{d}+(1-\varepsilon)\mathfrak{n},\psi),\mathfrak{z}),\Upsilon^*((\varepsilon\mathfrak{d}+(1-\varepsilon)\mathfrak{n},\psi),\mathfrak{z})],$$

It follows that

$$\Upsilon_\mathfrak{z}\left(\theta,\frac{\upsilon+d}{2}\right)\supseteq_I \Upsilon_\mathfrak{z}(\theta,\varepsilon\upsilon+(1-\varepsilon)d)+\Upsilon_\mathfrak{z}(\theta,(1-\varepsilon)\upsilon+\varepsilon d), \quad (32)$$

and

$$\Upsilon_\mathfrak{z}\left(\frac{\mathfrak{d}+\mathfrak{n}}{2},\psi\right)\supseteq_I \Upsilon_\mathfrak{z}(\varepsilon\mathfrak{d}+(1-\varepsilon)\mathfrak{n},\psi)+\Upsilon_\mathfrak{z}(\varepsilon\mathfrak{d}+(1-\varepsilon)\mathfrak{n},\psi). \quad (33)$$

Since $\Upsilon_\mathfrak{z}(\theta,.)$ and $\Upsilon_\mathfrak{z}(.,\psi)$, both are coordinated *UD*-convex-*IVMs*, then from inequality (15), for every $\mathfrak{z}\in[0,1]$, inequalities (32) and (43) we have

$$\Upsilon_{\mathfrak{z}_\theta}\left(\frac{\upsilon+d}{2}\right)\supseteq_I \frac{\Gamma(\beta+1)}{2(d-\upsilon)^\beta}\left[\mathcal{J}^\beta_{\upsilon^+}\Upsilon_{\mathfrak{z}_\theta}(d)+\mathcal{J}^\beta_{d^-}\Upsilon_{\mathfrak{z}_\theta}(\upsilon)\right]\supseteq_I \frac{\Upsilon_{\mathfrak{z}_\theta}(\upsilon)+\Upsilon_{\mathfrak{z}_\theta}(d)}{2}. \quad (34)$$

and

$$\Upsilon_{\mathfrak{z}_\psi}\left(\frac{\mathfrak{d}+\mathfrak{n}}{2}\right)\supseteq_I \frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\mathfrak{d})^\gamma}\left[\mathcal{J}^\gamma_{\mathfrak{d}^+}\Upsilon_{\mathfrak{z}_\psi}(\mathfrak{n})+\mathcal{J}^\gamma_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}_\psi}(\mathfrak{d})\right]\supseteq_I \frac{\Upsilon_{\mathfrak{z}_\psi}(\mathfrak{d})+\Upsilon_{\mathfrak{z}_\psi}(\mathfrak{n})}{2} \quad (35)$$

Since $\Upsilon_{\mathfrak{z}_\theta}(w)=\Upsilon_\mathfrak{z}(\theta,w)$, then (34) can be written as

$$\Upsilon_\mathfrak{z}\left(\theta,\frac{\upsilon+d}{2}\right)\supseteq_I \frac{\Gamma(\beta+1)}{2(d-\upsilon)^\beta}\left[\mathcal{J}^\gamma_{\upsilon^+}\Upsilon_\mathfrak{z}(\theta,d)+\mathcal{J}^\gamma_{d^-}\Upsilon_\mathfrak{z}(\theta,\upsilon)\right]\supseteq_I \frac{\Upsilon_\mathfrak{z}(\theta,\upsilon)+\Upsilon_\mathfrak{z}(\theta,d)}{2}. \quad (36)$$

That is

$$\Upsilon_\mathfrak{z}\left(\theta,\frac{\upsilon+d}{2}\right)\supseteq_I \frac{\beta}{2(d-\upsilon)^\beta}\left[\int_\upsilon^d (d-s)^{\beta-1}\Upsilon_\mathfrak{z}(\theta,s)ds + \int_\upsilon^d (s-\upsilon)^{\beta-1}\Upsilon_\mathfrak{z}(\theta,s)ds\right]\supseteq_I \frac{\Upsilon_\mathfrak{z}(\theta,\upsilon)+\Upsilon_\mathfrak{z}(\theta,d)}{2}.$$

Multiplying double inequality (36) by $\frac{\gamma(\mathfrak{n}-\theta)^{\gamma-1}}{2(\mathfrak{n}-\mathfrak{d})^\gamma}$ and integrating with respect to θ over $[\mathfrak{d},\mathfrak{n}]$, we have

$$\begin{aligned}&\frac{\gamma}{2(\mathfrak{n}-\mathfrak{d})^\gamma}\int_\mathfrak{d}^\mathfrak{n}\Upsilon_\mathfrak{z}\left(\theta,\frac{\upsilon+d}{2}\right)(\mathfrak{n}-\theta)^{\gamma-1}d\theta\\ &\supseteq_I \int_\mathfrak{d}^\mathfrak{n}\int_\upsilon^d(\mathfrak{n}-\theta)^{\gamma-1}(d-s)^{\beta-1}\Upsilon_\mathfrak{z}(\theta,s)\,dsd\theta+\int_\mathfrak{d}^\mathfrak{n}\int_\upsilon^d(\mathfrak{n}-\theta)^{\gamma-1}(s-\upsilon)^{\beta-1}\Upsilon_\mathfrak{z}(\theta,s)\,dsd\theta\\ &\supseteq_I \frac{\gamma}{4(\mathfrak{n}-\mathfrak{d})^\gamma}\left[\int_\mathfrak{d}^\mathfrak{n}(\mathfrak{n}-\theta)^{\gamma-1}\Upsilon_\mathfrak{z}(\theta,\upsilon)d\theta+\int_\mathfrak{d}^\mathfrak{n}(\mathfrak{n}-\theta)^{\gamma-1}\Upsilon_\mathfrak{z}(\theta,d)d\theta\right]. \end{aligned} \quad (37)$$

Again, multiplying double inequality (36) by $\frac{\gamma(\theta-\mathfrak{d})^{\gamma-1}}{2(\mathfrak{n}-\mathfrak{d})^\gamma}$ and integrating with respect to θ over $[\mathfrak{d},\mathfrak{n}]$, we have

$$\begin{aligned}&\frac{\gamma}{2(\mathfrak{n}-\mathfrak{d})^\gamma}\int_\mathfrak{d}^\mathfrak{n}\Upsilon_\mathfrak{z}\left(\theta,\frac{\upsilon+d}{2}\right)(\mathfrak{n}-\theta)^{\gamma-1}d\theta\\ &\supseteq_I \frac{\gamma\beta}{4(\mathfrak{n}-\mathfrak{d})^\gamma(d-\upsilon)^\beta}\int_\mathfrak{d}^\mathfrak{n}\int_\upsilon^d(\theta-\mathfrak{d})^{\gamma-1}(d-s)^{\beta-1}\Upsilon_\mathfrak{z}(\theta,s)\,dsd\theta\\ &+\frac{\gamma\beta}{4(\mathfrak{n}-\mathfrak{d})^\gamma(d-\upsilon)^\beta}\int_\mathfrak{d}^\mathfrak{n}\int_\upsilon^d(\theta-\mathfrak{d})^{\gamma-1}(s-\upsilon)^{\beta-1}\Upsilon_\mathfrak{z}(\theta,s)\,dsd\theta\\ &\supseteq_I \frac{\gamma}{4(\mathfrak{n}-\mathfrak{d})^\gamma}\left[\int_\mathfrak{d}^\mathfrak{n}(\theta-\mathfrak{d})^{\gamma-1}\Upsilon_\mathfrak{z}(\theta,\upsilon)d\theta+\int_\mathfrak{d}^\mathfrak{n}(\theta-\mathfrak{d})^{\gamma-1}\Upsilon_\mathfrak{z}(\theta,d)d\theta\right]. \end{aligned} \quad (38)$$

From (37), we have

$$\begin{aligned}&\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\mathfrak{d})^\gamma}\left[\mathcal{J}^\gamma_{\mathfrak{d}^+}\Upsilon_\mathfrak{z}\left(\mathfrak{n},\frac{\upsilon+d}{2}\right)\right]\\ &\supseteq_I \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^\gamma(d-\upsilon)^\beta}\left[\mathcal{J}^{\gamma,\beta}_{\mathfrak{d}^+,\upsilon^+}\Upsilon_\mathfrak{z}(\mathfrak{n},d)+\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\upsilon^+}\Upsilon_\mathfrak{z}(\mathfrak{n},\upsilon)\right]\\ &\supseteq_I \frac{\Gamma(\gamma+1)}{4(\mathfrak{n}-\mathfrak{d})^\gamma}\left[\mathcal{J}^\gamma_{\mathfrak{d}^+}\Upsilon_\mathfrak{z}(\mathfrak{n},\upsilon)+\mathcal{J}^\gamma_{\mathfrak{d}^+}\Upsilon_\mathfrak{z}(\mathfrak{n},d)\right]. \end{aligned} \quad (39)$$

From (38), we have

$$\begin{aligned}&\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\mathfrak{d})^\gamma}\left[\mathcal{J}^\gamma_{\mathfrak{n}^-}\Upsilon_\mathfrak{z}\left(\mathfrak{d},\frac{\upsilon+d}{2}\right)\right]\\ &\supseteq_I \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^\gamma(d-\upsilon)^\beta}\left[\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\upsilon^+}\Upsilon_\mathfrak{z}(\mathfrak{d},d)+\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,d^-}\Upsilon_\mathfrak{z}(\mathfrak{d},\upsilon)\right]\\ &\supseteq_I \frac{\Gamma(\gamma+1)}{4(\mathfrak{n}-\mathfrak{d})^\gamma}\left[\mathcal{J}^\gamma_{\mathfrak{n}^-}\Upsilon_\mathfrak{z}(\mathfrak{d},\upsilon)+\mathcal{J}^\gamma_{\mathfrak{n}^-}\Upsilon_\mathfrak{z}(\mathfrak{d},d)\right]. \end{aligned} \quad (40)$$

Since from $\mathfrak{z}$-cuts, we obtain the collection of *IVMs* $\Upsilon_\mathfrak{z}:\Delta\to\mathbb{R}^+_I$, then we have

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$$\frac{\Gamma(ɤ+1)}{2(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{ƌ^+}\widetilde{Y}\left(n,\frac{υ+ȡ}{2}\right)\right] \supseteq_{\mathbb{F}} \frac{\Gamma(ɤ+1)\Gamma(\beta+1)}{4(n-ƌ)^{ɤ}(ȡ-υ)^{\beta}}\left[\mathcal{J}^{ɤ,\beta}_{ƌ^+,υ^+}\widetilde{Y}(n,ȡ)\oplus \mathcal{J}^{ɤ,\beta}_{n^-,υ^+}\widetilde{Y}(n,υ)\right] \supseteq_{\mathbb{F}} \frac{\Gamma(ɤ+1)}{4(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{ƌ^+}\widetilde{Y}(n,υ)\oplus \mathcal{J}^{ɤ}_{ƌ^+}\widetilde{Y}(n,ȡ)\right]. \tag{41}$$

And

$$\frac{\Gamma(ɤ+1)}{2(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{n^-}\widetilde{Y}\left(ƌ,\frac{υ+ȡ}{2}\right)\right] \supseteq_{\mathbb{F}} \frac{\Gamma(ɤ+1)\Gamma(\beta+1)}{4(n-ƌ)^{ɤ}(ȡ-υ)^{\beta}}\left[\mathcal{J}^{ɤ,\beta}_{n^-,υ^+}\widetilde{Y}(ƌ,ȡ)\oplus \mathcal{J}^{ɤ,\beta}_{n^-,ȡ^-}\widetilde{Y}(ƌ,υ)\right] \supseteq_{\mathbb{F}} \frac{\Gamma(ɤ+1)}{4(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{n^-}\widetilde{Y}(ƌ,υ)\oplus \mathcal{J}^{ɤ}_{n^-}\widetilde{Y}(ƌ,ȡ)\right]. \tag{42}$$

Similarly, since $\widetilde{Y}_{\psi}(z)=\widetilde{Y}(z,\psi)$ then, from the (35), (41) and (42), we have

$$\frac{\Gamma(\beta+1)}{2(ȡ-υ)^{\beta}}\left[\mathcal{J}^{\beta}_{υ^+}\widetilde{Y}\left(\frac{ƌ+n}{2},ȡ\right)\right] \supseteq_{\mathbb{F}} \frac{\Gamma(ɤ+1)\Gamma(\beta+1)}{4(n-ƌ)^{ɤ}(ȡ-υ)^{\beta}}\left[\mathcal{J}^{ɤ,\beta}_{ƌ^+,υ^+}\widetilde{Y}(n,ȡ)\oplus \mathcal{J}^{ɤ,\beta}_{n^-,υ^+}\widetilde{Y}(ƌ,ȡ)\right] \supseteq_{\mathbb{F}} \frac{\Gamma(\beta+1)}{4(ȡ-υ)^{\beta}}\left[\mathcal{J}^{\beta}_{υ^+}\widetilde{Y}(ƌ,ȡ)\oplus \mathcal{J}^{\beta}_{υ^+}\widetilde{Y}(n,ȡ)\right]. \tag{43}$$

And

$$\frac{\Gamma(\beta+1)}{2(ȡ-υ)^{ɤ}}\left[\mathcal{J}^{\beta}_{ȡ^-}\widetilde{Y}\left(\frac{ƌ+n}{2},υ\right)\right] \supseteq_{\mathbb{F}} \frac{\Gamma(ɤ+1)\Gamma(\beta+1)}{4(n-ƌ)^{ɤ}(ȡ-υ)^{\beta}}\left[\mathcal{J}^{ɤ,\beta}_{ƌ^+,ȡ^-}\widetilde{Y}(n,υ)\oplus \mathcal{J}^{ɤ,\beta}_{n^-,ȡ^-}\widetilde{Y}(ƌ,υ)\right] \supseteq_{\mathbb{F}} \frac{\Gamma(\beta+1)}{4(ȡ-υ)^{\beta}}\left[\mathcal{J}^{\beta}_{ȡ^-}\widetilde{Y}(ƌ,υ)\oplus \mathcal{J}^{\beta}_{ȡ^-}\widetilde{Y}(n,υ)\right]. \tag{44}$$

The second, third, and fourth inequalities of (28) will be the consequence of adding the inequalities (41), (42), (43) and (44).

Now, for any $ʒ\in[0,1]$, we have inequality (15)'s left portion.

$$Y_{ʒ}\left(\frac{ƌ+n}{2},\frac{υ+ȡ}{2}\right)\supseteq_{I}\frac{\Gamma(\beta+1)}{2(ȡ-υ)^{\beta}}\left[\mathcal{J}^{\beta}_{υ^+}Y_{ʒ}\left(\frac{ƌ+n}{2},ȡ\right)+\mathcal{J}^{\beta}_{ȡ^-}Y_{ʒ}\left(\frac{ƌ+n}{2},υ\right)\right] \tag{45}$$

And

$$Y_{ʒ}\left(\frac{ƌ+n}{2},\frac{υ+ȡ}{2}\right)\supseteq_{I}\frac{\Gamma(ɤ+1)}{2(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{ƌ^+}Y_{ʒ}\left(n,\frac{υ+ȡ}{2}\right)+\mathcal{J}^{ɤ}_{n^-}Y_{ʒ}\left(ƌ,\frac{υ+ȡ}{2}\right)\right]. \tag{46}$$

The following inequality is created by adding the two inequalities (45 and 46):

$$Y_{ʒ}\left(\frac{ƌ+n}{2},\frac{υ+ȡ}{2}\right)\supseteq_{I}\frac{\Gamma(ɤ+1)}{4(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{ƌ^+}Y_{ʒ}\left(n,\frac{υ+ȡ}{2}\right)+\mathcal{J}^{ɤ}_{n^-}Y_{ʒ}\left(ƌ,\frac{υ+ȡ}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(ȡ-υ)^{\beta}}\left[\mathcal{J}^{\beta}_{υ^+}Y_{ʒ}\left(\frac{ƌ+n}{2},ȡ\right)+\mathcal{J}^{\beta}_{ȡ^-}Y_{ʒ}\left(\frac{ƌ+n}{2},υ\right)\right].$$

Similarly, since we obtain the set of *IVMs* $Y_{ʒ}:\Delta\to\mathbb{R}^{+}_{I}$ for for $ʒ\in[0,1]$, the inequality can be expressed as follows:

$$\widetilde{Y}\left(\frac{ƌ+n}{2},\frac{υ+ȡ}{2}\right)\supseteq_{\mathbb{F}}\frac{\Gamma(ɤ+1)}{4(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{ƌ^+}\widetilde{Y}\left(n,\frac{υ+ȡ}{2}\right)\oplus\mathcal{J}^{ɤ}_{n^-}\widetilde{Y}\left(ƌ,\frac{υ+ȡ}{2}\right)\right]\oplus\frac{\Gamma(\beta+1)}{4(ȡ-υ)^{\beta}}\left[\mathcal{J}^{\beta}_{υ^+}\widetilde{Y}\left(\frac{ƌ+n}{2},ȡ\right)\oplus\mathcal{J}^{\beta}_{ȡ^-}\widetilde{Y}\left(\frac{ƌ+n}{2},υ\right)\right]. \tag{47}$$

The first inequality of (28) is this one.

Now, for any $ʒ\in[0,1]$, we have inequality (15)'s right portion:

$$\frac{\Gamma(\beta+1)}{2(ȡ-υ)^{\beta}}\left[\mathcal{J}^{\beta}_{υ^+}Y_{ʒ}(ƌ,ȡ)+\mathcal{J}^{\beta}_{ȡ^-}Y_{ʒ}(ƌ,υ)\right]\supseteq_{I}\frac{Y_{ʒ}(ƌ,υ)+Y_{ʒ}(ƌ,ȡ)}{2}. \tag{48}$$

$$\frac{\Gamma(\beta+1)}{2(ȡ-υ)^{\beta}}\left[\mathcal{J}^{\beta}_{υ^+}Y_{ʒ}(n,ȡ)+\mathcal{J}^{\beta}_{ȡ^-}Y_{ʒ}(n,υ)\right]\supseteq_{I}\frac{Y_{ʒ}(n,υ)+Y_{ʒ}(n,ȡ)}{2}. \tag{49}$$

$$\frac{\Gamma(ɤ+1)}{2(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{ƌ^+}Y_{ʒ}(n,υ)+\mathcal{J}^{ɤ}_{n^-}Y_{ʒ}(ƌ,υ)\right]\supseteq_{I}\frac{Y_{ʒ}(ƌ,υ)+Y_{ʒ}(n,υ)}{2} \tag{50}$$

$$\frac{\Gamma(ɤ+1)}{2(n-ƌ)^{ɤ}}\left[\mathcal{J}^{ɤ}_{ƌ^+}Y_{ʒ}(n,ȡ)+\mathcal{J}^{ɤ}_{n^-}Y_{ʒ}(ƌ,ȡ)\right]\supseteq_{I}\frac{Y_{ʒ}(ƌ,ȡ)+Y_{ʒ}(n,ȡ)}{2} \tag{51}$$

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Summing inequalities (48), (49), (50) and (51), and then taking multiplication of the resultant with $\frac{1}{4}$, we have

$$\frac{\Gamma(\gamma+1)}{8(n-\delta)^{\gamma}}\left[\mathcal{I}_{\delta^+}^{\gamma}\mathrm{Y}_{\mathfrak{z}}(n,\upsilon)+\mathcal{I}_{n^-}^{\gamma}\mathrm{Y}_{\mathfrak{z}}(\delta,\upsilon)+\mathcal{I}_{\delta^+}^{\gamma}\mathrm{Y}_{\mathfrak{z}}(n,d)+\mathcal{I}_{n^-}^{\gamma}\mathrm{Y}_{\mathfrak{z}}(\delta,d)\right]$$
$$+\frac{\Gamma(\beta+1)}{8(d-\upsilon)^{\beta}}\left[\mathcal{I}_{\upsilon^+}^{\beta}\mathrm{Y}_{\mathfrak{z}}(\delta,d)+\mathcal{I}_{d^-}^{\beta}\mathrm{Y}_{\mathfrak{z}}(\delta,\upsilon)+\mathcal{I}_{\upsilon^+}^{\beta}\mathrm{Y}_{\mathfrak{z}}(n,d)+\mathcal{I}_{d^-}^{\beta}\mathrm{Y}_{\mathfrak{z}}(n,\upsilon)\right]$$
$$\supseteq_I\frac{\mathrm{Y}_{\mathfrak{z}}(\delta,\upsilon)+\mathrm{Y}_{\mathfrak{z}}(\delta,d)+\mathrm{Y}_{\mathfrak{z}}(n,\upsilon)+\mathrm{Y}_{\mathfrak{z}}(n,d)}{4}.$$

Since we receive the collection of $IVMs$ $\mathrm{Y}_{\mathfrak{z}}:\Delta\rightarrow\mathbb{R}_I^+$ from $\mathfrak{z}$-cuts, we have

$$\frac{\Gamma(\gamma+1)}{8(n-\delta)^{\gamma}}\left[\mathcal{I}_{\delta^+}^{\gamma}\widetilde{\mathrm{Y}}(n,\upsilon)\oplus\mathcal{I}_{n^-}^{\gamma}\widetilde{\mathrm{Y}}(\delta,\upsilon)\oplus\mathcal{I}_{\delta^+}^{\gamma}\widetilde{\mathrm{Y}}(n,d)\oplus\mathcal{I}_{n^-}^{\gamma}\widetilde{\mathrm{Y}}(\delta,d)\right]$$
$$\oplus\frac{\Gamma(\beta+1)}{8(d-\upsilon)^{\beta}}\left[\mathcal{I}_{\upsilon^+}^{\beta}\widetilde{\mathrm{Y}}(\delta,d)\oplus\mathcal{I}_{d^-}^{\beta}\widetilde{\mathrm{Y}}(\delta,\upsilon)\oplus\mathcal{I}_{\upsilon^+}^{\beta}\widetilde{\mathrm{Y}}(n,d)\oplus\mathcal{I}_{d^-}^{\beta}\widetilde{\mathrm{Y}}(n,\upsilon)\right]$$
$$\supseteq_{\mathbb{F}}\frac{\widetilde{\mathrm{Y}}(\delta,\upsilon)\oplus\widetilde{\mathrm{Y}}(\delta,d)\oplus\widetilde{\mathrm{Y}}(n,\upsilon)\oplus\widetilde{\mathrm{Y}}(n,d)}{4}. \tag{52}$$

This is the final inequality of (28) and the conclusion has been established.

Example 3. We assume the $FNVMs$ $\widetilde{\mathrm{Y}}:[0,2]\times[0,2]\rightarrow\mathbb{F}_0$ defined by,

$$\mathrm{Y}(\theta,\psi)(\sigma)=\begin{cases}\frac{\sigma-(2-\sqrt{\theta})(2-\sqrt{\psi})}{4-(2-\sqrt{\theta})(2-\sqrt{\psi})}, & \sigma\in\left[(2-\sqrt{\theta})(2-\sqrt{\psi}),4\right]\\ \frac{(2+\sqrt{\theta})(2+\sqrt{\psi})-\sigma}{(2+\sqrt{\theta})(2+\sqrt{\psi})-4}, & \sigma\in\left(4,(2+\sqrt{\theta})(2+\sqrt{\psi})\right]\\ 0, & \text{otherwise,}\end{cases} \tag{53}$$

then, for each $\mathfrak{z}\in[0,1]$, we have $\mathrm{Y}_{\mathfrak{z}}(\theta,\psi)=\left[(1-\mathfrak{z})(2-\sqrt{\theta})(2-\sqrt{\psi})+4\mathfrak{z},(1-\mathfrak{z})(2+\sqrt{\theta})(2+\sqrt{\psi})+4\mathfrak{z}\right]$. Since end point functions $\mathrm{Y}_*((\theta,\psi),\mathfrak{z})$, $\mathrm{Y}^*((\theta,\psi),\mathfrak{z})$ are coordinate concave functions for each $\mathfrak{z}\in[0,1]$. Hence $\widetilde{\mathrm{Y}}(\theta,\psi)$ is coordinate concave $FNVM$.

$$\mathrm{Y}_{\mathfrak{z}}\left(\frac{\delta+n}{2},\frac{\upsilon+d}{2}\right)=[(1-\mathfrak{z})+4\mathfrak{z},9(1-\mathfrak{z})+4\mathfrak{z}],$$
$$\frac{\Gamma(\gamma+1)}{4(n-\delta)^{\gamma}}\left[\mathcal{I}_{\delta^+}^{\gamma}\widetilde{\mathrm{Y}}\left(n,\frac{\upsilon+d}{2}\right)\oplus\mathcal{I}_{n^-}^{\gamma}\widetilde{\mathrm{Y}}\left(\delta,\frac{\upsilon+d}{2}\right)\right]\oplus\frac{\Gamma(\beta+1)}{4(d-\upsilon)^{\beta}}\left[\mathcal{I}_{\upsilon^+}^{\beta}\widetilde{\mathrm{Y}}\left(\frac{\delta+n}{2},d\right)\oplus\mathcal{I}_{d^-}^{\beta}\widetilde{\mathrm{Y}}\left(\frac{\delta+n}{2},\upsilon\right)\right]$$
$$=\left[(1-\mathfrak{z})\left(2-\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{8}\pi\right)+4\mathfrak{z},(1-\mathfrak{z})\left(2+\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{8}\pi\right)+4\mathfrak{z}\right]$$
$$\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\delta)^{\gamma}(d-\upsilon)^{\beta}}\left[\mathcal{I}_{\delta^+,\upsilon^+}^{\gamma,\beta}\mathrm{Y}_{\mathfrak{z}}(n,d)\oplus\mathcal{I}_{\delta^+,d^-}^{\gamma,\beta}\mathrm{Y}_{\mathfrak{z}}(n,\upsilon)\oplus\mathcal{I}_{n^-,\upsilon^+}^{\gamma,\beta}\mathrm{Y}_{\mathfrak{z}}(\delta,d)\oplus\mathcal{I}_{n^-,d^-}^{\gamma,\beta}\mathrm{Y}_{\mathfrak{z}}(\delta,\upsilon)\right]$$
$$=\left[(1-\mathfrak{z})\left(\frac{33}{8}-\sqrt{2}-\frac{\sqrt{2}}{2}\pi+\frac{\pi}{8}+\frac{\pi^2}{32}\right)+4\mathfrak{z},(1-\mathfrak{z})\left(\frac{33}{8}+\sqrt{2}+\frac{\sqrt{2}}{2}\pi+\frac{\pi}{8}+\frac{\pi^2}{32}\right)+4\mathfrak{z}\right]$$
$$\frac{\Gamma(\gamma+1)}{8(n-\delta)^{\gamma}}\left[\mathcal{I}_{\delta^+}^{\gamma}\widetilde{\mathrm{Y}}(n,\upsilon)\oplus\mathcal{I}_{\delta^+}^{\gamma}\widetilde{\mathrm{Y}}(n,d)\oplus\mathcal{I}_{n^-}^{\gamma}\widetilde{\mathrm{Y}}(\delta,\upsilon)\oplus\mathcal{I}_{n^-}^{\gamma}\widetilde{\mathrm{Y}}(\delta,d)\right]$$
$$\oplus\frac{\Gamma(\beta+1)}{8(d-\upsilon)^{\beta}}\left[\mathcal{I}_{\upsilon^+}^{\beta}\widetilde{\mathrm{Y}}(\delta,d)\oplus\mathcal{I}_{\upsilon^+}^{\beta}\widetilde{\mathrm{Y}}(n,d)\oplus\mathcal{I}_{d^-}^{\beta}\widetilde{\mathrm{Y}}(\delta,\upsilon)\oplus\mathcal{I}_{d^-}^{\beta}\widetilde{\mathrm{Y}}(n,\upsilon)\right]$$
$$=\left[\frac{34\sqrt{2}+(\sqrt{2}-4)\pi-24}{8\sqrt{2}}(1-\mathfrak{z})+4\mathfrak{z},\frac{34\sqrt{2}+(\sqrt{2}+4)\pi+24}{8\sqrt{2}}(1-\mathfrak{z})+4\mathfrak{z}\right]$$
$$\frac{\mathrm{Y}_{\mathfrak{z}}(\upsilon,n)+\mathrm{Y}_{\mathfrak{z}}(\sigma,n)+\mathrm{Y}_{\mathfrak{z}}(\upsilon,d)+\mathrm{Y}_{\mathfrak{z}}(\sigma,d)}{4}=\left[(1-\mathfrak{z})\left(\frac{9}{2}-2\sqrt{2}\right)+4\mathfrak{z},(1-\mathfrak{z})\left(\frac{9}{2}+2\sqrt{2}\right)+4\mathfrak{z}\right].$$

That is

$$[(1-\mathfrak{z})+4\mathfrak{z},9(1-\mathfrak{z})+4\mathfrak{z}]\supseteq_I\left[(1-\mathfrak{z})\left(2-\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{8}\pi\right)+4\mathfrak{z},(1-\mathfrak{z})\left(2+\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{8}\pi\right)+4\mathfrak{z}\right]$$
$$\supseteq_I\left[(1-\mathfrak{z})\left(\frac{33}{8}-\sqrt{2}-\frac{\sqrt{2}}{2}\pi+\frac{\pi}{8}+\frac{\pi^2}{32}\right)+4\mathfrak{z},(1-\mathfrak{z})\left(\frac{33}{8}+\sqrt{2}+\frac{\sqrt{2}}{2}\pi+\frac{\pi}{8}+\frac{\pi^2}{32}\right)+4\mathfrak{z}\right]$$
$$\supseteq_I\left[\frac{34\sqrt{2}+(\sqrt{2}-4)\pi-24}{8\sqrt{2}}(1-\mathfrak{z})+4\mathfrak{z},\frac{34\sqrt{2}+(\sqrt{2}+4)\pi+24}{8\sqrt{2}}(1-\mathfrak{z})+4\mathfrak{z}\right]$$
$$\supseteq_I\frac{34\sqrt{2}+(\sqrt{2}-4)\pi-24}{8\sqrt{2}}(1-\mathfrak{z})+4\mathfrak{z},.$$

Hence, Theorem 9 has been verified.

Remark 2. If one assumes that $\curlyvee = 1$ and $\beta = 1$, then from (28), as a result, there will be inequity, see [27]:

$$\widetilde{\mathrm{Y}}\left(\frac{\eth+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right)$$
$$\supseteq_{\mathbb{F}} \frac{1}{2}\left[\frac{1}{\mathrm{n}-\eth}\int_{\eth}^{\mathrm{n}}\widetilde{\mathrm{Y}}\left(\vartheta,\frac{\upsilon+\mathrm{d}}{2}\right)d\vartheta \oplus \frac{1}{\mathrm{d}-\upsilon}\int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}\left(\frac{\eth+\mathrm{n}}{2},\psi\right)d\psi\right] \supseteq_{\mathbb{F}} \frac{1}{(\mathrm{n}-\eth)(\mathrm{d}-\upsilon)}\int_{\eth}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}(\vartheta,\psi)d\psi d\vartheta$$
$$\supseteq_{\mathbb{F}} \frac{1}{4(\mathrm{n}-\eth)}\left[\int_{\eth}^{\mathrm{n}}\widetilde{\mathrm{Y}}(\vartheta,\upsilon)\,d\vartheta \oplus \int_{\eth}^{\mathrm{n}}\widetilde{\mathrm{Y}}(\vartheta,\mathrm{d})d\vartheta\right] \oplus \frac{1}{4(\mathrm{d}-\upsilon)}\left[\int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}(\eth,\psi)d\psi \oplus \int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}(\mathrm{n},\psi)d\psi\right]$$
$$\supseteq_{\mathbb{F}} \frac{\widetilde{\mathrm{Y}}(\eth,\upsilon)\oplus\widetilde{\mathrm{Y}}(\mathrm{n},\upsilon)\oplus\widetilde{\mathrm{Y}}(\eth,\mathrm{d})\oplus\widetilde{\mathrm{Y}}(\mathrm{n},\mathrm{d})}{4}. \tag{54}$$

If one assumes that $\curlyvee = 1$ and $\beta = 1$ and $\widetilde{\mathrm{Y}}$ is coordinated left-*UD*-convex, then from (28), as a result, there will be inequity, see [22]:

$$\widetilde{\mathrm{Y}}\left(\frac{\eth+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right)$$
$$\leq_{\mathbb{F}} \frac{1}{2}\left[\frac{1}{\mathrm{n}-\eth}\int_{\eth}^{\mathrm{n}}\widetilde{\mathrm{Y}}\left(\vartheta,\frac{\upsilon+\mathrm{d}}{2}\right)d\vartheta \oplus \frac{1}{\mathrm{d}-\upsilon}\int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}\left(\frac{\eth+\mathrm{n}}{2},\psi\right)d\psi\right] \leq_{\mathbb{F}} \frac{1}{(\mathrm{n}-\eth)(\mathrm{d}-\upsilon)}\int_{\eth}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}(\vartheta,\psi)d\psi d\vartheta$$
$$\leq_{\mathbb{F}} \frac{1}{4(\mathrm{n}-\eth)}\left[\int_{\eth}^{\mathrm{n}}\widetilde{\mathrm{Y}}(\vartheta,\upsilon)\,d\vartheta \oplus \int_{\eth}^{\mathrm{n}}\widetilde{\mathrm{Y}}(\vartheta,\mathrm{d})d\vartheta\right] \oplus \frac{1}{4(\mathrm{d}-\upsilon)}\left[\int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}(\eth,\psi)d\psi \oplus \int_{\upsilon}^{\mathrm{d}}\widetilde{\mathrm{Y}}(\mathrm{n},\psi)d\psi\right]$$
$$\leq_{\mathbb{F}} \frac{\widetilde{\mathrm{Y}}(\eth,\upsilon)\oplus\widetilde{\mathrm{Y}}(\mathrm{n},\upsilon)\oplus\widetilde{\mathrm{Y}}(\eth,\mathrm{d})\oplus\widetilde{\mathrm{Y}}(\mathrm{n},\mathrm{d})}{4}. \tag{55}$$

If $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z}) \neq \mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z} = 1$, then from (28), we succeed in bringing about the upcoming inequity, see [21]:

$$\mathrm{Y}\left(\frac{\eth+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right)$$
$$\supseteq \frac{\Gamma(\curlyvee+1)}{4(\mathrm{n}-\eth)^{\curlyvee}}\left[\mathcal{J}_{\eth^+}^{\curlyvee}\mathrm{Y}\left(\mathrm{n},\frac{\upsilon+\mathrm{d}}{2}\right)+\mathcal{J}_{\mathrm{n}^-}^{\curlyvee}\mathrm{Y}\left(\eth,\frac{\upsilon+\mathrm{d}}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\mathrm{Y}\left(\frac{\eth+\mathrm{n}}{2},\mathrm{d}\right)+\mathcal{J}_{\mathrm{d}^-}^{\beta}\mathrm{Y}\left(\frac{\eth+\mathrm{n}}{2},\upsilon\right)\right]$$
$$\supseteq \frac{\Gamma(\curlyvee+1)\Gamma(\beta+1)}{4(\mathrm{n}-\eth)^{\curlyvee}(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\eth^+,\upsilon^+}^{\curlyvee,\beta}\mathrm{Y}(\mathrm{n},\mathrm{d})+\mathcal{J}_{\eth^+,\mathrm{d}^-}^{\curlyvee,\beta}\mathrm{Y}(\mathrm{n},\upsilon)+\mathcal{J}_{\mathrm{n}^-,\upsilon^+}^{\curlyvee,\beta}\mathrm{Y}(\eth,\mathrm{d})+\mathcal{J}_{\mathrm{n}^-,\mathrm{d}^-}^{\curlyvee,\beta}\mathrm{Y}(\eth,\upsilon)\right]$$
$$\supseteq \frac{\Gamma(\curlyvee+1)}{8(\mathrm{n}-\eth)^{\curlyvee}}\left[\mathcal{J}_{\eth^+}^{\curlyvee}\mathrm{Y}(\mathrm{n},\upsilon)+\mathcal{J}_{\eth^+}^{\curlyvee}\mathrm{Y}(\mathrm{n},\mathrm{d})+\mathcal{J}_{\mathrm{n}^-}^{\curlyvee}\mathrm{Y}(\eth,\upsilon)+\mathcal{J}_{\mathrm{n}^-}^{\curlyvee}\mathrm{Y}(\eth,\mathrm{d})\right].$$
$$+\frac{\Gamma(\beta+1)}{8(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\mathrm{Y}(\eth,\mathrm{d})+\mathcal{J}_{\mathrm{d}^-}^{\beta}\mathrm{Y}(\eth,\upsilon)+\mathcal{J}_{\upsilon^+}^{\beta}\mathrm{Y}(\mathrm{n},\mathrm{d})+\mathcal{J}_{\mathrm{d}^-}^{\beta}\mathrm{Y}(\mathrm{n},\upsilon)\right]$$
$$\supseteq \frac{\mathrm{Y}(\eth,\upsilon)+\mathrm{Y}(\mathrm{n},\upsilon)+\mathrm{Y}(\eth,\mathrm{d})+\mathrm{Y}(\mathrm{n},\mathrm{d})}{4}. \tag{56}$$

If $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z}) \neq \mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z} = 1$, the by (28), we succeed in bringing about the upcoming inequity, see [20]:

$$\mathrm{Y}\left(\frac{\eth+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right)$$
$$\supseteq \frac{1}{2}\left[\frac{1}{\mathrm{n}-\eth}\int_{\eth}^{\mathrm{n}}\mathrm{Y}\left(\vartheta,\frac{\upsilon+\mathrm{d}}{2}\right)d\vartheta+\frac{1}{\mathrm{d}-\upsilon}\int_{\upsilon}^{\mathrm{d}}\mathrm{Y}\left(\frac{\eth+\mathrm{n}}{2},\psi\right)d\psi\right]$$
$$\subseteq \frac{1}{(\mathrm{n}-\eth)(\mathrm{d}-\upsilon)}\int_{\eth}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}\mathrm{Y}(\vartheta,\psi)d\psi d\vartheta$$
$$\supseteq \frac{1}{4(\mathrm{n}-\eth)}\left[\int_{\eth}^{\mathrm{n}}\mathrm{Y}(\vartheta,\upsilon)\,d\vartheta+\int_{\eth}^{\mathrm{n}}\mathrm{Y}(\vartheta,\mathrm{d})d\vartheta\right]+\frac{1}{4(\mathrm{d}-\upsilon)}\left[\int_{\upsilon}^{\mathrm{d}}\mathrm{Y}(\eth,\psi)d\psi+\int_{\upsilon}^{\mathrm{d}}\mathrm{Y}(\mathrm{n},\psi)d\psi\right]$$
$$\supseteq \frac{\mathrm{Y}(\eth,\upsilon)+\mathrm{Y}(\mathrm{n},\upsilon)+\mathrm{Y}(\eth,\mathrm{d})+\mathrm{Y}(\mathrm{n},\mathrm{d})}{4}. \tag{57}$$

If $\widetilde{\mathrm{Y}}$ is coordinated right-*UD*-convex and $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z}) = \mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z} = 1$, then from (28), we succeed in bringing about the upcoming inequity, see [23]:

$$\mathrm{Y}\left(\frac{\eth+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right)$$
$$\leq \frac{\Gamma(\curlyvee+1)}{4(\mathrm{n}-\eth)^{\curlyvee}}\left[\mathcal{J}_{\eth^+}^{\curlyvee}\mathrm{Y}\left(\mathrm{n},\frac{\upsilon+\mathrm{d}}{2}\right)+\mathcal{J}_{\mathrm{n}^-}^{\curlyvee}\mathrm{Y}\left(\eth,\frac{\upsilon+\mathrm{d}}{2}\right)\right]+\frac{\Gamma(\beta+1)}{4(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\mathrm{Y}\left(\frac{\eth+\mathrm{n}}{2},\mathrm{d}\right)+\mathcal{J}_{\mathrm{d}^-}^{\beta}\mathrm{Y}\left(\frac{\eth+\mathrm{n}}{2},\upsilon\right)\right]$$
$$\leq \frac{\Gamma(\curlyvee+1)\Gamma(\beta+1)}{4(\mathrm{n}-\eth)^{\curlyvee}(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\eth^+,\upsilon^+}^{\curlyvee,\beta}\mathrm{Y}(\mathrm{n},\mathrm{d})+\mathcal{J}_{\eth^+,\mathrm{d}^-}^{\curlyvee,\beta}\mathrm{Y}(\mathrm{n},\upsilon)+\mathcal{J}_{\mathrm{n}^-,\upsilon^+}^{\curlyvee,\beta}\mathrm{Y}(\eth,\mathrm{d})+\mathcal{J}_{\mathrm{n}^-,\mathrm{d}^-}^{\curlyvee,\beta}\mathrm{Y}(\eth,\upsilon)\right]$$
$$\leq \frac{\Gamma(\curlyvee+1)}{8(\mathrm{n}-\eth)^{\curlyvee}}\left[\mathcal{J}_{\eth^+}^{\curlyvee}\mathrm{Y}(\mathrm{n},\upsilon)\mathrm{Y}\mathcal{J}_{\eth^+}^{\curlyvee}\mathrm{Y}(\mathrm{n},\mathrm{d})+\mathcal{J}_{\mathrm{n}^-}^{\curlyvee}\mathrm{Y}(\eth,\upsilon)+\mathcal{J}_{\mathrm{n}^-}^{\curlyvee}\mathrm{Y}(\eth,\mathrm{d})\right].$$
$$+\frac{\Gamma(\beta+1)}{8(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\mathrm{Y}(\eth,\mathrm{d})\tilde{+}\mathcal{J}_{\mathrm{d}^-}^{\beta}\mathrm{Y}(\eth,\upsilon)+\mathcal{J}_{\upsilon^+}^{\beta}\mathrm{Y}(\mathrm{n},\mathrm{d})+\mathcal{J}_{\mathrm{d}^-}^{\beta}\mathrm{Y}(\mathrm{n},\upsilon)\right]$$
$$\leq \frac{\mathrm{Y}(\eth,\upsilon)+\mathrm{Y}(\mathrm{n},\upsilon)+\mathrm{Y}(\eth,\mathrm{d})+\mathrm{Y}(\mathrm{n},\mathrm{d})}{4}. \tag{58}$$

Theorem 10. Let $\widetilde{\mathrm{Y}},\widetilde{\mathcal{J}}:\Delta\rightarrow\mathbb{F}_0$ be a coordinate *UD*-convex *FNVM*s on Δ. Then, from $\mathfrak{z}$-cuts, we set up the sequence of *IVM* s $\mathrm{Y}_{\mathfrak{z}},\mathcal{J}_{\mathfrak{z}}:\Delta\rightarrow\mathbb{R}_I^+$ are given by $\mathrm{Y}_{\mathfrak{z}}(\vartheta,\psi) = [\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z}),\mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})]$ and

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$\mathcal{J}_{\mathfrak{z}}(\theta,\psi) = [\mathcal{J}_*((\theta,\psi),\mathfrak{z}), \mathcal{J}^*((\theta,\psi),\mathfrak{z})]$ for all $(\theta,\psi) \in \Delta$ and for all $\mathfrak{z} \in [0,1]$. If $\tilde{\mathfrak{Y}} \otimes \tilde{\mathcal{J}} \in \mathcal{FD}_\Delta$, then following inequalities holds:

$$\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^{\gamma}(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\mathfrak{d}^+,\upsilon^+}\tilde{\mathfrak{Y}}(\mathfrak{n},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{n},\mathfrak{d})\oplus\mathcal{J}^{\gamma,\beta}_{\mathfrak{d}^+,\mathfrak{d}^-}\tilde{\mathfrak{Y}}(\mathfrak{n},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{n},\upsilon)\right]$$
$$\oplus\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^{\gamma}(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\upsilon^+}\tilde{\mathfrak{Y}}(\mathfrak{d},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{d},\mathfrak{d})\oplus\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\mathfrak{d}^-}\tilde{\mathfrak{Y}}(\mathfrak{d},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{d},\upsilon)\right]$$
$$\supseteq_{\mathbb{F}}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\tilde{K}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\oplus\frac{\gamma}{(\gamma+1)(\gamma+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\tilde{L}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$$
$$\oplus\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\tilde{\mathcal{M}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\oplus\frac{\beta}{(\beta+1)(\beta+2)}\frac{\gamma}{(\gamma+1)(\gamma+2)}\tilde{\mathcal{N}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}).\quad(59)$$

If $\tilde{\mathfrak{Y}}$ and $\tilde{\mathcal{J}}$ are both coordinated concave *FNVM*s on Δ, then inequality above can be expressed as follows:

$$\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^{\gamma}(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\mathfrak{d}^+,\upsilon^+}\tilde{\mathfrak{Y}}(\mathfrak{n},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{n},\mathfrak{d})\oplus\mathcal{J}^{\gamma,\beta}_{\mathfrak{d}^+,\mathfrak{d}^-}\tilde{\mathfrak{Y}}(\mathfrak{n},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{n},\upsilon)\right]$$
$$\oplus\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^{\gamma}(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\upsilon^+}\tilde{\mathfrak{Y}}(\mathfrak{d},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{d},\mathfrak{d})\oplus\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\mathfrak{d}^-}\tilde{\mathfrak{Y}}(\mathfrak{d},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{d},\upsilon)\right]$$
$$\subseteq_{\mathbb{F}}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\tilde{K}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\oplus\frac{\gamma}{(\gamma+1)(\gamma+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\tilde{L}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$$
$$\oplus\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\tilde{\mathcal{M}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\oplus\frac{\beta}{(\beta+1)(\beta+2)}\frac{\gamma}{(\gamma+1)(\gamma+2)}\tilde{\mathcal{N}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\quad(60)$$

where

$$\tilde{K}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}) = \tilde{\mathfrak{Y}}(\mathfrak{d},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{d},\upsilon)\oplus\tilde{\mathfrak{Y}}(\mathfrak{n},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{n},\upsilon)\oplus\tilde{\mathfrak{Y}}(\mathfrak{d},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{d},\mathfrak{d})\oplus\tilde{\mathfrak{Y}}(\mathfrak{n},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{n},\mathfrak{d}),$$
$$\tilde{L}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}) = \tilde{\mathfrak{Y}}(\mathfrak{d},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{n},\upsilon)\oplus\tilde{\mathfrak{Y}}(\mathfrak{n},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{d},\mathfrak{d})\oplus\tilde{\mathfrak{Y}}(\mathfrak{n},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{d},\upsilon)\oplus\tilde{\mathfrak{Y}}(\mathfrak{d},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{n},\mathfrak{d}),$$
$$\tilde{\mathcal{M}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}) = \tilde{\mathfrak{Y}}(\mathfrak{d},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{d},\mathfrak{d})\oplus\tilde{\mathfrak{Y}}(\mathfrak{n},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{n},\mathfrak{d})\oplus\tilde{\mathfrak{Y}}(\mathfrak{d},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{d},\upsilon)\oplus\tilde{\mathfrak{Y}}(\mathfrak{n},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{n},\upsilon),$$
$$\tilde{\mathcal{N}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}) = \tilde{\mathfrak{Y}}(\mathfrak{d},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{n},\mathfrak{d})\oplus\tilde{\mathfrak{Y}}(\mathfrak{n},\upsilon)\otimes\tilde{\mathcal{J}}(\mathfrak{d},\mathfrak{d})\oplus\tilde{\mathfrak{Y}}(\mathfrak{d},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{n},\upsilon)\oplus\tilde{\mathfrak{Y}}(\mathfrak{n},\mathfrak{d})\otimes\tilde{\mathcal{J}}(\mathfrak{d},\upsilon),$$

and for each $\mathfrak{z} \in [0,1]$, $\tilde{K}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$, $\tilde{L}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$, $\tilde{\mathcal{M}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$ and $\tilde{\mathcal{N}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$ are defined as follows:

$$K_{\mathfrak{z}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}) = [K_*((\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\mathfrak{z}), K^*((\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\mathfrak{z})],$$
$$L_{\mathfrak{z}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}) = [L_*((\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\mathfrak{z}), L^*((\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\mathfrak{z})],$$
$$\mathcal{M}_{\mathfrak{z}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}) = [\mathcal{M}_*((\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\mathfrak{z}), \mathcal{M}^*((\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\mathfrak{z})],$$
$$\mathcal{N}_{\mathfrak{z}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}) = [\mathcal{N}_*((\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\mathfrak{z}), \mathcal{N}^*((\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}),\mathfrak{z})].$$

Proof. Let $\tilde{\mathfrak{Y}}$ and $\tilde{\mathcal{J}}$ be two coordinated *UD*-convex *FNVM*s on $[\mathfrak{d},\mathfrak{n}]\times[\upsilon,\mathfrak{d}]$. Then

$$\tilde{\mathfrak{Y}}(\varepsilon\mathfrak{d}+(1-\varepsilon)\mathfrak{n}, s\upsilon+(1-s)\mathfrak{d})$$
$$\supseteq_{\mathbb{F}} \varepsilon s\tilde{\mathfrak{Y}}(\mathfrak{d},\upsilon)\oplus\varepsilon(1-s)\tilde{\mathfrak{Y}}(\mathfrak{d},\mathfrak{d})\oplus(1-\varepsilon)s\tilde{\mathfrak{Y}}(\mathfrak{n},\upsilon)\oplus(1-\varepsilon)(1-s)\tilde{\mathfrak{Y}}(\mathfrak{n},\mathfrak{d}),$$

and

$$\tilde{\mathcal{J}}(\varepsilon\mathfrak{d}+(1-\varepsilon)\mathfrak{n}, s\upsilon+(1-s)\mathfrak{d})$$
$$\supseteq_{\mathbb{F}} \varepsilon s\tilde{\mathcal{J}}(\mathfrak{d},\upsilon)\oplus\varepsilon(1-s)\tilde{\mathcal{J}}(\mathfrak{d},\mathfrak{d})\oplus(1-\varepsilon)s\tilde{\mathcal{J}}(\mathfrak{n},\upsilon)\oplus(1-\varepsilon)(1-s)\tilde{\mathcal{J}}(\mathfrak{n},\mathfrak{d}).$$

Since $\tilde{\mathfrak{Y}}$ and $\tilde{\mathcal{J}}$ both are coordinated *UD*-convex *FNVM*s, Lemma 1 states that

$$\tilde{\mathfrak{Y}}_\theta: [\upsilon,\mathfrak{d}] \to \mathbb{F}_0,\ \tilde{\mathfrak{Y}}_\theta(\psi) = \tilde{\mathfrak{Y}}(\theta,\psi),\quad \tilde{\mathcal{J}}_\theta: [\upsilon,\mathfrak{d}] \to \mathbb{F}_0,\ \tilde{\mathcal{J}}_\theta(\psi) = \tilde{\mathcal{J}}(\theta,\psi),$$

Since $\tilde{\mathfrak{Y}}_\theta$, and $\tilde{\mathcal{J}}_\theta$ are *FNVM*s, then by inequality (16), we have

$$\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\upsilon^+}\tilde{\mathfrak{Y}}_\theta(\mathfrak{d})\otimes\tilde{\mathcal{J}}_\theta(\mathfrak{d})\oplus\mathcal{J}^{\beta}_{\mathfrak{d}^-}\tilde{\mathfrak{Y}}_\theta(\upsilon)\otimes\tilde{\mathcal{J}}_\theta(\upsilon)\right]$$
$$\supseteq_{\mathbb{F}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\tilde{\mathfrak{Y}}_\theta(\upsilon)\otimes\tilde{\mathcal{J}}_\theta(\upsilon)\oplus\tilde{\mathfrak{Y}}_\theta(\mathfrak{d})\otimes\tilde{\mathcal{J}}_\theta(\mathfrak{d})\right)$$
$$\oplus\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\tilde{\mathfrak{Y}}_\theta(\upsilon)\otimes\tilde{\mathcal{J}}_\theta(\mathfrak{d})\oplus\tilde{\mathfrak{Y}}_\theta(\mathfrak{d})\otimes\tilde{\mathcal{J}}_\theta(\upsilon)\right).$$

Now for all for all $\mathfrak{z} \in [0,1]$, we have

$$\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\upsilon^+}\mathfrak{Y}_{\mathfrak{z}_\theta}(\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}_\theta}(\mathfrak{d})+\mathcal{J}^{\beta}_{\mathfrak{d}^-}\mathfrak{Y}_{\mathfrak{z}_\theta}(\upsilon)\times\mathcal{J}_{\mathfrak{z}_\theta}(\upsilon)\right]$$
$$\supseteq_I\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{Y}_{\mathfrak{z}_\theta}(\upsilon)\times\mathcal{J}_{\mathfrak{z}_\theta}(\upsilon)+\mathfrak{Y}_{\mathfrak{z}_\theta}(\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}_\theta}(\mathfrak{d})\right)$$
$$+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathfrak{Y}_{\mathfrak{z}_\theta}(\upsilon)\times\mathcal{J}_{\mathfrak{z}_\theta}(\mathfrak{d})+\mathfrak{Y}_{\mathfrak{z}_\theta}(\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}_\theta}(\upsilon)\right).$$

That is

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$$\frac{\beta}{2(\mathfrak{d}-\upsilon)^{\beta}}\left[\int_{\upsilon}^{\mathfrak{d}}(\mathfrak{d}-\psi)^{\beta-1}\Upsilon_{\mathfrak{z}}(\vartheta,\psi)\times\mathcal{J}_{\mathfrak{z}}(\vartheta,\psi)\,d\psi+\int_{\upsilon}^{\mathfrak{d}}(\psi-\upsilon)^{\beta-1}\Upsilon_{\mathfrak{z}}(\vartheta,\psi)\times\mathcal{J}_{\mathfrak{z}}(\vartheta,\psi)\,d\psi\right]$$
$$\supseteq_I\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\vartheta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\vartheta,\upsilon)+\Upsilon_{\mathfrak{z}}(\vartheta,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\vartheta,\mathfrak{d})\right)$$
$$+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\vartheta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\vartheta,\mathfrak{d})+\Upsilon_{\mathfrak{z}}(\vartheta,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\vartheta,\upsilon)\right). \quad (61)$$

Multiplying double inequality (61) by $\frac{\gamma(\mathfrak{n}-\vartheta)^{\gamma-1}}{2(\mathfrak{n}-\eth)^{\gamma}}$ and integrating with respect to ϑ over $[\eth,\mathfrak{n}]$, we get

$$\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\eth)^{\gamma}(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\eth^+,\upsilon^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\gamma,\beta}_{\eth^+,\mathfrak{d}^-}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)\right]$$
$$\supseteq_I\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)+\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\right)$$
$$+\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)\right). \quad (62)$$

Again, multiplying double inequality (61) by $\frac{\gamma(\vartheta-\eth)^{\gamma-1}}{2(\mathfrak{n}-\eth)^{\gamma}}$ and integrating with respect to ϑ over $[\eth,\mathfrak{n}]$, we gain

$$\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\eth)^{\gamma}(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\upsilon^+}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\mathfrak{d}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right]$$
$$\supseteq_I\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right)$$
$$+\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right). \quad (63)$$

Summing (62) and (63), we have

$$\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\eth)^{\gamma}(\mathfrak{d}-\upsilon)^{\beta}}\begin{bmatrix}\mathcal{J}^{\gamma,\beta}_{\eth^+,\upsilon^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\gamma,\beta}_{\eth^+,\mathfrak{d}^-}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)\\+\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\upsilon^+}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\mathcal{J}^{\gamma,\beta}_{\mathfrak{n}^-,\mathfrak{d}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\end{bmatrix}$$
$$\supseteq_I\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right)$$
$$+\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right)$$
$$+\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right)$$
$$+\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\frac{\beta}{(\beta+1)(\beta+2)}\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right). \quad (64)$$

Now, once more with the aid of integral inequality (16), we obtain the following relationship for the first two integrals on the right-hand side of (64):

$$\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right)$$
$$\supseteq_I\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)\right)$$
$$+\left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)+\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right). \quad (65)$$

$$\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right)$$
$$\supseteq_I\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\right)$$
$$+\left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right). \quad (66)$$

$$\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right)$$
$$\supseteq_I\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\right)$$
$$+\left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})+\Upsilon_{\mathfrak{z}}(\mathfrak{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right). \quad (67)$$

And

$$\frac{\Gamma(\gamma+1)}{2(\mathfrak{n}-\eth)^{\gamma}}\left(\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathfrak{n},\upsilon)+\mathcal{J}^{\gamma}_{\mathfrak{n}^-}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right)$$

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$$\supseteq_I \left(\frac{1}{2}-\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\right)\left(\mathfrak{Y}_\mathfrak{z}(\mathfrak{d},\mathfrak{d})\times\mathcal{J}_\mathfrak{z}(\mathfrak{d},\upsilon)+\mathfrak{Y}_\mathfrak{z}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_\mathfrak{z}(\mathfrak{n},\upsilon)\right)$$
$$+\left(\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\right)\left(\mathfrak{Y}_\mathfrak{z}(\mathfrak{d},\mathfrak{d})\times\mathcal{J}_\mathfrak{z}(\mathfrak{n},\upsilon)+\mathfrak{Y}_\mathfrak{z}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_\mathfrak{z}(\mathfrak{d},\upsilon)\right). \quad (68)$$

From (65)-(68), inequality (64) we have

$$\frac{\Gamma(\curlyvee+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^\curlyvee(\mathfrak{d}-\upsilon)^\beta}\begin{bmatrix}\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{d}^+,\upsilon^+}\,\mathfrak{Y}_\mathfrak{z}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}_\mathfrak{z}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{d}^+,\mathfrak{d}^-}\mathfrak{Y}_\mathfrak{z}(\mathfrak{n},\upsilon)\times\mathcal{J}_\mathfrak{z}(\mathfrak{n},\upsilon)\\+\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{n}^-,\upsilon^+}\,\mathfrak{Y}_\mathfrak{z}(\mathfrak{d},\mathfrak{d})\times\mathcal{J}_\mathfrak{z}(\mathfrak{d},\mathfrak{d})+\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{n}^-,\mathfrak{d}^-}\mathfrak{Y}_\mathfrak{z}(\mathfrak{d},\upsilon)\times\mathcal{J}_\mathfrak{z}(\mathfrak{d},\upsilon)\end{bmatrix}$$
$$\supseteq_I \left(\frac{1}{2}-\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K_\mathfrak{z}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})+\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L_\mathfrak{z}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$$
$$+\left(\frac{1}{2}-\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}_\mathfrak{z}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\mathcal{N}_\mathfrak{z}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}).$$

Since we get the collection of *IVM*s $\mathfrak{Y}_\mathfrak{z},\mathcal{J}_\mathfrak{z}:\Delta\to\mathbb{R}_I^+$ from $\mathfrak{z}$-cuts, the aforementioned inequality can be expressed as an inequality (59). The conclusion has therefore been established.

Remark 3. If one assumes that $\curlyvee=1$ and $\beta=1$, then from (59), as a result, there will be inequity, see [28]:

$$\frac{1}{(\mathfrak{n}-\mathfrak{d})(\mathfrak{d}-\upsilon)}\int_\mathfrak{d}^\mathfrak{n}\int_\upsilon^\mathfrak{d}\widetilde{\mathfrak{Y}}(\vartheta,\psi)\otimes\tilde{\mathcal{J}}(\vartheta,\psi)d\psi d\vartheta$$
$$\supseteq_\mathbb{F}\frac{1}{9}\widetilde{K}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\oplus\frac{1}{18}\left[\widetilde{L}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\oplus\widetilde{\mathcal{M}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\right]\oplus\frac{1}{36}\widetilde{\mathcal{N}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}). \quad (69)$$

If $\widetilde{\mathfrak{Y}}$ is coordinated left-*UD*-convex and one assumes that $\curlyvee=1$ and $\beta=1$, then from (59), as a result, there will be inequity, see [22]:

$$\frac{1}{(\mathfrak{n}-\mathfrak{d})(\mathfrak{d}-\upsilon)}\int_\mathfrak{d}^\mathfrak{n}\int_\upsilon^\mathfrak{d}\widetilde{\mathfrak{Y}}(\vartheta,\psi)\otimes\tilde{\mathcal{J}}(\vartheta,\psi)d\psi d\vartheta$$
$$\leq_\mathbb{F}\frac{1}{9}\widetilde{K}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\oplus\frac{1}{18}\left[\widetilde{L}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\oplus\widetilde{\mathcal{M}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\right]\oplus\frac{1}{36}\widetilde{\mathcal{N}}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}). \quad (70)$$

If $\mathfrak{Y}_*((\vartheta,\psi),\mathfrak{z})\neq\mathfrak{Y}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z}=1$ then, by (59), we succeed in bringing about the upcoming inequity, see [21]:

$$\frac{\Gamma(\curlyvee+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^\curlyvee(\mathfrak{d}-\upsilon)^\beta}\left[\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{d}^+,\upsilon^+}\mathfrak{Y}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{d}^+,\mathfrak{d}^-}\mathfrak{Y}(\mathfrak{n},\upsilon)\times\mathcal{J}(\mathfrak{n},\upsilon)\right]$$
$$+\frac{\Gamma(\curlyvee+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^\curlyvee(\mathfrak{d}-\upsilon)^\beta}\left[\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{n}^-,\upsilon^+}\mathfrak{Y}(\mathfrak{d},\mathfrak{d})\times\mathcal{J}(\mathfrak{d},\mathfrak{d})+\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{n}^-,\mathfrak{d}^-}\mathfrak{Y}(\mathfrak{d},\upsilon)\times\mathcal{J}(\mathfrak{d},\upsilon)\right]$$
$$\supseteq\left(\frac{1}{2}-\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})+\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$$
$$+\left(\frac{1}{2}-\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\mathcal{N}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}). \quad (71)$$

If $\mathfrak{Y}_*((\vartheta,\psi),\mathfrak{z})\neq\mathfrak{Y}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z}=1$, then by (59), we succeed in bringing about the upcoming inequity, see [20]:

$$\frac{1}{(\mathfrak{n}-\mathfrak{d})(\mathfrak{d}-\upsilon)}\int_\mathfrak{d}^\mathfrak{n}\int_\upsilon^\mathfrak{d}\mathfrak{Y}(\vartheta,\psi)\times\mathcal{J}(\vartheta,\psi)d\psi d\vartheta$$
$$\supseteq\frac{1}{9}K(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})+\frac{1}{18}\left[L(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})+\mathcal{M}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})\right]+\frac{1}{36}\mathcal{N}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}). \quad (72)$$

If $\mathfrak{Y}_*((\vartheta,\psi),\mathfrak{z})=\mathfrak{Y}^*((\vartheta,\psi),\mathfrak{z})$ and $\mathcal{J}_*((\vartheta,\psi),\mathfrak{z})=\mathcal{J}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z}=1$, then from (59), we succeed in bringing about the upcoming inequity, see [27]:

$$\frac{\Gamma(\curlyvee+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^\curlyvee(\mathfrak{d}-\upsilon)^\beta}\left[\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{d}^+,\upsilon^+}\mathfrak{Y}(\mathfrak{n},\mathfrak{d})\times\mathcal{J}(\mathfrak{n},\mathfrak{d})+\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{d}^+,\mathfrak{d}^-}\mathfrak{Y}(\mathfrak{n},\upsilon)\times\mathcal{J}(\mathfrak{n},\upsilon)\right]$$
$$+\frac{\Gamma(\curlyvee+1)\Gamma(\beta+1)}{4(\mathfrak{n}-\mathfrak{d})^\curlyvee(\mathfrak{d}-\upsilon)^\beta}\left[+\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{n}^-,\upsilon^+}\mathfrak{Y}(\mathfrak{d},\mathfrak{d})\times\mathcal{J}(\mathfrak{d},\mathfrak{d})+\mathcal{J}^{\curlyvee,\beta}_{\mathfrak{n}^-,\mathfrak{d}^-}\mathfrak{Y}(\mathfrak{d},\upsilon)\times\mathcal{J}(\mathfrak{d},\upsilon)\right]$$
$$\leq\left(\frac{1}{2}-\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\right)\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)K(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})+\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)L(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})$$
$$+\left(\frac{1}{2}-\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\right)\frac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d})+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\curlyvee}{(\curlyvee+1)(\curlyvee+2)}\mathcal{N}(\mathfrak{d},\mathfrak{n},\upsilon,\mathfrak{d}). \quad (73)$$

Theorem 11. Let $\widetilde{\mathfrak{Y}},\tilde{\mathcal{J}}:\Delta\to\mathbb{F}_0$ be a coordinated *UD*-convex *FNVM*s on Δ. Then, from $\mathfrak{z}$-cuts, we set up the sequence of *IVM*s $\mathfrak{Y}_\mathfrak{z},\mathcal{J}_\mathfrak{z}:\Delta\to\mathbb{R}_I^+$ are given by $\mathfrak{Y}_\mathfrak{z}(\vartheta,\psi)=\left[\mathfrak{Y}_*((\vartheta,\psi),\mathfrak{z}),\mathfrak{Y}^*((\vartheta,\psi),\mathfrak{z})\right]$ and $\mathcal{J}_\mathfrak{z}(\vartheta,\psi)=\left[\mathcal{J}_*((\vartheta,\psi),\mathfrak{z}),\mathcal{J}^*((\vartheta,\psi),\mathfrak{z})\right]$ for all $(\vartheta,\psi)\in\Delta$ and for all $\mathfrak{z}\in[0,1]$. If $\widetilde{\mathfrak{Y}}\otimes\tilde{\mathcal{J}}\in\mathcal{FD}_\Delta$, then following inequalities holds:

$$4\widetilde{\mathfrak{Y}}\left(\frac{\mathfrak{d}+\mathfrak{n}}{2},\frac{\upsilon+\mathfrak{d}}{2}\right)\otimes\tilde{\mathcal{J}}\left(\frac{\mathfrak{d}+\mathfrak{n}}{2},\frac{\upsilon+\mathfrak{d}}{2}\right)$$

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$$\begin{aligned}
&\supseteq_{\mathbb{F}} \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\eth)^{\gamma}(d-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\eth^+,\upsilon^+}\tilde{Y}(n,d)\otimes\tilde{\mathcal{J}}(n,d)\oplus\mathcal{J}^{\gamma,\beta}_{\eth^+,d^-}\tilde{Y}(n,\upsilon)\otimes\tilde{\mathcal{J}}(n,\upsilon)\right]\\
&\oplus\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\eth)^{\gamma}(d-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{n^-,\upsilon^+}\tilde{Y}(\eth,d)\otimes\tilde{\mathcal{J}}(\eth,d)\oplus\mathcal{J}^{\gamma,\beta}_{n^-,d^-}\tilde{Y}(\eth,\upsilon)\otimes\tilde{\mathcal{J}}(\eth,\upsilon)\right]\\
&\oplus\left[\frac{\gamma}{2(\gamma+1)(\gamma+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\right]\tilde{K}(\eth,n,\upsilon,d)\\
&\oplus\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)+\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{L}(\eth,n,\upsilon,d)\\
&\oplus\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{\mathcal{M}}(\eth,n,\upsilon,d)\\
&\oplus\left[\frac{1}{4}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{\mathcal{N}}(\eth,n,\upsilon,d).
\end{aligned}\tag{74}$$

If $\tilde{Y}$ and $\tilde{\mathcal{J}}$ both are coordinate concave $FNVM$s on Δ, then the inequality above can be expressed as follows

$$\begin{aligned}
&4\tilde{Y}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\otimes\tilde{\mathcal{J}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\\
&\subseteq_{\mathbb{F}} \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\eth)^{\gamma}(d-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\eth^+,\upsilon^+}\tilde{Y}(n,d)\otimes\tilde{\mathcal{J}}(n,d)\oplus\mathcal{J}^{\gamma,\beta}_{\eth^+,d^-}\tilde{Y}(n,\upsilon)\otimes\tilde{\mathcal{J}}(n,\upsilon)\right]\\
&\oplus\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\eth)^{\gamma}(d-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{n^-,\upsilon^+}\tilde{Y}(\eth,d)\otimes\tilde{\mathcal{J}}(\eth,d)\oplus\mathcal{J}^{\gamma,\beta}_{n^-,d^-}\tilde{Y}(\eth,\upsilon)\otimes\tilde{\mathcal{J}}(\eth,\upsilon)\right]\\
&\oplus\left[\frac{\gamma}{2(\gamma+1)(\gamma+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\right]\tilde{K}(\eth,n,\upsilon,d)\\
&\oplus\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)+\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{L}(\eth,n,\upsilon,d)\\
&\oplus\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{\mathcal{M}}(\eth,n,\upsilon,d)\\
&\oplus\left[\frac{1}{4}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{\mathcal{N}}(\eth,n,\upsilon,d),
\end{aligned}\tag{75}$$

where $\tilde{K}(\eth,n,\upsilon,d)$, $\tilde{L}(\eth,n,\upsilon,d)$, $\tilde{\mathcal{M}}(\eth,n,\upsilon,d)$ and $\tilde{\mathcal{N}}(\eth,n,\upsilon,d)$ are given in Theorem 10.

Proof. Since $\tilde{Y},\tilde{\mathcal{J}}:\Delta\to\mathbb{F}_0$ be two UD-convex $FNVM$s, then from inequality (17) and for each $\mathfrak{z}\in[0,1]$, we have

$$\begin{aligned}
&2Y_{\mathfrak{z}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\\
&\supseteq_I \frac{\gamma}{2(n-\eth)^{\gamma}}\left[\begin{array}{l}\int_{\eth}^{n}(n-\theta)^{\gamma-1}Y_{\mathfrak{z}}\left(\theta,\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\theta,\frac{\upsilon+d}{2}\right)d\theta\\+\int_{\eth}^{n}(\theta-\eth)^{\gamma-1}Y_{\mathfrak{z}}\left(\theta,\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\theta,\frac{\upsilon+d}{2}\right)d\theta\end{array}\right]\\
&+\left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(Y_{\mathfrak{z}}\left(\eth,\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\eth,\frac{\upsilon+d}{2}\right)+Y_{\mathfrak{z}}\left(n,\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(n,\frac{\upsilon+d}{2}\right)\right)\\
&+\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(Y_{\mathfrak{z}}\left(\eth,\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(n,\frac{\upsilon+d}{2}\right)+Y_{\mathfrak{z}}\left(n,\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\eth,\frac{\upsilon+d}{2}\right)\right),
\end{aligned}\tag{76}$$

and

$$\begin{aligned}
&2Y_{\mathfrak{z}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\\
&\supseteq_I \frac{\beta}{2(d-\upsilon)^{\beta}}\left[\begin{array}{l}\int_{\upsilon}^{d}(d-\psi)^{\beta-1}Y_{\mathfrak{z}}\left(\frac{\eth+n}{2},\psi\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\psi\right)d\psi\\+\int_{\upsilon}^{d}(\psi-\upsilon)^{\beta-1}Y_{\mathfrak{z}}\left(\frac{\eth+n}{2},\psi\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\psi\right)d\psi\end{array}\right]\\
&+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(Y_{\mathfrak{z}}\left(\frac{\eth+n}{2},\upsilon\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\upsilon\right)+Y_{\mathfrak{z}}\left(\frac{\eth+n}{2},d\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},d\right)\right)\\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(Y_{\mathfrak{z}}\left(\frac{\eth+n}{2},\upsilon\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},d\right)+Y_{\mathfrak{z}}\left(\frac{\eth+n}{2},d\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\upsilon\right)\right).
\end{aligned}\tag{77}$$

Adding (76) and (77), and then taking the multiplication of the resultant one by 2, we obtain: By adding (76) and (77), multiplying the result by 2, we arrive at:

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$$
\begin{aligned}
&8\Upsilon_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right)\\
&\supseteq_I \frac{\gamma}{2(\mathrm{n}-\delta)^{\gamma}}\left[\begin{array}{l}\int_{\delta}^{\mathrm{n}}2(\mathrm{n}-\theta)^{\gamma-1}\Upsilon_{\mathfrak{z}}\left(\theta,\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\theta,\frac{\upsilon+\mathrm{d}}{2}\right)d\theta\\ +\int_{\delta}^{\mathrm{n}}2(\theta-\delta)^{\gamma-1}\Upsilon_{\mathfrak{z}}\left(\theta,\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\theta,\frac{\upsilon+\mathrm{d}}{2}\right)d\theta\end{array}\right]\\
&+\frac{\beta}{2(\mathrm{d}-\upsilon)^{\beta}}\left[\begin{array}{l}\int_{\upsilon}^{\mathrm{d}}2(\mathrm{d}-\psi)^{\beta-1}\Upsilon_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\psi\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\psi\right)d\psi\\ +\int_{\upsilon}^{\mathrm{d}}2(\psi-\upsilon)^{\beta-1}\Upsilon_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\psi\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\psi\right)d\psi\end{array}\right]\\
&+\left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(2\Upsilon_{\mathfrak{z}}\left(\delta,\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\delta,\frac{\upsilon+\mathrm{d}}{2}\right)+2\Upsilon_{\mathfrak{z}}\left(\mathrm{n},\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\mathrm{n},\frac{\upsilon+\mathrm{d}}{2}\right)\right)\\
&+\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(2\Upsilon_{\mathfrak{z}}\left(\delta,\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\mathrm{n},\frac{\upsilon+\mathrm{d}}{2}\right)+2\Upsilon_{\mathfrak{z}}\left(\mathrm{n},\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\delta,\frac{\upsilon+\mathrm{d}}{2}\right)\right)\\
&+\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(2\Upsilon_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\upsilon\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\upsilon\right)+2\Upsilon_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\mathrm{d}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\mathrm{d}\right)\right)\\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(2\Upsilon_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\upsilon\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\mathrm{d}\right)+2\Upsilon_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\mathrm{d}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\upsilon\right)\right).
\end{aligned}
\tag{78}
$$

By Lemma 1, for each integral on the right-hand side of (78) and integral inequality (17) once more lead us to arrive at:

$$
\begin{aligned}
&\frac{\gamma}{2(\mathrm{n}-\delta)^{\gamma}}\int_{\delta}^{\mathrm{n}}2(\mathrm{n}-\theta)^{\gamma-1}\Upsilon_{\mathfrak{z}}\left(\theta,\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\theta,\frac{\upsilon+\mathrm{d}}{2}\right)d\theta\\
&\supseteq_I \frac{\gamma\beta}{4(\mathrm{n}-\delta)^{\gamma}(\mathrm{d}-\upsilon)^{\beta}}\left[\int_{\delta}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}(\mathrm{n}-\theta)^{\gamma-1}(\mathrm{d}-\psi)^{\beta-1}\Upsilon_{\mathfrak{z}}(\theta,\psi)\,d\psi d\theta\right]\\
&+\frac{\gamma\beta}{4(\mathrm{n}-\delta)^{\gamma}(\mathrm{d}-\upsilon)^{\beta}}\left[\int_{\delta}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}(\mathrm{n}-\theta)^{\gamma-1}(\psi-\upsilon)^{\beta-1}\Upsilon_{\mathfrak{z}}(\theta,\psi)\,d\psi d\theta\right]\\
&+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\gamma}{2(\mathrm{n}-\delta)^{\gamma}}\int_{\delta}^{\mathrm{n}}(\mathrm{n}-\theta)^{\gamma-1}\left(\Upsilon_{\mathfrak{z}}(\theta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\theta,\upsilon)+\Upsilon_{\mathfrak{z}}(\theta,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\theta,\mathrm{d})\right)d\theta\\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\gamma}{2(\mathrm{n}-\delta)^{\gamma}}\int_{\delta}^{\mathrm{n}}(\mathrm{n}-\theta)^{\gamma-1}\left(\Upsilon_{\mathfrak{z}}(\theta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\theta,\mathrm{d})+\Upsilon_{\mathfrak{z}}(\theta,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\theta,\upsilon)\right)d\theta,\\
&=\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathrm{n}-\delta)^{\gamma}(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\delta^+,\upsilon^+}^{\gamma,\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathrm{d})+\mathcal{J}_{\delta^+,\mathrm{d}^-}^{\gamma,\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right]\\
&+\frac{\Gamma(\gamma+1)}{2(\mathrm{n}-\delta)^{\gamma}}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}_{\delta^+}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)+\mathcal{J}_{\delta^+}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathrm{d})\right)\\
&+\frac{\Gamma(\gamma+1)}{2(\mathrm{n}-\delta)^{\gamma}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}_{\delta^+}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathrm{d})+\mathcal{J}_{\delta^+}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right).
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
&\frac{\gamma}{2(\mathrm{n}-\delta)^{\gamma}}\int_{\delta}^{\mathrm{n}}2(\theta-\delta)^{\gamma-1}\Upsilon_{\mathfrak{z}}\left(\theta,\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\theta,\frac{\upsilon+\mathrm{d}}{2}\right)d\theta\\
&\supseteq_I \frac{\gamma\beta}{4(\mathrm{n}-\delta)^{\gamma}(\mathrm{d}-\upsilon)^{\beta}}\left[\int_{\delta}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}(\theta-\delta)^{\gamma-1}(\mathrm{d}-\psi)^{\beta-1}\Upsilon_{\mathfrak{z}}(\theta,\psi)\,d\psi d\theta\right]\\
&+\frac{\gamma\beta}{4(\mathrm{n}-\delta)^{\gamma}(\mathrm{d}-\upsilon)^{\beta}}\left[\int_{\delta}^{\mathrm{n}}\int_{\upsilon}^{\mathrm{d}}(\theta-\delta)^{\gamma-1}(\psi-\upsilon)^{\beta-1}\Upsilon_{\mathfrak{z}}(\theta,\psi)\,d\psi d\theta\right]\\
&+\frac{\beta}{(\beta+1)(\beta+2)}\frac{\gamma}{2(\mathrm{n}-\delta)^{\gamma}}\int_{\delta}^{\mathrm{n}}(\theta-\delta)^{\gamma-1}\left(\Upsilon_{\mathfrak{z}}(\theta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\theta,\upsilon)+\Upsilon_{\mathfrak{z}}(\theta,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\theta,\mathrm{d})\right)d\theta\\
&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\frac{\gamma}{2(\mathrm{n}-\delta)^{\gamma}}\int_{\delta}^{\mathrm{n}}(\theta-\delta)^{\gamma-1}\left(\Upsilon_{\mathfrak{z}}(\theta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\theta,\mathrm{d})+\Upsilon_{\mathfrak{z}}(\theta,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\theta,\upsilon)\right)d\theta,\\
&=\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathrm{n}-\delta)^{\gamma}(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\mathrm{n}^-,\upsilon^+}^{\gamma,\beta}\,\Upsilon_{\mathfrak{z}}(\delta,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\delta,\mathrm{d})+\mathcal{J}_{\mathrm{n}^-,\mathrm{d}^-}^{\gamma,\beta}\,\Upsilon_{\mathfrak{z}}(\delta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\delta,\upsilon)\right]\\
&+\frac{\Gamma(\gamma+1)}{2(\mathrm{n}-\delta)^{\gamma}}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}_{\mathrm{n}^-}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\delta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\delta,\upsilon)+\mathcal{J}_{\mathrm{n}^-}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\delta,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\delta,\mathrm{d})\right)\\
&+\frac{\Gamma(\gamma+1)}{2(\mathrm{n}-\delta)^{\gamma}}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\mathcal{J}_{\mathrm{n}^-}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\delta,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\delta,\mathrm{d})+\mathcal{J}_{\mathrm{n}^-}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\delta,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\delta,\upsilon)\right).
\end{aligned}
\tag{80}
$$

$$
\frac{\beta}{2(\mathrm{d}-\upsilon)^{\beta}}\left[\int_{\upsilon}^{\mathrm{d}}2(\mathrm{d}-\psi)^{\beta-1}\Upsilon_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\psi\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\delta+\mathrm{n}}{2},\psi\right)d\psi\right]
$$

$$\begin{aligned}&\supseteq_I \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\eth)^{\gamma}(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\eth^+,\upsilon^+}\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})+\mathcal{J}^{\gamma,\beta}_{n^-,\upsilon^+}\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})\right]\\&+\frac{\Gamma(\beta+1)}{2(\mathrm{d}-\upsilon)^{\beta}}\left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\mathcal{J}^{\beta}_{\upsilon^+}\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})+\mathcal{J}^{\beta}_{\upsilon^+}\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})\right)\\&+\frac{\Gamma(\beta+1)}{2(\mathrm{d}-\upsilon)^{\beta}}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\mathcal{J}^{\beta}_{\upsilon^+}\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})+\mathcal{J}^{\beta}_{\upsilon^+}\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})\right).\end{aligned}\tag{81}$$

$$\begin{aligned}&\frac{\beta}{2(\mathrm{d}-\upsilon)^{\beta}}\left[\int_{\upsilon}^{\mathrm{d}}2(\psi-\upsilon)^{\beta-1}\Upsilon_{\mathfrak{z}}\left(\frac{\eth+n}{2},\psi\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\psi\right)d\psi\right]\\&\supseteq_I \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\eth)^{\gamma}(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\gamma,\beta}_{\eth^+,\mathrm{d}^-}\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)+\mathcal{J}^{\gamma,\beta}_{n^-,\mathrm{d}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right]\\&+\frac{\Gamma(\beta+1)}{2(\mathrm{d}-\upsilon)^{\beta}}\left(\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\mathcal{J}^{\beta}_{\mathrm{d}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\mathcal{J}^{\beta}_{\mathrm{d}^-}\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)\right)\\&+\frac{\Gamma(\beta+1)}{2(\mathrm{d}-\upsilon)^{\beta}}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\mathcal{J}^{\beta}_{\mathrm{d}^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)+\mathcal{J}^{\beta}_{\mathrm{d}^-}\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)\right).\end{aligned}\tag{82}$$

And

$$\begin{aligned}&2\Upsilon_{\mathfrak{z}}\left(\frac{\eth+n}{2},\upsilon\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\upsilon\right)\\&\supseteq_I \frac{\Gamma(\gamma+1)}{2(n-\eth)^{\gamma}}\left[\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)+\mathcal{J}^{\gamma}_{n^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right]\\&+\frac{\gamma}{(\gamma+1)(\gamma+2)}\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)\right)\\&+\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)+\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right),\end{aligned}\tag{83}$$

$$\begin{aligned}&2\Upsilon_{\mathfrak{z}}\left(\frac{\eth+n}{2},\mathrm{d}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\mathrm{d}\right)\\&\supseteq_I \frac{\Gamma(\gamma+1)}{2(n-\eth)^{\gamma}}\left[\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})+\mathcal{J}^{\gamma}_{n^-}\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})\right]\\&+\frac{\gamma}{(\gamma+1)(\gamma+2)}\left(\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})+\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})\right)\\&+\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})+\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})\right),\end{aligned}\tag{84}$$

$$\begin{aligned}&2\Upsilon_{\mathfrak{z}}\left(\frac{\eth+n}{2},\upsilon\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\mathrm{d}\right)\\&\supseteq_I \frac{\Gamma(\gamma+1)}{2(n-\eth)^{\gamma}}\left[\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})+\mathcal{J}^{\gamma}_{n^-}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})\right]\\&+\frac{\gamma}{(\gamma+1)(\gamma+2)}\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})+\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})\right)\\&+\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})+\Upsilon_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})\right),\end{aligned}\tag{85}$$

$$\begin{aligned}&2\Upsilon_{\mathfrak{z}}\left(\frac{\eth+n}{2},\mathrm{d}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\upsilon\right)\\&\supseteq_I \frac{\Gamma(\gamma+1)}{2(n-\eth)^{\gamma}}\left[\mathcal{J}^{\gamma}_{\eth^+}\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)+\mathcal{J}^{\gamma}_{n^-}\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right]\\&+\frac{\gamma}{(\gamma+1)(\gamma+2)}\left(\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)\right)\\&+\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)+\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right),\end{aligned}\tag{86}$$

$$\begin{aligned}&2\Upsilon_{\mathfrak{z}}\left(\eth,\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\eth,\frac{\upsilon+\mathrm{d}}{2}\right)\\&\supseteq_I \frac{\Gamma(\beta+1)}{2(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\upsilon^+}\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})+\mathcal{J}^{\beta}_{\mathrm{d}^-}\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right]\\&+\frac{\beta}{(\beta+1)(\beta+2)}\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})\right)\\&+\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathrm{d})+\Upsilon_{\mathfrak{z}}(\eth,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right),\end{aligned}\tag{87}$$

$$\begin{aligned}&2\Upsilon_{\mathfrak{z}}\left(n,\frac{\upsilon+\mathrm{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(n,\frac{\upsilon+\mathrm{d}}{2}\right)\\&\supseteq_I \frac{\Gamma(\beta+1)}{2(\mathrm{d}-\upsilon)^{\beta}}\left[\mathcal{J}^{\beta}_{\upsilon^+}\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\mathrm{d})+\mathcal{J}^{\beta}_{\mathrm{d}^-}\Upsilon_{\mathfrak{z}}(n,\mathrm{d})\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)\right]\end{aligned}$$

$$+\frac{\beta}{(\beta+1)(\beta+2)}\left(\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)+\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\right)$$
$$+\left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})+\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right), \quad (88)$$

$$2\Upsilon_{\mathfrak{z}}\left(\eth,\frac{\upsilon+\mathfrak{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\mathrm{n},\frac{\upsilon+\mathfrak{d}}{2}\right)$$
$$\supseteq_I \frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})+\mathcal{J}_{\mathfrak{d}^-}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right]$$
$$+\frac{\beta}{(\beta+1)(\beta+2)}\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)+\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\right)$$
$$+\left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})+\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right), \quad (89)$$

and

$$2\Upsilon_{\mathfrak{z}}\left(\mathrm{n},\frac{\upsilon+\mathfrak{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\eth,\frac{\upsilon+\mathfrak{d}}{2}\right)$$
$$\supseteq_I \frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left[\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\mathcal{J}_{\mathfrak{d}^-}^{\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right]$$
$$+\frac{\beta}{(\beta+1)(\beta+2)}\left(\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right)$$
$$+\left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\left(\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right). \quad (90)$$

From inequalities (79) to (90), inequality (78) we have

$$8\Upsilon_{\mathfrak{z}}\left(\frac{\eth+\mathrm{n}}{2},\frac{\upsilon+\mathfrak{d}}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+\mathrm{n}}{2},\frac{\upsilon+\mathfrak{d}}{2}\right)$$
$$\supseteq_I \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{2(\mathrm{n}-\eth)^{\gamma}(\mathfrak{d}-\upsilon)^{\beta}}\begin{bmatrix}\mathcal{J}_{\eth^+,\upsilon^+}^{\gamma,\beta}\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})+\mathcal{J}_{\eth^+,\mathfrak{d}^-}^{\gamma,\beta}\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\\+\mathcal{J}_{\mathrm{n}^-,\upsilon^+}^{\gamma,\beta}\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\mathcal{J}_{\mathrm{n}^-,\mathfrak{d}^-}^{\gamma,\beta}\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\end{bmatrix}$$
$$+\left(\frac{2\gamma}{(\gamma+1)(\gamma+2)}\right)\begin{bmatrix}\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left(\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\mathcal{J}_{\upsilon^+}^{\beta}\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\right)\\+\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left(\mathcal{J}_{\mathfrak{d}^-}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\mathcal{J}_{\mathfrak{d}^-}^{\beta}\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right)\end{bmatrix}$$
$$+2\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\begin{bmatrix}\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left(\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})+\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\right)\\+\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left(\mathcal{J}_{\mathfrak{d}^-}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)+\mathcal{J}_{\mathfrak{d}^-}^{\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right)\end{bmatrix}$$
$$+2\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\begin{bmatrix}\frac{\Gamma(\gamma+1)}{2(\mathrm{n}-\eth)^{\gamma}}\left(\mathcal{J}_{\eth^+}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)+\mathcal{J}_{\eth^+}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\right)\\+\frac{\Gamma(\gamma+1)}{2(\mathrm{n}-\eth)^{\gamma}}\left(\mathcal{J}_{\mathrm{n}^-}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\mathcal{J}_{\mathrm{n}^-}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})\right)\end{bmatrix}$$
$$+2\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\begin{bmatrix}\frac{\Gamma(\gamma+1)}{2(\mathrm{n}-\eth)^{\gamma}}\left(\mathcal{J}_{\eth^+}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})+\mathcal{J}_{\eth^+}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right)\\+\frac{\Gamma(\gamma+1)}{2(\mathrm{n}-\eth)^{\gamma}}\left(\mathcal{J}_{\mathrm{n}^-}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\mathcal{J}_{\mathrm{n}^-}^{\gamma}\,\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right)\end{bmatrix}$$
$$+\frac{2\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}K_{\mathfrak{z}}(\eth,\mathrm{n},\upsilon,\mathfrak{d})++\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\frac{2\beta}{(\beta+1)(\beta+2)}L_{\mathfrak{z}}(\eth,\mathrm{n},\upsilon,\mathfrak{d})$$
$$+\frac{2\gamma}{(\gamma+1)(\gamma+2)}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{M}_{\mathfrak{z}}(\eth,\mathrm{n},\upsilon,\mathfrak{d})$$
$$+2\left(\tfrac{1}{2}-\tfrac{\gamma}{(\gamma+1)(\gamma+2)}\right)\left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)\mathcal{N}_{\mathfrak{z}}(\eth,\mathrm{n},\upsilon,\mathfrak{d}). \quad (91)$$

By Lemma 1, for each integral on the right-hand side of (91) and integral inequality (16) once more lead us to arrive at:

$$\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left(\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\eth,\mathfrak{d})+\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\right)$$
$$+\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left(\mathcal{J}_{\mathfrak{d}^-}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\mathcal{J}_{\mathfrak{d}^-}^{\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\upsilon)\right)$$
$$\supseteq_I \left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)K_{\mathfrak{z}}(\eth,\mathrm{n},\upsilon,\mathfrak{d})+\tfrac{\beta}{(\beta+1)(\beta+2)}\mathcal{M}_{\mathfrak{z}}(\eth,\mathrm{n},\upsilon,\mathfrak{d}). \quad (92)$$

$$\frac{\Gamma(\beta+1)}{2(\mathfrak{d}-\upsilon)^{\beta}}\left(\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\eth,\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})+\mathcal{J}_{\upsilon^+}^{\beta}\,\Upsilon_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\times\mathcal{J}_{\mathfrak{z}}(\mathrm{n},\mathfrak{d})\right)$$

$$+\frac{\Gamma(\beta+1)}{2(d-\upsilon)^{\beta}}\left(\mathcal{J}_{d^-}^{\beta}\,\mathrm{Y}_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)+\mathcal{J}_{d^-}^{\beta}\,\mathrm{Y}_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)\right)$$
$$\supseteq_I\left(\tfrac{1}{2}-\tfrac{\beta}{(\beta+1)(\beta+2)}\right)L_{\mathfrak{z}}(\eth,n,\upsilon,d)+\tfrac{\beta}{(\beta+1)(\beta+2)}\mathcal{N}_{\mathfrak{z}}(\eth,n,\upsilon,d). \quad (93)$$

$$\frac{\Gamma(\gamma+1)}{2(n-\eth)^{\gamma}}\left(\mathcal{J}_{\eth^+}^{\gamma}\,\mathrm{Y}_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)+\mathcal{J}_{\eth^+}^{\gamma}\,\mathrm{Y}_{\mathfrak{z}}(n,d)\times\mathcal{J}_{\mathfrak{z}}(n,d)\right)$$
$$+\frac{\Gamma(\gamma+1)}{2(n-\eth)^{\gamma}}\left(\mathcal{J}_{n^-}^{\gamma}\,\mathrm{Y}_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)+\mathcal{J}_{n^-}^{\gamma}\,\mathrm{Y}_{\mathfrak{z}}(\eth,d)\times\mathcal{J}_{\mathfrak{z}}(\eth,d)\right)$$
$$\supseteq_I\left(\tfrac{1}{2}-\tfrac{\gamma}{(\gamma+1)(\gamma+2)}\right)K_{\mathfrak{z}}(\eth,n,\upsilon,d)+\tfrac{\gamma}{(\gamma+1)(\gamma+2)}L_{\mathfrak{z}}(\eth,n,\upsilon,d). \quad (94)$$

$$\frac{\Gamma(\gamma+1)}{2(n-\eth)^{\gamma}}\left(\mathcal{J}_{n^-}^{\gamma}\,\mathrm{Y}_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,d)+\mathcal{J}_{n^-}^{\gamma}\,\mathrm{Y}_{\mathfrak{z}}(\eth,d)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right)$$
$$+\frac{\Gamma(\gamma+1)}{2(n-\eth)^{\gamma}}\left(\mathcal{J}_{n^-}^{\gamma}\,\mathrm{Y}_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,d)+\mathcal{J}_{n^-}^{\gamma}\,\mathrm{Y}_{\mathfrak{z}}(\eth,d)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\right)$$
$$\supseteq_I\left(\tfrac{1}{2}-\tfrac{\gamma}{(\gamma+1)(\gamma+2)}\right)\mathcal{M}_{\mathfrak{z}}(\eth,n,\upsilon,d)+\tfrac{\gamma}{(\gamma+1)(\gamma+2)}\mathcal{N}_{\mathfrak{z}}(\eth,n,\upsilon,d). \quad (95)$$

From (88) to (95), (91) we have

$$4\mathrm{Y}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\times\mathcal{J}_{\mathfrak{z}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)$$
$$\supseteq_I\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\eth)^{\gamma}(d-\upsilon)^{\beta}}\begin{bmatrix}\mathcal{J}_{\eth^+,\upsilon^+}^{\gamma,\beta}\mathrm{Y}_{\mathfrak{z}}(n,d)\times\mathcal{J}_{\mathfrak{z}}(n,d)+\mathcal{J}_{\eth^+,d^-}^{\gamma,\beta}\mathrm{Y}_{\mathfrak{z}}(n,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(n,\upsilon)\\+\mathcal{J}_{n^-,\upsilon^+}^{\gamma,\beta}\mathrm{Y}_{\mathfrak{z}}(\eth,d)\times\mathcal{J}_{\mathfrak{z}}(\eth,d)+\mathcal{J}_{n^-,d^-}^{\gamma,\beta}\mathrm{Y}_{\mathfrak{z}}(\eth,\upsilon)\times\mathcal{J}_{\mathfrak{z}}(\eth,\upsilon)\end{bmatrix}$$
$$+\left[\frac{\gamma}{2(\gamma+1)(\gamma+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\right]K_{\mathfrak{z}}(\eth,n,\upsilon,d)$$
$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)+\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]L_{\mathfrak{z}}(\eth,n,\upsilon,d)$$
$$+\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{M}_{\mathfrak{z}}(\eth,n,\upsilon,d)$$
$$+\left[\tfrac{1}{4}-\tfrac{\gamma}{(\gamma+1)(\gamma+2)}\tfrac{\beta}{(\beta+1)(\beta+2)}\right]\mathcal{N}_{\mathfrak{z}}(\eth,n,\upsilon,d). \quad (96)$$

That is

$$4\tilde{\mathrm{Y}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\otimes\tilde{\mathcal{J}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)$$
$$\supseteq_{\mathbb{F}}\frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(n-\eth)^{\gamma}(d-\upsilon)^{\beta}}\begin{bmatrix}\mathcal{J}_{\eth^+,\upsilon^+}^{\gamma,\beta}\tilde{\mathrm{Y}}(n,d)\otimes\tilde{\mathcal{J}}(n,d)\oplus\mathcal{J}_{\eth^+,d^-}^{\gamma,\beta}\tilde{\mathrm{Y}}(n,\upsilon)\otimes\tilde{\mathcal{J}}(n,\upsilon)\\\oplus\mathcal{J}_{n^-,\upsilon^+}^{\gamma,\beta}\tilde{\mathrm{Y}}(\eth,d)\otimes\tilde{\mathcal{J}}(\eth,d)\oplus\mathcal{J}_{n^-,d^-}^{\gamma,\beta}\tilde{\mathrm{Y}}(\eth,\upsilon)\otimes\tilde{\mathcal{J}}(\eth,\upsilon)\end{bmatrix}$$
$$\oplus\left[\frac{\gamma}{2(\gamma+1)(\gamma+2)}+\frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\right]\tilde{K}(\eth,n,\upsilon,d)$$
$$\oplus\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\gamma}{(\gamma+1)(\gamma+2)}\right)+\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{L}(\eth,n,\upsilon,d)$$
$$\oplus\left[\frac{1}{2}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)+\frac{\gamma}{(\gamma+1)(\gamma+2)}\frac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{\mathcal{M}}(\eth,n,\upsilon,d)$$
$$\oplus\left[\tfrac{1}{4}-\tfrac{\gamma}{(\gamma+1)(\gamma+2)}\tfrac{\beta}{(\beta+1)(\beta+2)}\right]\tilde{\mathcal{N}}(\eth,n,\upsilon,d).$$

The conclusion has therefore been established.

Remark 4. If one assumes that $\gamma=1$ and $\beta=1$, then from (74), as a result, there will be inequity, see [28]:

$$4\tilde{\mathrm{Y}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\otimes\tilde{\mathcal{J}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)$$
$$\supseteq_{\mathbb{F}}\frac{1}{(n-\eth)(d-\upsilon)}\int_{\eth}^{n}\int_{\upsilon}^{d}\tilde{\mathrm{Y}}(\vartheta,\psi)\otimes\tilde{\mathcal{J}}(\vartheta,\psi)d\psi d\vartheta\oplus\frac{5}{36}\tilde{K}(\eth,n,\upsilon,d)$$
$$\oplus\tfrac{7}{36}\left[\tilde{L}(\eth,n,\upsilon,d)\tilde{+}\tilde{\mathcal{M}}(\eth,n,\upsilon,d)\right]\oplus\tfrac{2}{9}\tilde{\mathcal{N}}(\eth,n,\upsilon,d). \quad (97)$$

If $\tilde{\mathrm{Y}}$ is coordinated left-*UD*-convex and one assumes that $\gamma=1$ and $\beta=1$, then from (74), as a result, there will be inequity, see [22]:

$$4\tilde{\mathrm{Y}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)\otimes\tilde{\mathcal{J}}\left(\frac{\eth+n}{2},\frac{\upsilon+d}{2}\right)$$

$$\leq_{\mathbb{F}} \frac{1}{(\mathrm{n}-\mathrm{ð})(\mathrm{d}-\upsilon)} \int_{\mathrm{ð}}^{\mathrm{n}} \int_{\upsilon}^{\mathrm{d}} \tilde{\mathrm{Y}}(\vartheta,\psi) \otimes \tilde{\mathcal{J}}(\vartheta,\psi) d\psi d\vartheta \oplus \frac{5}{36}\tilde{K}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) \oplus \frac{7}{36}\left[\tilde{L}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) \tilde{+} \widetilde{\mathcal{M}}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d})\right] \oplus \frac{2}{9}\widetilde{\mathcal{N}}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}). \quad (98)$$

If $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z}) \neq \mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z}=1$, then from (74), we succeed in bringing about the upcoming inequity, see [20]:

$$4\,\mathrm{Y}\left(\frac{\mathrm{ð}+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right) \times \mathcal{J}\left(\frac{\mathrm{ð}+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right) \supseteq \frac{1}{(\mathrm{n}-\mathrm{ð})(\mathrm{d}-\upsilon)} \int_{\mathrm{ð}}^{\mathrm{n}} \int_{\upsilon}^{\mathrm{d}} \mathrm{Y}(\vartheta,\psi) \times \mathcal{J}(\vartheta,\psi) d\psi d\vartheta + \frac{5}{36}K(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) + \frac{7}{36}\left[L(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) + \mathcal{M}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d})\right] + \frac{2}{9}\mathcal{N}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}). \quad (99)$$

If $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z}) \neq \mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z}=1$, then from (74), we succeed in bringing about the upcoming inequity, see [21]:

$$4\mathrm{Y}\left(\frac{\mathrm{ð}+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right) \times \mathcal{J}\left(\frac{\mathrm{ð}+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right) \supseteq \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathrm{n}-\mathrm{ð})^{\gamma}(\mathrm{d}-\upsilon)^{\beta}} \begin{bmatrix} \mathcal{J}^{\gamma,\beta}_{\mathrm{ð}^+,\upsilon^+}\, \mathrm{Y}(\mathrm{n},\mathrm{d}) \times \mathcal{J}(\mathrm{n},\mathrm{d}) + \mathcal{J}^{\gamma,\beta}_{\mathrm{ð}^+,\mathrm{d}^-}\, \mathrm{Y}(\mathrm{n},\upsilon) \times \mathcal{J}(\mathrm{n},\upsilon) \\ + \mathcal{J}^{\gamma,\beta}_{\mathrm{n}^-,\upsilon^+}\, \mathrm{Y}(\mathrm{ð},\mathrm{d}) \times \mathcal{J}(\mathrm{ð},\mathrm{d}) + \mathcal{J}^{\gamma,\beta}_{\mathrm{n}^-,\mathrm{d}^-}\, \mathrm{Y}(\mathrm{ð},\upsilon) \times \mathcal{J}(\mathrm{ð},\upsilon) \end{bmatrix} + \left[\frac{\gamma}{2(\gamma+1)(\gamma+2)} + \frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2} - \frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\right] K(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) + \left[\frac{1}{2}\left(\frac{1}{2} - \frac{\gamma}{(\gamma+1)(\gamma+2)}\right) + \frac{\gamma}{(\gamma+1)(\gamma+2)} \frac{\beta}{(\beta+1)(\beta+2)}\right] L(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) + \left[\frac{1}{2}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right) + \frac{\gamma}{(\gamma+1)(\gamma+2)} \frac{\beta}{(\beta+1)(\beta+2)}\right] \mathcal{M}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) + \left[\frac{1}{4} - \frac{\gamma}{(\gamma+1)(\gamma+2)} \frac{\beta}{(\beta+1)(\beta+2)}\right] \mathcal{N}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}). \quad (100)$$

If $\mathrm{Y}_*((\vartheta,\psi),\mathfrak{z}) = \mathrm{Y}^*((\vartheta,\psi),\mathfrak{z})$ and $\mathcal{J}_*((\vartheta,\psi),\mathfrak{z}) = \mathcal{J}^*((\vartheta,\psi),\mathfrak{z})$ with $\mathfrak{z}=1$, then from (74), we succeed in bringing about the upcoming inequity, see [27]:

$$4\mathrm{Y}\left(\frac{\mathrm{ð}+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right) \times \mathcal{J}\left(\frac{\mathrm{ð}+\mathrm{n}}{2},\frac{\upsilon+\mathrm{d}}{2}\right) \leq \frac{\Gamma(\gamma+1)\Gamma(\beta+1)}{4(\mathrm{n}-\mathrm{ð})^{\gamma}(\mathrm{d}-\upsilon)^{\beta}} \begin{bmatrix} \mathcal{J}^{\gamma,\beta}_{\mathrm{ð}^+,\upsilon^+}\, \mathrm{Y}(\mathrm{n},\mathrm{d}) \times \mathcal{J}(\mathrm{n},\mathrm{d}) + \mathcal{J}^{\gamma,\beta}_{\mathrm{ð}^+,\mathrm{d}^-}\, \mathrm{Y}(\mathrm{n},\upsilon) \times \mathcal{J}(\mathrm{n},\upsilon) \\ + \mathcal{J}^{\gamma,\beta}_{\mathrm{n}^-,\upsilon^+}\, \mathrm{Y}(\mathrm{ð},\mathrm{d}) \times \mathcal{J}(\mathrm{ð},\mathrm{d}) + \mathcal{J}^{\gamma,\beta}_{\mathrm{n}^-,\mathrm{d}^-}\, \mathrm{Y}(\mathrm{ð},\upsilon) \times \mathcal{J}(\mathrm{ð},\upsilon) \end{bmatrix} + \left[\frac{\gamma}{2(\gamma+1)(\gamma+2)} + \frac{\beta}{(\beta+1)(\beta+2)}\left(\frac{1}{2} - \frac{\gamma}{(\gamma+1)(\gamma+2)}\right)\right] K(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) + \left[\frac{1}{2}\left(\frac{1}{2} - \frac{\gamma}{(\gamma+1)(\gamma+2)}\right) + \frac{\gamma}{(\gamma+1)(\gamma+2)} \frac{\beta}{(\beta+1)(\beta+2)}\right] L(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) + \left[\frac{1}{2}\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right) + \frac{\gamma}{(\gamma+1)(\gamma+2)} \frac{\beta}{(\beta+1)(\beta+2)}\right] \mathcal{M}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}) + \left[\frac{1}{4} - \frac{\gamma}{(\gamma+1)(\gamma+2)} \frac{\beta}{(\beta+1)(\beta+2)}\right] \mathcal{N}(\mathrm{ð},\mathrm{n},\upsilon,\mathrm{d}). \quad (101)$$

4. Conclusions and Future Plans

In this study, Hermite-Hadamard type inequalities for coordinated UD-convex $FNVM$ were established. These inequalities are very important in the field of inequalities because the findings in this research constitute an expansion of a number of earlier findings. A coordinated fuzzy-number-valued convexity is a novel type of class and by using this class and other fractional integrals, new fractional inequalities can be found that is available to interested authors.

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