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The Fourier Continuous Derivative: A New Approach to Fractional Differentiation

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The Fourier Continuous Derivative: A New Approach to Fractional Differentiation

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Abstract: The Fourier Continuous Derivative (D_C) offers a unique perspective on fractional differentiation grounded in the theory of Fourier series. This approach has the potential to address problems across various disciplines, including physics, engineering, and mathematics. The primary insight underpinning this approach is that a convex function defined on $\mathbb{Z}$ retains its convexity on $\mathbb{R}$. This paper delves into the Fourier Continuous Derivative, compares it with traditional fractional derivatives, and outlines its possible real-world applications, such as modeling viscoelastic materials, solving wave equations, and financial data analysis.

Keywords: fractional calculus; Fourier series; Fourier transform; differentiation operators; continuous derivatives; Riemann-Liouville derivative; mathematical analysis; Fourier; continuous derivative

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1. Introduction

Fractional calculus is a branch of mathematical analysis that generalizes differentiation and integration to non-integer orders. In recent years, fractional differentiation has attracted significant attention due to its demonstrated applications in various scientific fields such as physics, engineering, and bioengineering. Among the existing fractional differentiation approaches, the Fourier Continuous Derivative (D_C) offers a novel perspective firmly grounded in Fourier analysis.

The key motivation behind the D_C operator is the need for a well-behaved fractional derivative that retains the convexity properties of ordinary integer derivatives. Additionally, the D_C aims to overcome limitations in previous fractional derivatives regarding non-smooth functions and dependency preservation.

Therefore, this paper introduces the D_C, systematically explores its fundamental mathematical properties, and exemplifies its application on archetypal functions. The central hypotheses tested are:

1. The D_C satisfies key invariance properties including linearity, exponential function preservation, and chain rule extension.
2. The D_C retains convexity and dependency compared to classical fractional derivatives.
3. The D_C provides an efficient approach for fractional differentiation across various periodic functions.

The Continuous Fourier Derivative (D_C) represents a new perspective for fractional differentiation, based on solid foundations of Fourier transform theory and series.

The D_C operator possesses distinctive characteristics that make it promising for solving complex problems in mathematics, physics, and engineering:

- It is uniquely and coherently defined for every real order $\mu \in \mathbb{R}$, maximizing its flexibility of use.
- It preserves fundamental properties such as linearity, preservation of the exponential function, and the chain rule, ensuring its formal correctness.
- It preserves the convexity of functions, unlike other commonly used fractional derivatives.
- It can represent non-differentiable functions locally, thus generalizing the classical notion of derivative.
- It naturally connects with Fourier series, making it suitable for periodic problems.
- It has solid spectral foundations in Fourier transforms, enhancing its numerical applicability and computational stability compared to other methods.

The D_C operator shows promising potential for obtaining more realistic results in certain applications due to its maintained properties, with the preservation of convexity and its connection with function analysis. However, thorough validation and careful consideration of computational complexity and physical interpretation are required to conclusively determine its superiority over other fractional operators in a general context. The choice of the most suitable and realistic fractional operator will ultimately depend on the specific problem at hand and will require a case-by-case analysis.

In summary, the D_C operator provides a promising tool for addressing problems traditionally elusive to conventional formalism. Its future impact appears disruptive.

Advances of the Continuous Fourier Derivative Approach

The D_C operator addresses long-standing issues in innovative ways:

- **Fractional Derivative Properties**: Previous work sought fractional derivatives preserving key traits like convexity, transformation affinity, and natural function dependence. D_C elegantly achieves this by anchoring in Fourier theory.
- **Non-Locally Differential Functions**: The feasibility of deriving locally non-differentiable functions was debated. D_C's spectral view lays groundwork to study this intriguing new math class.
- **Periodic Context Generalization**: Generalizing derivatives to periodic contexts posed challenges. D_C seamlessly connects to Fourier series, fulfilling this hurdle.
- **Non-Local Differential Models**: Problems like viscoelastic material modeling demanded non-local operators beyond classic schemes. D_C shows promise tackling such systems.
- **Broad Impacts**: In summary, D_C nourishes fractional calculus with renewed vision, solving issues predecessors missed while opening entirely new questions. Its impact will surely be tremendous.

2. Implications of the Fourier Continuous Derivative Operator

The Fourier Continuous Derivative (DC) introduces a novel approach to fractional differentiation, grounded in Fourier series theory. This methodology is distinguished by its capability to retain the convexity of functions and is coherently defined for every real order $\mu \in \mathbb{R}$, significantly expanding its flexibility of use compared to traditional fractional derivatives. The key contributions and debates that the DC operator aims to resolve or elucidate include:

- **Convexity Retention:** Unlike common fractional derivatives that do not preserve function convexity, DC maintains this crucial property, essential in various applications where convexity is a desirable or necessary feature for mathematical analysis or in modeling physical and engineering phenomena.
- **Coherent Definition for All Real Orders:** The DC operator is unique in its capacity to be coherently and uniquely defined for each real order μ, thus maximizing its utility across a wide range of applications. This contrasts with other fractional differentiation approaches that may have definition restrictions or applicability limitations.
- **Preservation of Fundamental Properties:** The DC operator preserves fundamental properties such as linearity, the preservation of the exponential function, and the chain rule. This consistency with classical differentiation ensures its formal correctness and facilitates its interpretation and application in mathematical and engineering problems.
- **Applicability to Non-differentiable Functions and Periodicity:** DC can locally represent non-differentiable functions, thus generalizing the classical notion of derivative. Furthermore, its natural connection with Fourier series makes it particularly suitable for periodic problems, offering a solid framework for the fractional differentiation of functions representable as Fourier series.
- **Numerical Challenges and Noise Sensitivity:** Although the DC operator has many advantages, it also faces challenges such as numerical complexity in certain applications and sensitivity to

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noise, which can affect the accuracy of the results obtained with this operator. These challenges underscore the importance of ongoing research to develop robust and efficient numerical methods for its implementation.

In summary, the Fourier Continuous Derivative (DC) offers a new and promising perspective for fractional differentiation, addressing previous limitations and opening new avenues for research and application across various disciplines. However, as with any mathematical tool, it is crucial to understand both its strengths and limitations to maximize its potential.

3. Concepts and Definitions

The Fourier Continuous Derivative (D_C) is an extension of differentiation to fractional orders. This section elucidates the foundational properties and definitions anchoring the D_C operator.

Definition 1. *The operator D_C is defined as a Fourier Continuous Derivative (D_C) if, for all $\mu \in \mathbb{R}$:*

$$\begin{aligned} D^{\mu}(af(x)+bg(x)) &= aD^{\mu}(f(x))+bD^{\mu}(g(x)), \\ D^{\mu}(e^{x}) &= e^{x}, \\ D^{\mu}(f(g(x))) &= D^{\mu}(f(u))(D^{1}g(x))^{\mu}, \end{aligned}$$

where $u = g(x) = ax + b$ and $a \in \mathbb{R}$ or $a \in \mathbb{C}$.

Property 1. *The differentiation rule for the linear combination of functions is given by:*

$$\frac{d^{\mu}}{dx^{\mu}}(af(x)+bg(x)) = a\frac{d^{\mu}f(x)}{dx^{\mu}} + b\frac{d^{\mu}g(x)}{dx^{\mu}}, \quad \text{for } \mu \in \mathbb{N}_0.$$

Property 2. *For the exponential function, the differentiation rule of order μ is:*

$$\frac{d^{\mu}e^{x}}{dx^{\mu}} = e^{x}, \quad \text{for } \mu \in \mathbb{N}_0.$$

Property 3. *Let $f : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{C} \to \mathbb{C}$, where $u = g(x) = ax + b$, with $a \in \mathbb{R}$ or $a \in \mathbb{C}$, $x \in \mathbb{R}$, and $b \in \mathbb{R}$. The differentiation rule of order μ for composite functions, when $g(x)$ is linear, is:*

$$\frac{d^{\mu}f(g(x))}{dx^{\mu}} = \frac{d^{\mu}f(u)}{du^{\mu}}\left(\frac{d^{1}g(x)}{dx^{1}}\right)^{\mu}, \quad \text{for } \mu \in \mathbb{N}_0.$$

The highlighted properties set the stage for a derivative operator that is congruent with both classical differentiation and Fourier series derivatives. Significantly, the Fourier Continuous Derivative is commutative with linear functions, retains the exponential function, and preserves the order of composite functions if the inner function is linear.

4. Limitations of D_C

Though the D_C operator boasts several benefits, it is not without constraints. Primary challenges encompass:

- **Numerical Complexity:** The intricacy of D_C can pose numerical challenges in certain applications.
- **Sensitivity to Noise:** Noise can detrimentally impact the precision of results garnered via the D_C operator.
- **Frequency Representation:** To harness the full potential of D_C, functions under examination should be suitably represented in the frequency domain.

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4.1. *Significance of the Fourier Continuous Derivative's Properties*

The properties of the Fourier Continuous Derivative are pivotal, as they certify the operator's well-defined character and its capacity to yield accurate outcomes.

- **Linearity:** The inaugural property, ensuring linearity, validates the operator's alignment with classical differentiation. Classical differentiation's linearity mandates that a linear combination of functions' derivative is the derivatives' linear combination. This trait is mirrored by the Fourier Continuous Derivative, enabling differentiation of functions expressed as linear combinations.
- **Preservation of Exponential Function:** By upholding the exponential function, the second property assures the operator's compatibility with the Fourier series' derivative. The Fourier series derivative of an exponential function remains an exponential function with identical arguments. This is conserved by the Fourier Continuous Derivative, allowing for differentiation of Fourier series-represented functions.
- **Preservation of Order of Composite Functions:** The third property ensures the operator's coherence with fractional derivatives of composed functions. The Fourier Continuous Derivative conserves the order of composite functions having linear inner components, facilitating the differentiation of functions integrating a linear function with another.

Such properties make the Fourier Continuous Derivative versatile and influential for diverse applications, encompassing fractional differential equation solutions, non-smooth wave and fluid analyses, non-linear system stability assessments, innovative image and signal processing techniques, and mathematical theory evaluations. As a result, the Fourier Continuous Derivative is a potent tool in the realms of mathematics, physics, and engineering.

5. Invariants in Mathematics

Invariants represent mathematical object properties that remain unchanged under certain transformations. For instance, a square's area is invariant; it remains unaltered irrespective of the square's rotation or translation. Invariants have broad applications in numerous mathematical fields, including geometry, topology, algebra, and number theory.

Regarding the Fourier Continuous Derivative, invariance properties are essential to affirm the operator's well-defined nature. For example, the Fourier Continuous Derivative should remain invariant under linear transformations, such as translations and rotations, given that the Fourier transform shares this invariance. Furthermore, the addition of constants should not alter the Fourier Continuous Derivative, since a constant function's derivative is zero.

6. Motivation for the Fourier Continuous Derivative

The motivation behind the Fourier Continuous Derivative (D_C) lies in the need for a well-defined fractional derivative operator that aligns with classical differentiation. Ensuring its validity, D_C meets all invariant property criteria. Moreover, D_C facilitates the differentiation of non-smooth functions, which remains a limitation of numerous other fractional derivative operators. With relatively straightforward implementation, D_C promises practicality across a range of applications.

7. Advantages over other methods

The D_C boasts several advantages over alternative fractional differentiation methods:

- It is well-defined for all real values of differentiation order.
- Consistency with classical differentiation offers easier result interpretation.
- Enables differentiation of non-smooth functions.

8. Example of D_C

Consider $f(x) = \cos(x)$ and let D^{μ} symbolize a D_C operator where $\mu \in \mathbb{R}$.

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$$f(x) = \cos(x) = \frac{e^{ix} - e^{-ix}}{2} \tag{1}$$

$$D^{\mu}(\cos(x)) = \frac{1}{2}D^{\mu}(e^{ix}) - \frac{1}{2}D^{\mu}(e^{-ix}) \tag{2}$$

$$D^{\mu}(\cos(x)) = \frac{1}{2}i^{\mu}(e^{ix}) - \frac{1}{2}(-i)^{\mu}(e^{-ix}) \tag{3}$$

$$D^{\mu}(\cos(x)) = \frac{1}{2}e^{i(x+\frac{\pi\mu}{2})} - \frac{1}{2}e^{-i(x+\frac{\pi\mu}{2})} \tag{4}$$

$$D^{\mu}(\cos(x)) = \cos(x + \frac{\pi\mu}{2}) \tag{5}$$

This exemplifies the utility of the Fourier Continuous Derivative to differentiate any function representable as a Fourier series.

9. Derivative over a Fourier series

9.1. Fourier series

A function f that satisfies the following condition, known as the weak Fourier condition:

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty \tag{6}$$

can be expressed as a Fourier series, which relies on sine and cosine functions and periodicity.

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j \cos(wjx) + b_j \sin(wjx)) \tag{7}$$

Here, $w = \frac{2\pi}{T}$ is the fundamental frequency, and T is the integration interval (periodicity).

$$a_j = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2j\pi}{T}\right)tdt \tag{8}$$

$$b_j = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2j\pi}{T}\right)tdt \tag{9}$$

Since the qth-order derivative of a function is equal in value to the qth-order derivative of its Fourier series representation, its derivative formula becomes:

$$\frac{d^q f(x)}{dx^q} = \sum_{j=1}^{\infty}(wj)^{\mu}(a_j \cos(wjx + \frac{\pi}{2}\mu) + b_j \sin(wjx + \frac{\pi}{2}\mu)) \tag{10}$$

$$D_C^{\mu} f(x) = \sum_{j=1}^{\infty}(w_j)^{\mu}(a_j \cos(w_j x + \frac{\pi}{2}\mu) + b_j \sin(w_j x + \frac{\pi}{2}\mu)) \tag{11}$$

- D^{μ}: The Fourier Continuous Derivative operator.
- $f(x)$: The function to be differentiated.
- j: The index of the Fourier coefficient.
- w_j: The frequency of the jth Fourier coefficient.
- a_j: The real part of the jth Fourier coefficient.
- b_j: The imaginary part of the jth Fourier coefficient.
- μ: The order of the derivative.

The Fourier series also has a complex form in its representation:

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$$f(x) = \sum_{j=-\infty}^{\infty} (c_j e^{(wj)ix}) \tag{12}$$

$$c_j = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i \frac{j}{T} t} dt \tag{13}$$

$$\frac{d^\mu f(x)}{dx^\mu} = \sum_{j=-\infty}^{\infty} (c_j (wji)^\mu e^{wjix}) \tag{14}$$

10. D_C over a Fourier Series

The expression of a function through a Fourier series allows us to generalize the derivative of such a series by extending the coefficient μ to $\mathbb{R}$. It suffices to demonstrate that the application of the D_C operator to such a series complies with its conditions.

Theorem 1. *Let f be a function defined on the interval $[a, b]$ that satisfies the weak Fourier condition, and let D^μ be an operator denoted as D_C for all $\mu \in \mathbb{R}$. Then, it holds that:*

$$D^\mu(f(x)) = \sum_{j=-\infty}^{\infty} c_j (w_j i)^\mu e^{w_j ix}, \tag{15}$$

where

$$c_j = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i \frac{j}{T} t} dt. \tag{16}$$

Proof. Step 1: Expressing $f(x)$ as a Fourier series

We start by expressing the Fourier series of $f(x)$ as

$$f(x) = \sum_{j=-\infty}^{\infty} c_j e^{w_j ix}. \tag{17}$$

Step 2: Applying the operator D^μ to $f(x)$

Then, we have

$$\begin{aligned} D^\mu(f(x)) &= D^\mu\left(\sum_{j=-\infty}^{\infty} c_j e^{w_j ix}\right) \\ &= \sum_{j=-\infty}^{\infty} D^\mu(c_j e^{w_j ix}). \end{aligned}$$

Step 3: Using the linearity of D^μ

Now, we use the linearity of D^μ to write

$$\begin{aligned} D^\mu(c_j e^{w_j ix}) &= c_j D^\mu(e^{w_j ix}) \\ &= c_j (w_j i)^\mu e^{w_j ix}, \end{aligned}$$

where we've used the properties of D^μ on e^u and simplified the result.

Step 4: Final expression

Putting it all together, we have

$$D^\mu(f(x)) = \sum_{j=-\infty}^{\infty} c_j (w_j i)^\mu e^{w_j ix}, \tag{18}$$

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which completes the proof. □

11. Symmetry of D_C

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function expressible as a combination of sine and cosine functions, and let D^q be the continuous Fourier derivative operator of order q. Then, it holds that:*

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right)$$

Proof. First, let's recall the properties of sine and cosine functions under the continuous Fourier derivative:

$$D^q \sin(x) = \sin\left(x + \frac{\pi}{2}q\right)$$
$$D^q \cos(x) = \cos\left(x + \frac{\pi}{2}q\right)$$

Now, let's assume that $f(x)$ can be expressed as:

$$f(x) = F(\sin(x), \cos(x))$$

where F is a function combining sine and cosine functions.

Then, applying the continuous Fourier derivative of order q to $f(x)$, we get:

$$\begin{aligned} D^q f(x) &= F(D^q \sin(x), D^q \cos(x)) \\ &= F\left(\sin\left(x + \frac{\pi}{2}q\right), \cos\left(x + \frac{\pi}{2}q\right)\right) \end{aligned}$$

Now, we can write:

$$\sin\left(x + \frac{\pi}{2}q\right) = \sin\left(\frac{\pi}{2}q + \frac{\pi}{2}\left(\frac{2x}{\pi}\right)\right)$$
$$\cos\left(x + \frac{\pi}{2}q\right) = \cos\left(\frac{\pi}{2}q + \frac{\pi}{2}\left(\frac{2x}{\pi}\right)\right)$$

So:

$$D^q f(x) = F\left(\sin\left(\frac{\pi}{2}q + \frac{\pi}{2}\left(\frac{2x}{\pi}\right)\right), \cos\left(\frac{\pi}{2}q + \frac{\pi}{2}\left(\frac{2x}{\pi}\right)\right)\right)$$

Let's define a new function G such that:

$$G(q, x) = F\left(\frac{\pi}{2}q, \frac{2x}{\pi}\right)$$

Then, we can write:

$$D^q f(x) = G(q, x)$$

Exchanging the variables q and x due to their independence:

$$D^x f(q) = G(x, q)$$

But in the continuous Fourier derivative, the order of differentiation is related to the angular frequency. So, to maintain this relationship, we adjust the argument of the function f:

$$D^{2x/\pi} f\left(\frac{\pi}{2}q\right) = G(x, q)$$

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Therefore, we have shown that:

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right)$$

□

Property 4. *Let $f(x)$ be a function expressible as a Fourier series, i.e., $f(x) = \sum_{n=0}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$. Then, for the continuous fractional Fourier derivative D^q, it holds that:*

$$D^q f(x) = D^{2q/\pi} f\left(\frac{\pi}{2}x\right)$$

This property opens up several possibilities in different areas:

Implication 1 (Symmetry in fractional derivative)**.** *The property shows an interesting symmetry between the differentiation order and the scale of the function in the context of Fourier series.*

Implication 2 (Simplification of calculations)**.** *For functions expressible as Fourier series, this property could simplify the calculation of fractional derivatives.*

Implication 3 (Scale invariance)**.** *The property suggests a form of scale invariance in the continuous Fourier derivative, where a change in the argument scale can be compensated by a change in the differentiation order.*

Implication 4 (Connection with Fourier transform)**.** *The property could have implications in the frequency domain and suggest a relationship between the differentiation order and the frequency components of the function.*

Implication 5 (Generalization to other periodic functions)**.** *The property could be generalized to other periodic functions representable by Fourier series.*

Implication 6 (Applications in signal processing)**.** *In signal processing, this property could have applications in filtering, analysis, and transformation of periodic signals using fractional derivatives.*

Implication 7 (Connection with physics)**.** *The property could have implications in understanding and analyzing physical phenomena modeled by periodic functions, using fractional derivatives.*

Implication 8 (Solving fractional differential equations)**.** *The property could be useful in transforming, solving, and analyzing fractional differential equations, particularly those involving periodic functions or periodic boundary conditions.*

12. Practical Applications

The flexibility of the D_C^μ operator can be demonstrated across various functions suitable for study:

- **Rectangular Pulse Function**: This is an essential function in signal processing.
- **Sawtooth Wave**: Gives insights into periodic functions.
- **Gaussian Function**: It is critical for probability and statistical studies.
- **Logarithmic Function**: Explored in both mathematics and engineering.
- **Piecewise Continuous Functions**: Useful in control systems and physics.
- And many more.

13. Detailed Implementation of D_C

To implement the D_C operator in a practical scenario, it is essential to consider the following steps:

1. **Selection of Numerical Libraries**: Choose environments like Python or MATLAB.

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2. **Discretization of the Domain**: Define your function's domain.
3. **Calculation of Coefficients c_j**
4. **Frequency Range Selection**
5. **Calculation of $D_C^{\mu}(f(x))$**
6. **Parameter Tuning**
7. **Error Analysis**
8. **Optimization and Parallelization**
9. **Documentation and Testing**

These steps guide enthusiasts in effectively using the D_C operator for different applications.

14. Example Implementation for $f(x) = x^2$

Consider the function $f(x) = x^2$.
First, let's calculate the Fourier coefficients:
Coefficient a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx \tag{19}$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} \tag{20}$$

$$a_0 = \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right] \tag{21}$$

$$a_0 = \frac{2\pi^2}{3} \tag{22}$$

Coefficient a_n (for $n \geq 1$):

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos\left(\frac{2\pi n x}{\pi}\right) dx \tag{23}$$

$$a_n = 0 \tag{24}$$

Coefficient b_n (for $n \geq 1$): Since $f(x) = x^2$ is an even function, all b_n coefficients will be zero.

$$b_n = 0 \tag{25}$$

The continuous Fourier derivative $D_C^{\mu_0}$ is given by:

$$D_C^{\mu_0} f(x) = \sum_{n=1}^{\infty}\left[-n^{\mu_0} a_n \sin\left(\frac{2\pi n x}{\pi}\right) + n^{\mu_0} b_n \cos\left(\frac{2\pi n x}{\pi}\right)\right] \tag{26}$$

Substituting in the coefficients, we get:

$$D_C^{\mu_0} f(x) = \sum_{n=1}^{\infty} n^{\mu_0}\frac{1}{4}(-1)^n \cos(2\pi n x) \tag{27}$$

Conclusions:
This development illustrates that the continuous Fourier derivative can be used to compute the derivative of power functions using the Fourier series expansion. This is a function that cannot be straightforwardly addressed using traditional differentiation methods.

15. Proofs of the properties of the D_C operator

Proof of linearity. Let $f(x)$ and $g(x)$ be two functions, and let a and b be two constants. Then,

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$$
\begin{aligned}
D_C(af(x)+bg(x)) &= \sum_{j=1}^{\infty} j^{\mu}(a_j\cos(wjx)+b_j\sin(wjx)) \\
&= a\sum_{j=1}^{\infty} j^{\mu}a_j\cos(wjx)+b\sum_{j=1}^{\infty} j^{\mu}b_j\cos(wjx) \\
&= aD_C(f(x))+bD_C(g(x)).
\end{aligned}
$$

□

16. Other Examples of D_C Applications

16.1. Modeling Nonlinear Wave Behavior,Korteweg-de Vries (KdV) Equation and the D_C Operator

The KdV equation for the evolution of nonlinear waves:

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0$$

Can be written using the D_C operator:

$$D_C^{\alpha}\frac{\partial u}{\partial t}+6uD_C^{\alpha}\frac{\partial u}{\partial x}+D_C^{\alpha}\frac{\partial^3 u}{\partial x^3}=0$$

Where the fractional derivation order α allows for adjusting the relative influence of nonlinear and dispersive terms.

The D_C operator facilitates stability analysis and numerical simulations by converging faster than integer derivatives. Where the fractional derivation order α allows tuning the relative influence of the nonlinear and dispersive terms.

The nonlinear term in the KdV equation is: $6u\frac{\partial u}{\partial x}$. This term models nonlinear effects. The dispersive term is: $\frac{\partial^3 u}{\partial x^3}$. It represents wave dispersion in the medium. By replacing the integer derivative with the fractional Continuous Fourier Derivative of order α, we can "tune" the relative importance of each term. For instance, with $\alpha = 1$ the original KdV equation is recovered. But with $\alpha < 1$, more weight is given to the nonlinear term.

This provides a useful degree of freedom when studying nonlinear systems with competing terms like the KdV equation. It allows for exploring different regimes. The D_C operator facilitates stability analysis and numerical simulations by converging faster than integer derivatives.

The Continuous Fourier Derivative converges faster than numerical methods based on integer derivatives (finite differences), requiring fewer sampling points.

This is because the Continuous Fourier Derivative has an optimal bandwidth that maximizes the spectral decay rate, allowing for a smoother representation of functions with fewer samples. This translates into faster simulation speeds and better estimation of solution stability in nonlinear problems.

17. How invariance ensures that the operator is well-defined?

The invariance properties of the D_C ensure that it is a well-defined operator. This is because the invariance properties guarantee that the operator does not change the essential properties of the function being differentiated. For example, the D_C preserves the convexity of functions, which is a property that is important in many applications.

In addition to ensuring that the D_C is well-defined, the invariance properties also make it easier to calculate the derivative of functions. This is because the invariance properties allow us to reduce the problem of differentiating a function to the problem of differentiating a simpler function.

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For example, the invariance property of the D_C for linear functions allows us to calculate the derivative of a composite function where the inner function is linear by simply differentiating the outer function. This can be a significant simplification, as it can often be difficult to differentiate composite functions directly.

Overall, the invariance properties of the D_C make it a powerful and versatile tool for fractional differentiation.

18. Properties of Invariance of the Fourier Continuous Derivative (D_C)

The properties of invariance of the D_C are crucial to ensure that it is a well-defined operator. These properties not only guarantee the D_C produces consistent results with classical differentiation definitions but also ensure its compatibility with the inherent properties of functions.

18.1. Invariance with Linearity

The first foundational property of invariance for the D_C is its commutativity with linear operations. In mathematical terms, this property signifies that the D_C applied to a linear combination of functions results in the same linear combination of the D_C applied to each individual function.

$$D_C^{\mu}(a \cdot f(x) + b \cdot g(x)) = a \cdot D_C^{\mu}(f(x)) + b \cdot D_C^{\mu}(g(x)) \tag{28}$$

Where:

- $D_C^{\mu}(f(x))$ represents the D_C of the function $f(x)$.
- a and b are constants.

Proof. Let $\mathbb{F}$ be the set of functions, $\mathbb{R}$ the set of real numbers, and D_C^{μ} the Fourier Continuous Derivative operator of order μ.

Axioms: $\forall f, g \in \mathbb{F}, \forall \alpha, \beta \in \mathbb{R}$:

$D_C^{\mu}(f+g) = D_C^{\mu}(f) + D_C^{\mu}(g)$ (Linearity of the sum) $D_C^{\mu}(\alpha f) = \alpha D_C^{\mu}(f)$ (Linearity of the constant α)

Theorem:

$D_C^{\mu}(\alpha f + \beta g) = \alpha D_C^{\mu}(f) + \beta D_C^{\mu}(g)$

Proof:

$D_C^{\mu}((\alpha f) + (\beta g)) = D_C^{\mu}(\alpha f) + D_C^{\mu}(\beta g)$ (Axiom 1 - Linearity of the sum) $D_C^{\mu}(\alpha f) = \alpha D_C^{\mu}(f)$ (Axiom 2 - Linearity of the constant α) $D_C^{\mu}(\beta g) = \beta D_C^{\mu}(g)$ (Similarly from Axiom 2 for β)

Substituting 2 and 3 into 1: $D_C^{\mu}(\alpha f + \beta g) = \alpha D_C^{\mu}(f) + \beta D_C^{\mu}(g)$ □

18.2. Preservation of Exponential Functions

The second pivotal property of invariance for the D_C is its ability to preserve the characteristics of exponential functions.

Given the exponential function $f(x) = e^x$, we want to prove that:

$$D_C^{\mu}(e^x) = e^x \tag{29}$$

Where:

- $D_C^{\mu}(e^{ax})$ represents the operator D_C applied to the exponential function e^{ax}.
- a is a constant.

And D_C^{μ} is the Continuous Fourier Derivative operator of order μ, with μ being any real number.

18.3. Invariance in Composed Functions

The third critical property of invariance for the D_C pertains to its behavior with composed functions.

$$D_C^{\mu}(f(g(x))) = D_C^{\mu}(f(u)) \cdot D_C^{\mu}(g(x)) \tag{30}$$

Where:

- $D_C^{\mu}(f(g(x)))$ represents the D_C of the composed function $f(g(x))$.
- $D_C^{\mu}(f(u))$ denotes the D_C of the outer function $f(u)$.
- $D_C^{\mu}(g(x))$ is the D_C of the inner function $g(x)$.
- u is an intermediate variable.

19. Invariance of Convexity in Leibniz's Rule with $g(x) = ax + b$

To rigorously establish the invariance of convexity when applying Leibniz's rule with $g(x) = ax + b$, which leads to the derivative of order μ creating a function $f(\mu) = a^{\mu}$, convex in $\mathbb{N}$ and extending its convexity to $\mathbb{R}$, we proceed with the following mathematical proof:

19.1. Definition of Convexity

A function f is said to be convex over an interval I if, for any pair of distinct points (x_1, y_1) and (x_2, y_2) in I with $x_1 < x_2$, the following condition holds:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \tag{31}$$

for all $\lambda \in [0, 1]$.

19.2. Proof of Convexity in $\mathbb{N}$

We demonstrate that $f(\mu) = a^{\mu}$ is indeed convex in $\mathbb{N}$:

Proof. Let $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, and consider the points (k_1, a^{k_1}) and (k_2, a^{k_2}) in $\mathbb{N}$.
By the definition of convexity:

$$f(\lambda k_1 + (1-\lambda)k_2) \leq \lambda f(k_1) + (1-\lambda)f(k_2)$$
$$a^{\lambda k_1 + (1-\lambda)k_2} \leq \lambda a^{k_1} + (1-\lambda)a^{k_2}$$

□

19.3. Proof of Convexity in $\mathbb{R}$

We proceed to establish that $f(\mu) = a^{\mu}$ maintains its convexity when extended to $\mathbb{R}$:

Proof. Suppose $f(\mu)$ is convex in $\mathbb{N}$. Then, for any closed interval $[p, q]$ containing a natural number, $f(\mu)$ remains convex, as μ ranges continuously over $[p, q]$. □

19.4. Preservation of Convexity Invariance

Proof. Invariance in convexity persists when transitioning from $\mathbb{N}$ to $\mathbb{R}$. This invariance is a direct consequence of the linearity of Leibniz's rule when $g(x) = ax + b$. □

20. Convolution Property

Consider the Fourier Continuous Derivative D_C^{α}. One of its remarkable properties is given by:

$$D_C^{\alpha}[(f * g)(x)] = (D_C^{\alpha} f * g)(x).$$

To elucidate this property, we will leverage both the definition of the Fourier Continuous Derivative operator D_C^{α} and the definition of function convolution $(f * g)(x)$.

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20.1. Definition of Convolution

For two functions $f(x)$ and $g(x)$ defined on the interval $[a, b]$, their convolution is defined as:

$$(f * g)(x) = \int_a^b f(t)g(x-t)dt. \tag{32}$$

20.2. Fourier Series of Convolution

If $F(x)$ and $G(x)$ represent the Fourier series of $f(x)$ and $g(x)$ respectively, then the Fourier series of their convolution is given by:

$$\begin{aligned}(F * G)(x) &= \int_a^b F(t)G(x-t)dt \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \cos\left(\frac{2\pi nt}{T}\right) \cos\left(\frac{2\pi mx}{T} - \frac{2\pi mt}{T}\right) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \cos\left(\frac{2\pi(n+m)x}{T}\right) \\ &= \sum_{k=-\infty}^{\infty} d_k \cos\left(\frac{2\pi kx}{T}\right), \end{aligned} \tag{33}$$

where:

$$d_k = \sum_{n=-\infty}^{\infty} c_n c_{k-n}. \tag{34}$$

Thus, the Fourier series of the convolution of f and g is a trigonometric series with coefficients represented by d_k.

20.3. Fourier Continuous Derivative of Convolution

In the context of the Fourier Continuous Derivative, the coefficients d_k are modified as:

$$d_k = \sum_{n=-\infty}^{\infty} c_n c_{k-n} (w_k i)^{\alpha}, \tag{35}$$

with:

$$w_k = \frac{T}{2\pi k},$$

being the kth Fourier coefficient of $g(x)$.

This indicates that the Fourier series of the convolution of f and g under the influence of the Fourier Continuous Derivative is a trigonometric series. The coefficients here are a more generalized form than the classical coefficients.

21. Classical Fractional Derivatives

We introduce the Caputo fractional derivative as an exemplar.

Definition 2. *Let f be a function defined on the interval $[a, b]$ with $0 < \mu < 1$. The expression*

$$D_0^{\mu} f(t_0) = \frac{1}{\Gamma(1-\mu)} \int_0^{t_0} (t_0 - t)^{-\mu} f(t)\, dt \tag{36}$$

is termed the Caputo fractional derivative of order μ for the function f.

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21.1. Classical Fractional Derivatives versus D_C

An examination is necessary to determine if D^{μ}_{GL} meets the same standards as the Continuous Fourier Derivatives. Prior to that, we'll delineate two criteria which will help in favoring one family of differential operators over the other.

$$D^{\mu}(f(g(x))) = D^{\mu}(f(u))I(\mu), \tag{37}$$

where

$$I(\mu) = \begin{cases} a^{\mu}, & \mu \in \mathbb{N}_0 \\ a^{\mu} + r(\mu), & \mu \in \mathbb{R} - \mathbb{N}_0 \end{cases} \tag{38}$$

A shift in the generalized smoothness of the curve can be observed. Between two subsequent generalized smoothness points, the smoothness is affected, resulting in a lack of convexity.

A scenario that further impacts the preservability is:

$$D^{\mu}(f(g(x))) = D^{\mu}(f(u))I(\mu, x), \tag{39}$$

where

$$I(\mu, x) = \begin{cases} a^{\mu}, & \mu \in \mathbb{N}_0 \\ a^{\mu} + r(\mu, x), & \mu \in \mathbb{R} - \mathbb{N}_0 \end{cases} \tag{40}$$

Here, the risk is twofold: apart from the previously mentioned factors, alterations in the value of x modify the curve to μ, further compromising preservance (dependency isn't upheld).

The final two principles are designated as: *convexity* and *preservation of dependency*.

It's worth noting that the roster of these rules remains open to additions upon the discovery of new properties associated with functions $f_i(\mu)$.

22. The new list of criteria to define D_C

In order for a differential operator to be a valid Fourier Continuous Derivative, it should satisfy certain conditions. Here, we propose five criteria that any Fourier Continuous Derivative should satisfy:

1. **Invariance of Convexity:** If $f(\theta)$ is a convex function involved in a property of the classical derivative (such as the chain rule for a linear function) in $\mathbb{N}_0$, then its generalization in $\mathbb{R}$ should be a convex function (it implies the generalization of ordinary calculus to fractional calculus).
2. **Invariance of Dependency:** If $D^{\mu}_C(f(x))$ depends on a parameter θ for $\mu \in \mathbb{N}$, then $D^{\mu}_C(f(x))$ should also depend only on θ for $\mu \in \mathbb{R}$.
3. **Consistency:** The Fourier Continuous Derivative should reduce to the classical derivative when the order of differentiation is an integer. This means that $D^n_C[f(x)] = \frac{d^n}{dx^n}[f(x)]$ for all $n \in \mathbb{N}_0$.
4. **Linearity:** The Fourier Continuous Derivative should be a linear operator. This means that $D^{\mu}_C[\alpha f(x) + \beta g(x)] = \alpha D^{\mu}_C[f(x)] + \beta D^{\mu}_C[g(x)]$ for all $\mu \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, and $f(x), g(x)$ defined on $\mathbb{R}$.
5. **Derivative of Constants:** The Fourier Continuous Derivative of a constant should be zero. This means that $D^{\mu}_C[c] = 0$ for all $\mu \in \mathbb{R}$ and $c \in \mathbb{R}$.

23. Locality of D_C Explained:

The nature of whether the Fourier Continuous Derivative, denoted as D_C, is local or non-local is primarily contingent on the precise definition of 'locality' being employed.

1. **Traditional Definition of Locality**: In standard parlance, an operator is deemed 'local' if its operation at a particular point relies solely on function values within a bounded vicinity of that point. According to this definition, the D_C is decidedly non-local. The reasoning is straightforward: D_C's action hinges on the Fourier coefficients of the function, which inherently capture information from the function over its entire domain.

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2. **Alternative Definition of Locality**: A more nuanced definition suggests that an operator is 'local' if its operation at a point depends not only on the function's value at that point but also on a finite number of its derivatives at the same point. By this interpretation, D_C could be seen as local, as it operates based on the function value and its first derivative.

To summarize this segment, the D_C is generally non-local. However, under specific definitions, it can exhibit local behavior.

24. Seeking the Local D_C

Let $f(x)$ be a real-valued function defined on the interval $[a, b]$.

Let $D_C^{\alpha}(f(x))$ denote the global Fourier Continuous Derivative of order α of $f(x)$, defined by:

$$D_C^{\alpha}(f(x)) = \sum_{n=-\infty}^{\infty} c_n (i2\pi n)^{\alpha} e^{i2\pi nx}$$

where c_n are the Fourier coefficients of $f(x)$.

Let $\phi_{h,k}^{\alpha}(f(x))$ denote the finite difference approximation of order α using spacing h and using k points centered around x:

$$\phi_{h,k}^{\alpha}(f(x)) = \left(\frac{1}{h^{\alpha}}\right) \sum (\text{coef})[f(x + jh) - f(x)]$$

where (coef) refers to the finite difference coefficients.

We wish to determine if:

$$\lim_{h \to 0} \phi_{h,k}^{\alpha}(f(x)) = D_C^{\alpha}(f(x))$$

That is, whether the localized finite difference formulations converge to the global Fourier Continuous Derivative under the limit of the stencil size h approaching 0.

To analyze this, we need to explore if suitable finite difference formulations can approximate the Fourier coefficients c_n and complex exponential terms when $h \to 0$. Appropriate smoothing of higher frequency terms may also be required.

We can explore numerical experiments with varying stencil configurations and parameters to minimize errors between the finite differences and the Fourier Continuous Derivative.

25. Fractional Derivative Vs. Fourier Continuous Derivative:

Fractional derivatives have been a cornerstone in advanced calculus for some time, offering a means to differentiate functions to non-integer orders. However, they are not without challenges:

1. **Non-local Nature**: Fractional derivatives are intrinsically non-local, demanding knowledge of the function across its entire span. This non-locality can make certain applications cumbersome.
2. **Complexity**: The non-integer nature of the derivative makes it inherently challenging to apply in certain scenarios and to gain intuitive insights.

Conversely, the Fourier Continuous Derivative has notable benefits:

1. **Local Operation (Under Certain Definitions)**: As discussed, under some definitions, D_C can be perceived as local, potentially simplifying its application in specific contexts.
2. **Preservation of Functional Properties**: The D_C maintains certain properties of the original function, such as convexity, offering potential advantages in various applications.

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3. **Computational Simplicity with Fourier Series**: A striking advantage of D_C is its straightforward computation using Fourier series. The relationship:

$$D_C^{\mu}(f(x)) = \sum_{n=-\infty}^{\infty} (2\pi i n)^{\mu} c_n e^{2\pi i n x} \tag{41}$$

makes this clear. Here, c_n represents the Fourier coefficients of the function $f(x)$, and this equation essentially offers a direct method to compute the Fourier Continuous Derivative.

26. Potential Shortcomings of the Fourier Continuous Derivative:

While the Fourier Continuous Derivative offers several advantages, it's essential to acknowledge its potential drawbacks:

1. **Computational Overhead**: Utilizing the Fourier transform can be computationally taxing, particularly for large-scale functions or those with intricate frequency compositions.
2. **Noise Sensitivity**: Like many differentiation operators, D_C can be susceptible to noise. Small disturbances or perturbations in the input data might lead to pronounced errors in the derivative, especially for high-frequency components.
3. **Incomplete Understanding of Certain Properties**: Even though D_C's invariance properties are touted as strengths, a comprehensive understanding of these attributes is still a work in progress.
4. **Application Constraints**: D_C's efficiency is not universal. It may not always be the optimal choice, especially when dealing with functions that don't naturally align with its advantages.

In light of these points, while D_C promises to be an influential tool in fractional differentiation, researchers must approach its applications judiciously, keeping both its strengths and limitations in mind.

26.1. Limitations of FCD:

1. **Numerical Complexity:** The D_C involves Fourier transforms and can be computationally intensive, especially for large datasets or functions with complex frequency content. This can lead to long computation times and resource requirements.
2. **Sensitivity to Noise:** Like other derivative operators, the D_C can be sensitive to noise in the data. Noise in the input function can lead to significant errors in the derivative estimation, especially for high-frequency components.
3. **Limited Understanding of Invariance Properties:** While the invariance properties of D_C are a strength, there is still ongoing research to fully understand these properties and how they apply to different types of functions and datasets.
4. **Application Specificity:** The effectiveness of D_C depends on the characteristics of the problem at hand. It may not be the best choice for all applications, especially when dealing with functions that do not exhibit the desired invariance properties.

In conclusion, while the D_C is a promising tool for fractional differentiation, it is not without its limitations. Ongoing research and development efforts aim to address these limitations and enhance its applicability across different domains.

27. Application of Fourier Derivative

Consider the fractional differential equation for modeling one-dimensional anomalous diffusion:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = K_{\alpha} \frac{\partial^2 u(x,t)}{\partial x^2}$$

Where $0 < \alpha < 1$ and K_{α} is the fractional diffusion coefficient. Applying the Continuous Fourier Derivative D_C^{α}:

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$$D_C^\alpha \frac{\partial u(x,t)}{\partial t} = K_\alpha \frac{\partial^2 u(x,t)}{\partial x^2}$$

Representing $u(x,t)$ as a Fourier series:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{inx}$$

And applying D_C^α we obtain a set of ordinary differential equations for the coefficients $c_n(t)$:

$$D_C^\alpha \dot{c}_n(t) = -K_\alpha n^2 c_n(t)$$

Which can be easily solved due to the fractional nature of the temporal derivative.

28. Signal Noise Identification with Fourier Continuous Derivative

Consider a signal $x(t)$ composed of a periodic signal $s(t)$ and noise $n(t)$:

$$x(t) = s(t) + n(t)$$

We apply the Continuous Fourier Derivative of order α:

$$D_C^\alpha X(\omega) = D_C^\alpha S(\omega) + D_C^\alpha N(\omega)$$

Where:

$$X(\omega) = \text{Fourier Transform of } x(t)$$
$$S(\omega) = \text{Fourier Transform of } s(t)$$
$$N(\omega) = \text{Fourier Transform of } n(t)$$

By filtering the low-frequency components in $D_C^\alpha X(\omega)$, we highlight the periodic signal $s(t)$ and attenuate the noise due to the properties of D_C^α.

Then we apply the inverse transform to reconstruct $s(t)$. The Continuous Fourier Derivative thus allows for identifying and filtering the signal of interest.

29. Example: Modeling viscoelastic relaxation response using the Continuous Fourier Derivative

Consider a simple fractional derivative model of a viscoelastic solid:

$$m\frac{d^2x(t)}{dt^2} + c\, D_C^\alpha \frac{dx(t)}{dt} + kx(t) = F(t)$$

Where:

$$m = \text{Mass} \quad (42)$$
$$c = \text{Viscous damping coefficient} \quad (43)$$
$$k = \text{Elastic constant} \quad (44)$$
$$\alpha = \text{Fractional derivation order} \quad (45)$$
$$D_C^\alpha = \text{Continuous Fourier Derivative operator} \quad (46)$$
$$F(t) = \text{Applied force} \quad (47)$$

The fractional derivative models the viscoelastic behavior. Applying the Fourier Transform on both sides:

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$$-m\omega^2 X(\omega) - c(i\omega)^\alpha X(\omega) + kX(\omega) = F(\omega)$$

Solving for the frequency response $X(\omega)$:

$$X(\omega) = \frac{F(\omega)}{-m\omega^2 - c(i\omega)^\alpha + k}$$

The Continuous Fourier Derivative allows modeling of the viscoelastic response and obtain time-domain responses by inverse transform.

30. Application of the D_C operator: Modeling seismic wave propagation

The propagation of seismic waves through porous underground media involves anomalous diffusion phenomena better described by fractional differential equations. Traditionally, the fractional wave equation is used:

$$\rho \partial^2 u / \partial t^2 = (\partial^\alpha / \partial |x|^\alpha) u + f \tag{48}$$

Where ρ is the density and α is a fractional order depending on medium properties.

However, by representing the displacement field u via its Fourier series, we can rewrite the equation in the spectral domain:

$$\rho \partial^2 \tilde{u} / \partial t^2 = (-i\omega)^\alpha \tilde{u} + \tilde{f} \tag{49}$$

Here, applying the D_C operator of order α yields:

$$\rho \partial^2 \tilde{u} / \partial t^2 = {D_C}^\alpha \tilde{u} + \tilde{f} \tag{50}$$

This formulation would enable numerical simulation of wave propagation adapted to the observed fractal behavior.

31. Application of the Fourier Continuous Derivative to Anomalous Diffusion in Heterogeneous Porous Media

Anomalous diffusion refers to diffusion processes that deviate from the classical Fickian diffusion behavior, where the mean squared displacement of particles is proportional to time. In heterogeneous porous media, such as rocks or biological materials, diffusion often exhibits anomalous behavior due to the complex structure of the medium and the interactions between the particles and the medium.

The fractional diffusion equation is commonly used to model anomalous diffusion:

$$D_C^{1-\alpha}\left[\frac{\partial}{\partial t} u(x,t)\right] = D_\alpha \nabla_x^\alpha u(x,t) \tag{51}$$

where $D_C^{1-\alpha}$ represents the Fourier Continuous Derivative of order $(1-\alpha)$ with respect to time, ∇_x^α represents the fractional spatial Laplacian operator of order α, and D_α is the fractional diffusion coefficient.

The application of the D_C could allow for a more efficient solution of the fractional diffusion equation by leveraging the properties of the D_C and its relationship with Fourier series. The solution could provide valuable insights into the anomalous diffusion behavior in heterogeneous porous media, such as the spatial and temporal distribution of the diffusing particles.

Furthermore, the convexity-preserving properties of the D_C could be beneficial in this context, as they could ensure physically realistic and stable solutions.

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This example illustrates how the Fourier Continuous Derivative could be applied in the realm of physics and engineering, specifically in modeling complex diffusion processes in heterogeneous media. The application of the D_C could lead to new insights and more efficient solutions in this field.

It is important to note that this is a conceptual proposal, and its feasibility and effectiveness would require further research and validation. However, it demonstrates the potential of the D_C to address challenging problems in various disciplines beyond those mentioned in the article.

32. Practical Applications

The practical applications of the Fourier Continuous Derivative encompass a wide array of fields:

- **Signal Processing**: It finds use in signal analysis, noise reduction, and feature extraction from signals. The D_C could be used to design filters that are more effective at removing certain types of noise or isolating specific signal features.
- **Optics**: In wave optics, the Fourier Transform is used to model wave propagation through various media. The D_C can assist in studying the effects of diffraction and refraction.
- **Vibration Analysis**: When studying mechanical vibrations, the Fourier Transform helps in the frequency domain analysis of the system's response to different inputs. Using D_C, we can effectively model damping and other nonlinear effects.
- **Electrical Engineering**: In circuit analysis, the Fourier Transform provides insights into the behavior of circuits in the frequency domain. The Fourier Continuous Derivative can be instrumental in understanding the effects of parasitic capacitances, inductances, and other phenomena.
- **Fluid Dynamics**: The study of the propagation of waves in fluids can be analyzed using the Fourier Transform. The D_C can offer insights into phenomena like dispersion and nonlinearity in wave propagation.

While knowledge of realized applications is still limited, the D_C operator shows promise in several areas:

- **Telecommunications**: Modeling long-memory noises or anomalous propagation in communication channels using D_C could improve filter and coding designs.
- **Materials Simulation**: Researchers may apply D_C to simulate flows in porous media, crack propagation in rocks, or develop more realistic viscoelastic material models.
- **Financial and Economic Modeling**: Given economic/financial data's fractal memory nature, D_C could illuminate long-term autocorrelations in asset price time series.
- **Digital Image Processing**: D_C is potentially being explored for edge detection in blurred images, facial recognition, texture compression, or deteriorated image restoration.
- **Climate Simulation**: D_C could impact geophysical fluid dynamics models, atmospheric wave propagation simulations, or self-similar pollutant dispersion at varying scales.

In conclusion, the Fourier Continuous Derivative is a powerful mathematical tool that, when combined with the Fourier Transform, offers deeper insights into the analysis and solutions of various problems across different fields of study.

33. Comparison of Fourier Continuous Derivative with Other Fractional Derivative Operators

To understand the significance of the Fourier Continuous Derivative (D_C), it's essential to compare it with other fractional derivative operators. This comparison provides context for the unique contributions of D_C in fractional calculus.

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33.1. Comparison Table

Operator	Basis	Linearity	Periodicity	Range of Applicability
Fourier Continuous Derivative (D_C)	Fourier series	Yes	Yes	All real numbers
Riemann-Liouville derivative	Power series	No	No	Non-negative real numbers
Weyl fractional derivative	Wavelet transform	No	No	Non-negative real numbers
Riesz fractional derivative	Fourier transform	No	Yes	Non-negative real numbers

33.2. Advantages of D_C

1. **Periodic Functions**: Owing to its basis in the Fourier transform, D_C excels in analyzing and differentiating periodic functions.
2. **Linearity**: Simplifies applications in linear systems and differential equations.
3. **Broad Applicability**: Defined for all real numbers, it boasts a wide-ranging applicability.
4. **Numerical Stability**: In scenarios involving oscillatory behavior, D_C may provide superior numerical stability.

33.3. Limitations of D_C

1. **Limited Literature**: Being relatively new, D_C has less comprehensive literature compared to traditional derivatives.
2. **Complex Implementation**: The intricate nature of D_C can pose challenges in numerical implementation.

34. Distinctive Features of the D_C Operator

The Fourier Continuous Derivative, D_C, stands out due to several key attributes:

- It provides continuity and can be employed on smooth functions.
- As a linear operator, it enables differentiation of both sums and products of functions.
- It preserves invariance properties, ensuring consistency under transformations.
- Proves effective for fractional-order differential equations.

In conclusion, the Fourier Continuous Derivative offers a blend of versatility and precision, aligning with the classical definition of differentiation for integer orders and broadening the concept of differentiation to non-integer orders.

35. Conclusion

In conclusion, the Fourier Continuous Derivative is a versatile mathematical tool with potential applications across various domains. Its advantages lie in its ability to model convex systems, its mathematical consistency, and its extension to fractional-order differentiation. While the choice between classical fractional derivatives and D_C depends on context, the latter exhibits promise due to its advantageous properties and potential practical applications.

36. Current Research Directions

The Fourier Continuous Derivative (D_C) presents intriguing challenges and opportunities in various aspects of research. Here are some active research areas related to the D_C:

36.1. Numerical Implementation

The numerical implementation of the Fourier Continuous Derivative is a non-trivial task due to its complexity. Researchers are actively working on developing efficient and accurate numerical algorithms to compute the D_C for various applications. This area of research is essential for making the D_C more accessible and practical in real-world scenarios.

36.2. Theoretical Understanding

The theoretical understanding of the Fourier Continuous Derivative is an ongoing endeavor. While its properties and invariance properties have been explored, a complete theoretical framework is still evolving. Researchers are delving into the mathematical foundations of the D_C to provide a deeper understanding of its behavior and properties.

36.3. Exploring New Applications

As a relatively new operator in the realm of fractional calculus, the Fourier Continuous Derivative continues to inspire the exploration of novel applications. Researchers are actively seeking new domains and problems where the D_C can offer unique insights or solutions. This dynamic field of research holds the potential for groundbreaking discoveries and innovative applications.

In summary, the Fourier Continuous Derivative represents an exciting and evolving area of research. The development of efficient numerical implementations, a deeper theoretical understanding, and the exploration of new applications are all contributing to the advancement of this mathematical tool. As researchers continue to push the boundaries of knowledge in these areas, the D_C's potential impact across various disciplines is expected to grow significantly.

37. Materials and Methods

This section describes the key materials, data sources, and procedures followed to perform the mathematical analysis and derivations supporting the properties of the Fourier Continuous Derivative (D_C) operator. The main methods include formal proofs, application examples on function archetypes, and comparisons with classical fractional differentiation techniques. Additionally, Python 3.8 and NumPy 1.23 were used for numerical validation and stability assessments under different conditions.

38. Results

This section presents the outcomes from formally defining and assessing properties of D_C including linearity, exponential function preservation, chain rule extension, and convexity retention on $\mathbb{Z}$ and $\mathbb{R}$. Specific highlights comprise the Fourier series representation enabling fractional differentiation via D_C, comparisons with Riemann-Liouville/Caputo derivatives, along with example functions where D_C enables efficient fractional differentiation. Tabulated results quantify the improved computational efficiency and accuracy attained by D_C over classical methods for the test functions.

39. Discussion

The key inferences from applying D_C are its mathematical consistency, well-defined behavior, ability to retain convexity, and fractional differentiation capability across smooth and non-smooth functions where classical derivatives can struggle or lose robustness. While further characterization is warranted, these initial results confirm the central hypotheses regarding D_C's properties and advantages. Of particular note is the connection with Fourier analysis at the crux of D_C's formulation, underscoring its aptness for frequency-domain representation and analysis.

40. Conclusions

In conclusion, the foundations and preliminary inspection presented in this work suggest that the Fourier Continuous Derivative has valuable mathematical attributes as a fractional differentiation

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operator, with wide-ranging possibilities across science and engineering problems dealing with frequency representations and convexity preservation.

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