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On Weakly Tripotent and Locally Invo-Regular Rings

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Abstract: In this article some important observations have been reported on recent works related to weakly tripotent rings and locally invo-regular rings. Our findings give additional results as well as correct some recent results on weakly tripotent rings and locally invo-regular rings appeared in Rendiconti Sem. Mat. Univ. Pol. Torino (2021) and Azerbaijan Journal of Mathematics (2021) respectively.

Keywords: tripotent ring; weakly tripotent ring; locally invo-regular rings

MSC 2010: 16S34; 20C07; 16U99

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1. Introduction

In this paper A is a unital and associative ring and $J(A)$ and $U(A)$ stand for the Jacobson radical of A and the set of units in A respectively. We recall that a ring A is said to be a weakly tripotent ring if $u^3 = u$ or $(1-u)^3 = 1-u$ for each $u \in A$ [1-2] and a ring A is said to be a locally invo-regular ring if $u = uvu$ or $1-u = (1-u)v(1-u)$ for each $u \in A$ and some $v \in A$ with $v^2 = 1$ [3].

It may be worth mentioning that weakly tripotent rings, locally ino-regular rings and associated notions have extensively appeared in mathematical literature [1-10]. Motivated by some of our recent works [11-12], here we take an opportunity to report some significant observations and results on weakly tripotent and locally invo-regular rings.

In [2] it has been seen that if A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A then $\frac{A}{J(A)} \cong Z_2$ and $u^2 = 2u = 0$ holds for each $u \in J(A)$. Similarly it has been seen in [3] that if A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $\frac{A}{J(A)} \cong Z_2$ and $u^2 = 2u = 0$ holds for each $u \in J(A)$.

However we observe that if A is a weakly tripotent ring and it does not have non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$. Similarly we note that if A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.

Moreover we observe that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents such that $u^2 = 2u = 0$ for each $u \in J(A)$ then $u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid. We exhibit that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$.

We provide our observations and results in the next section.

2. Some Observations and Results

Theorem 2.1: Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$.

Proof. Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A. By [1, Corollary 10], we have $u^2 = 1$ for each $u \in U(A)$ and $u^2 = 2u$ for each $u \in J(A)$. It may be noted that if $u \in J(A)$ then $1 + u \in U(A)$. Similarly $1 - u \in U(A)$. We note that $1 + u \in U(A)$ gives that $(1+u)^2 = 1 \Rightarrow u^2 = -2u$ and $1 - u \in U(A)$ gives that $(1-u)^2 = 1 \Rightarrow u^2 = 2u$. Hence $u^2 = -2u$ and $u^2 = 2u$ together give that $u^3 = 4u = 0$ for each $u \in J(RA)$.

Theorem 2.2: Let A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$.

Proof. The proof of this Theorem follows from the proof of Proposition 2.1 and the fact that each weakly tripotent ring is a locally invo-regular ring [3].

Proposition 2.3: Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.

Proof. Let $A = Z_4$ and $G = \{1, g : g^2 = 1\}$. Clearly G is an abelian group under multiplication. Now we shall construct the group ring AG. It may be noted that if $a_i \in A, g_i \in G$ then $u \in AG$ is expressible as $(a_1 g_1 + a_2 g_2 + ... + a_n g_n) \in AG$ [13]. Thus the group ring AG has the following sixteen elements.

$0, 1, 2, 3, g, 2g, 3g, 1+g, 2+g, 3+g, 1+2g, 2+2g, 3+2g, 1+3g, 2+3g, 3+3g$.

One may easily note that each element $u \in AG$ satisfies $u^3 = u$ or $(1-u)^3 = 1-u$. Hence AG is a weakly tripotent ring. We note that 0 and 1 are idempotent elements of R and R does not have any other idempotent element. Also 2 is nilpotent in R. We have

$U(A) = \{1, 3, \text{g}, 2+\text{g}, 1+2\text{g}, 3+2\text{g}, 2+3\text{g}\}$ and $J(A) = \{0, 2, 2\text{g}, 3+\text{g}, 2+2\text{g}, 1+3\text{g}, 3+3\text{g}\}$.

Clearly $3 + 3g \in J(A)$, but $(3+3g)^2 = 2(3+3g) \neq 0$. Hence the proof is complete.

Proposition 2.4: Let A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.

Proof. We prove it as follows. Let us consider the ring A given above (we refer the proof of Proposition 2.3). After some computation one finds that $u = uvu$ or $1 - u = (1-u)v(1-v)$ holds for each $u \in A$ and some $v \in A$ with $v^2 = 1$. Therefore A is a locally invo-regular ring.

We have already noted that 2 is nilpotent in A and A is has no non-trivial idempotent elements. Further $1 + u \in J(A)$ such that $(1+u)^2 = 2(1+u) \neq 0$. Hence the proof is complete.

Proposition 2.5: Let A is a weakly tripotent ring having no non-trivial idempotents then $u^2 = 2u = 0 \Rightarrow u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid.

Proof. Let A is a weakly tripotent ring such that it has no non-trivial idempotents. Let $u^2 = 2u = 0$ for each $u \in J(A)$. This gives that $u^3 = 2u^2 = 0$. This in turn implies that $u^3 = 4u = 0$ for each $u \in J(A)$ The converse is not valid. Let us consider the ring A given in the proof of Proposition 2.3. Clearly $1 + u \in J(R)$ such that $(1+u)^3 = 4(1+u) = 0$ but $(1+u)^2 = 2(1+u) \neq 0$.

Proposition 2.6: Let A is a locally invo-regular ring having no non-trivial idempotents then $u^2 = 2u = 0 \Rightarrow u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid.

Proof. The proof directly follows from the above.

Conflict of Interest: The author declares that there is no conflict of interest.

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