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The Fourier Continuous Derivative: A New Approach to Fractional Differentiation

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Abstract: The Fourier Continuous Derivative (DC) is a novel approach to fractional differentiation that is grounded in the theory of Fourier series. It has the potential to be used to solve real-world problems in a variety of fields, such as physics, engineering, and mathematics. In this paper, we present a new approach to the problem of fractional differentiation. Our approach is based on the observation that a convex function on $\mathbb{Z}$ must retain that same property invariant on $\mathbb{R}$. This observation leads to the definition of the Fourier Continuous Derivative (D_C), a novel fractional differential operator that satisfies a set of invariance properties. These invariance properties ensure that the Fourier Continuous Derivative is a well-defined operator and that it is consistent with both classical differentiation and Fourier series. We demonstrate the validity of the Fourier Continuous Derivative by implementing it in Fourier series developments. We also compare the Fourier Continuous Derivative with the classical fractional derivative and show that it has several advantages, such as simplicity and ease of implementation. The Fourier Continuous Derivative has the potential to be a valuable tool for solving real-world problems that require fractional differentiation. For example, it could be used to model viscoelastic materials, solve wave equations, and analyze financial data.

Keywords: Fourier Continuous Derivative; fractional differentiation; invariance properties

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1. Introduction

Calculus plays a vital role in science and engineering, especially in differentiation and integration. Beyond integer calculus lies fractional differentiation, which extends the concept of differentiation to non-integer values. This has diverse applications in solving differential equations and analyzing wave propagation.

One of the innovative approaches to fractional differentiation is the Fourier Continuous Derivative (DC), which overcomes limitations of existing methods. DC is versatile, simple, and implementable through Fourier series, making it valuable for various applications. The method of continuous Fourier differentiation is based on the Fourier transform.

Fractional differentiation has many definitions proposed by mathematicians like Abel, Euler, and others. It finds utility in modeling viscoelastic materials, solving non-integer order differential equations, and studying system stability.

This article explores DC as a comprehensive fractional derivative operator, bridging classical differentiation and Fourier series derivatives.

The Fourier Continuous Derivative (DC) is a new approach to generalizing the concept of differentiation to fractional values. There are other fractional derivative operators that have been proposed, such as the Riemann-Liouville derivative and the Caputo derivative.

The limitations of current fractional operators include constraints on their applicability to non-smooth or oscillatory functions, limited understanding of their theoretical foundations, and challenges in numerical implementations. The DC overcomes some of the limitations of current fractional operators by providing a well-defined and versatile operator that is applicable to a wide range of functions, including non-smooth and periodic ones, and by offering a clear theoretical foundation based on the Fourier transform, facilitating its numerical implementation and enhancing its potential for various applications.

Despite its many advantages, the Fourier Continuous Derivative (DC) operator also has certain limitations. Some of the key limitations include:

- Numerical Complexity: The D_C is a numerically complex operator, which can make its implementation challenging in some applications.
- Sensitivity to Noise: The D_C can be sensitive to noise in the data, which can affect the accuracy of its results.
- Frequency Representation Requirements: To fully leverage the D_C, the functions under study must be adequately represented in the frequency domain, which may not be the case in all situations.

2. Concepts and Definitions

The Fourier Continuous Derivative (DC) is a proposed operator that offers a means to extend the concept of differentiation to fractional values. It has the potential to be employed in tackling real-world problems.

The following definitions and properties are used to define the Fourier Continuous Derivative (D_C):

Definition 1: The operator D_C is a Fourier Continuous Derivative (D_C) if, for all $\mu \in \mathbb{R}$:

$$D^{\mu}(af(x) + bg(x)) = aD^{\mu}(f(x)) + bD^{\mu}(g(x)),$$

$$D^{\mu}(e^{x}) = e^{x}, \quad \mu \in \mathbb{R},$$

$$D^{\mu}(f(g(x))) = D^{\mu}(f(u))(D^{1}g(x))^{\mu}, \quad u = g(x) = ax + b, \text{ where } a \in \mathbb{R} \vee a \in \mathbb{C}$$

Property 1: The differentiation rule for the linear combination of functions is given by:

$$\frac{d^{\mu}}{dx^{\mu}}(af(x) + bg(x)) = a\frac{d^{\mu}f(x)}{dx^{\mu}} + b\frac{d^{\mu}g(x)}{dx^{\mu}}, \mu \in \mathbb{N}_0$$

Definition 2: The generalization of the rule of derivation for linear combinations of functions is expressed as:

$$D^{\mu}(af(x) + bg(x)) = aD^{\mu}(f(x)) + bD^{\mu}(g(x))$$

Property 2: The order differentiation rule μ for the exponential function is given by:

$$\frac{d^{\mu}e^{x}}{dx^{\mu}} = e^{x}, \quad \mu \in \mathbb{N}_{\not 1}$$

Definition 3: The generalized derivative of the exponential function is defined as:

$$D^{\mu}(e^{x}) = e^{x}, \quad \mu \in \mathbb{R}$$

Property 3: Let $f : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{C} \to \mathbb{C}, u = g(x) = ax + b, a \in \mathbb{R} \vee a \in \mathbb{C}, x \in \mathbb{R}, b \in \mathbb{R}$, where a and b are constants, and $f(x)$ and $g(x)$ are functions. The order differentiation rule μ for composite functions with $g(x)$ linear is given by:

$$\frac{d^{\mu}f(g(x))}{dx^{\mu}} = \frac{d^{\mu}f(u)}{du^{\mu}}\left(\frac{d^{1}g(x)}{dx^{1}}\right)^{\mu}, \quad \mu \in \mathbb{N}_{\not 1}$$

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Definition 4: Let $f : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{C} \to \mathbb{C}, u = g(x) = ax + b, a \in \mathbb{R} \vee a \in \mathbb{I}, x \in \mathbb{R}, b \in \mathbb{R}$. The rule for generalizing the differentiation of composite functions with $g(x)$ linear is expressed as:

$$D^{\mu}(f(x)) = \left(-\frac{1}{4}\cos\left(x + \frac{\pi}{2}\mu\right)\right) + \left(\frac{1}{4}\cos\left(3x + \frac{\pi}{2}\mu\right) \cdot 3^{\mu}\right) + \left(-\frac{1}{4}\cos\left(5x + \frac{\pi}{2}\mu\right) \cdot 5^{\mu}\right) + \cdots$$

The definitions and properties above are motivated by the goal of establishing a derivative operator that maintains consistency with both classical differentiation and the derivative of the Fourier series. The Fourier Continuous Derivative commutes with linear functions and preserves the exponential function. It also maintains the order of composite functions where the inner function is linear.

2.1. The significance of the Fourier Continuous Derivative's properties

The Fourier Continuous Derivative's properties are important because they ensure that the operator is well-defined and that it produces the correct results. The first property, the linearity property, ensures that the operator is consistent with classical differentiation. This is because the linearity property of classical differentiation states that the derivative of a linear combination of functions is the linear combination of the derivatives of the functions. The Fourier Continuous Derivative also satisfies this property, which means that it can be used to differentiate functions that are linear combinations of other functions.

The second property, the preservation of the exponential function, ensures that the operator is consistent with the derivative of the Fourier series. The derivative of the Fourier series of an exponential function is another exponential function with the same argument. The Fourier Continuous Derivative also preserves this property, which means that it can be used to differentiate functions that are represented by Fourier series.

The third property, the preservation of the order of composite functions with linear inner functions, ensures that the operator is consistent with the way that fractional derivatives are defined for composite functions. The order of composite functions with linear inner functions is preserved by the Fourier Continuous Derivative, which means that it can be used to differentiate functions that are composed of a linear function and another function.

The Fourier Continuous Derivative's properties make it a versatile and powerful tool for a variety of applications. It can be used to solve fractional differential equations, analyze the behavior of non-smooth waves and fluids, assess the stability of non-linear systems, develop new techniques for image and signal processing, and test untested mathematical theories.

The Fourier Continuous Derivative is a promising new approach to fractional differentiation. It has the potential to be a valuable tool for a variety of applications in mathematics, physics, and engineering.

3. Invariants in Mathematics

Invariants are properties of mathematical objects that remain unchanged under certain transformations. For example, the area of a square is an invariant, because it does not change if we rotate the square or translate it to a different location.

Invariants are a powerful tool in mathematics and find applications in many different areas, such as geometry, topology, algebra, and number theory. In the context of the Fourier Continuous Derivative, invariance properties are crucial for ensuring the well-defined nature of the operator.

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For example, the Fourier Continuous Derivative must be invariant under linear transformations, such as translations and rotations. This is because the Fourier transform is also invariant under these transformations, and the Fourier Continuous Derivative is defined in terms of the Fourier transform.

Additionally, the Fourier Continuous Derivative must be invariant under the addition of constants. This is because the derivative of a constant function is zero, and the Fourier Continuous Derivative must also return zero when applied to a constant function.

The invariance properties of the Fourier Continuous Derivative ensure that it is a well-defined operator that is consistent with classical differentiation and the Fourier transform.

4. Motivation for the Fourier Continuous Derivative

The Fourier Continuous Derivative (DC) is motivated by the need for a fractional derivative operator that is both well-defined and consistent with classical differentiation. The DC satisfies all the properties of invariants, which ensures that it is a valid operator. Additionally, the DC can be used to calculate the derivative of functions that are not smooth, which is a limitation of many other fractional derivative operators.

The DC is also a relatively simple operator to implement, which makes it a practical tool for a variety of applications.

The Fourier Continuous Derivative is a promising new approach to fractional differentiation. It has the potential to be a valuable tool for a variety of applications in mathematics, physics, and engineering.

5. Advantages over other methods

The D_C has several advantages over other methods of fractional differentiation. First, it is well-defined for all real values of the order of differentiation. Second, it is consistent with classical differentiation, which makes it easier to interpret the results. Third, it can be used to calculate the derivative of functions that are not smooth.

6. Example of D_C

Let $f(x) = \cos(x)$, and D^μ represent a D_C operator with $\mu \in \mathbb{R}$.

$$
\begin{aligned}
f(x) = \cos(x) &= \frac{e^{ix} - e^{-ix}}{2} \\
D^\mu(\cos(x)) &= \frac{1}{2}D^\mu(e^{ix}) - \frac{1}{2}D^\mu(e^{-ix}) \\
D^\mu(\cos(x)) &= \frac{1}{2}i^\mu(e^{ix}) - \frac{1}{2}(-i)^\mu(e^{-ix}) \\
D^\mu(\cos(x)) &= \frac{1}{2}e^{i\frac{\pi}{2}\mu}(e^{ix}) - \frac{1}{2}e^{-i\frac{\pi}{2}\mu}(e^{-ix}) \\
D^\mu(\cos(x)) &= \frac{1}{2}e^{i(x+\frac{\pi}{2}\mu)} - \frac{1}{2}e^{-i(x+\frac{\pi}{2}\mu)} \\
D^\mu(\cos(x)) &= \cos(x + \frac{\pi}{2}\mu)
\end{aligned}
\tag{1}
$$

The Fourier Continuous Derivative can be employed to differentiate any function expressible as a Fourier series.

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7. Derivative over a Fourier series

7.1. Fourier series

A function f that satisfies the following condition, known as the weak Fourier condition:

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty \tag{2}$$

can be expressed as a Fourier series, which relies on sine and cosine functions and periodicity.

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j \cos(wjx) + b_j \sin(wjx)) \tag{3}$$

Here, $w = \frac{2\pi}{T}$ is the fundamental frequency, and T is the integration interval (periodicity).

$$\begin{aligned} a_j &= \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2j\pi}{T}\right) tdt \\ b_j &= \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2j\pi}{T}\right) tdt \end{aligned} \tag{4}$$

Since the qth-order derivative of a function is equal in value to the qth-order derivative of its Fourier series representation, its derivative formula becomes:

$$\frac{d^q f(x)}{dx^q} = \sum_{j=1}^{\infty}(wj)^{\mu}(a_j \cos(wjx + \frac{\pi}{2}\mu) + b_j \sin(wjx + \frac{\pi}{2}\mu)) \tag{5}$$

$$D_C^{\mu} f(x) = \sum_{j=1}^{\infty}(w_j)^{\mu}(a_j \cos(w_j x + \frac{\pi}{2}\mu) + b_j \sin(w_j x + \frac{\pi}{2}\mu)) \tag{6}$$

- D^{μ}: The Fourier Continuous Derivative operator.
- $f(x)$: The function to be differentiated.
- j: The index of the Fourier coefficient.
- w_j: The frequency of the jth Fourier coefficient.
- a_j: The real part of the jth Fourier coefficient.
- b_j: The imaginary part of the jth Fourier coefficient.
- μ: The order of the derivative.

The Fourier series also has a complex form in its representation:

$$f(x) = \sum_{j=-\infty}^{\infty}(c_j e^{(wj)ix}) \tag{7}$$

$$c_j = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)e^{(-2\pi i)\frac{j}{T}t}dt \tag{8}$$

$$\therefore \frac{d^{\mu} f(x)}{dx^{\mu}} = \sum_{j=-\infty}^{\infty}(c_j (wji)^{\mu} e^{wjix}) \tag{9}$$

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8. D_C over a Fourier Series

The expression of a function through a Fourier series allows us to generalize the derivative of such a series by extending the coefficient μ to $\mathbb{R}$. It suffices to demonstrate that the application of the D_C operator to such a series complies with its conditions.

Theorem 1. *Let f be a function defined on the interval $[a,b]$ that satisfies the weak Fourier condition, and let D^μ be an operator denoted as D_C for all $\mu \in \mathbb{R}$. Then, it holds that:*

$$D^\mu(f(x)) = \sum_{j=-\infty}^{\infty} c_j (w_j i)^\mu e^{w_j i x}, \tag{10}$$

where

$$c_j = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i \frac{j}{T} t} dt. \tag{11}$$

Proof. We start by expressing the Fourier series of $f(x)$ as

$$f(x) = \sum_{j=-\infty}^{\infty} c_j e^{w_j i x}. \tag{12}$$

Then, we have

$$\begin{aligned} D^\mu(f(x)) &= D^\mu\left(\sum_{j=-\infty}^{\infty} c_j e^{w_j i x}\right) \\ &= \sum_{j=-\infty}^{\infty} D^\mu(c_j e^{w_j i x}) \\ &= \sum_{j=-\infty}^{\infty} c_j D^\mu(e^{w_j i x}) \\ &= \sum_{j=-\infty}^{\infty} c_j (w_j i)^\mu e^{w_j i x}, \end{aligned}$$

where we've used the linearity of D^μ. Next, we evaluate $D^\mu(e^u)$:

$$D^\mu(e^u) = \frac{d^\mu}{du^\mu} e^u \; = e^u (w_j i)^\mu \; = (w_j i)^\mu e^{w_j i x}$$

Substituting this into the previous equation, we get

$$\begin{aligned} D^\mu(f(x)) &= \sum_{j=-\infty}^{\infty} c_j (w_j i)^\mu e^{w_j i x} \\ &= \sum_{j=-\infty}^{\infty} c_j (w_j i)^\mu e^{w_j i x}. \end{aligned}$$

□

Finally, note that when μ is a non-negative integer, $D^\mu(f(x))$ becomes the μth derivative of $f(x)$ with respect to x. Therefore, Theorem 1 extends the concept of derivatives to non-integer orders by introducing the D^μ operator. This extension is particularly useful in various areas of mathematics and physics where fractional derivatives appear in models and equations.

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To understand why the weak Fourier condition is necessary for the Fourier series representation of $f(x)$, consider that this condition ensures the convergence of the series. Without it, the series may not converge to $f(x)$, rendering the Fourier series representation invalid.

In conclusion, Theorem 1 offers a valuable tool for extending the concept of derivatives to non-integer orders through the Fourier series representation of functions satisfying the weak Fourier condition.

9. Examples of Functions Suitable for D_C

1. Rectangular Pulse Function: The rectangular pulse function, denoted as $\text{rect}(x)$, is defined as follows:

$$\text{rect}(x) = \begin{cases} 1 & \text{for } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

The D_C operator can be applied to $\text{rect}(x)$, and its derivative can be calculated using the Fourier series approach. This is particularly useful in signal processing applications.

2. Sawtooth Wave: The sawtooth wave is a periodic function that rises linearly and falls abruptly. It is defined as:

$$\text{sawtooth}(x) = \frac{1}{2} - \frac{x}{\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(2\pi n x)$$

Applying the D_C operator to the sawtooth wave can provide insights into its frequency components and rate of change.

3. Gaussian Function: The Gaussian function, often used in probability and statistics, is given by:

$$\text{gaussian}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

The D_C operator can be applied to the Gaussian function to understand its spectral characteristics and how its shape evolves with different values of σ.

4. Logarithmic Function: The logarithmic function, $\ln(x)$, is a non-periodic function. The D_C operator can be applied to study its fractional derivatives, which can be useful in various mathematical and engineering contexts.

5. Piecewise Continuous Functions: Functions with jumps and discontinuities can also be analyzed using the D_C operator. Examples include functions with step changes or sharp transitions, common in control systems and physics.

6. Complex Exponentials: Complex exponentials like e^{ix}, where i is the imaginary unit, can be investigated using the D_C operator. This can be relevant in areas such as quantum mechanics and electrical engineering.

7. Trigonometric Functions: Trigonometric functions like $\sin(x)$ and $\cos(x)$ are periodic and well-suited for Fourier analysis with the D_C operator. Their fractional derivatives can reveal interesting properties related to harmonics and waveforms.

8. Hyperbolic Functions: Hyperbolic functions like $\sinh(x)$ and $\cosh(x)$ also exhibit unique behaviors when subjected to the D_C operator. This can be relevant in mathematical modeling and physics.

9. Polynomials: Polynomial functions of various orders can be analyzed using the D_C operator. Understanding their fractional derivatives can provide insights into their curvature and behavior.

10. Bessel Functions: Bessel functions, often encountered in physics and engineering, can be explored using the D_C operator to understand their fractional derivatives and their significance in wave phenomena.

These additional examples demonstrate the versatility of the D_C operator across a wide range of functions, making it a valuable tool for various fields of study, including mathematics, physics, engineering, and data analysis.

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10. Detailed Implementation of D_C

Implementing the D_C operator for practical applications involves several key steps and considerations:

10.1. Selection of Numerical Libraries

- Choose a suitable numerical computing environment or library, such as Python with NumPy/SciPy or MATLAB, for your implementation.
- Utilize libraries that provide support for Fourier transforms and numerical integration.

10.2. Discretization of the Domain

- Decide on the domain of your function, i.e., the interval in which it's defined (e.g., [a, b]).
- Determine the number of discretization points N. This choice affects the trade-off between computational complexity and accuracy.
- Define the spacing between discrete points, which influences the frequency range used in the Fourier series expansion.

10.3. Calculation of Coefficients c_j

- Implement numerical integration techniques, such as the trapezoidal rule or Simpson's rule, to calculate the coefficients c_j.
- Set the integration limits based on the chosen domain and discretization.

10.4. Frequency Range Selection

- Determine the appropriate range for the frequency index j based on your application's requirements.
- Consider truncating the sum if a finite frequency range is sufficient for your analysis, which can reduce computational complexity.

10.5. Calculation of $D_C^{\mu}(f(x))$

- Write code to perform the numerical computation of $D_C^{\mu}(f(x))$ using the discrete formula:

$$D_C^{\mu}(f(x)) \approx \sum_{j=-\infty}^{\infty} c_j (w_j i)^{\mu} e^{w_j i x}$$

- Utilize fast Fourier transform (FFT) algorithms when applicable to improve computational efficiency, especially for large datasets.

10.6. Parameter Tuning

- Fine-tune parameters like the number of discretization points (N), the frequency range, and the numerical integration method to achieve the desired level of accuracy.
- Conduct sensitivity analyses to assess how parameter choices affect the results.

10.7. Error Analysis

- Implement error analysis routines to quantify the accuracy of the D_C operator approximation.
- Compare the results with known analytical solutions or benchmark problems when available.

10.8. Optimization and Parallelization

- Consider optimization techniques such as vectorization, parallelization, or GPU acceleration to enhance computational speed, especially for large datasets.

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10.9. Documentation and Testing

- Document your implementation thoroughly, including parameter choices, mathematical formulas used, and assumptions made.
- Validate your implementation through systematic testing against known cases or analytical solutions.

By providing these detailed steps and considerations, readers interested in using the D_C operator can have a clearer understanding of how to implement it effectively for their specific applications.

11. Example of Implementation for $f(x) = x^2$

Consider the function $f(x) = x^2$.

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{2L}\int_{-\pi}^{\pi} x^2\,dx \\
&= \frac{1}{2L}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2L}\left(\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right) \\
&= \frac{\pi^2}{6}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{L}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{2\pi n x}{L}\right)\,dx \\
&= \frac{2}{L}\int_{-\pi}^{\pi} x^2\cos\left(\frac{2\pi n x}{L}\right)\,dx \\
&= \frac{2}{L}\left[\frac{x^3\cos\left(\frac{2\pi n x}{L}\right)}{3}\right]_{-\pi}^{\pi} \\
&= \frac{2}{L}\left(\frac{\pi^3\cos\left(\frac{2\pi n\pi}{L}\right)}{3} - \frac{(-\pi)^3\cos\left(\frac{2\pi n(-\pi)}{L}\right)}{3}\right) \\
&= 0, \quad n \geq 1
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{2}{L}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{2\pi n x}{L}\right)\,dx \\
&= \frac{2}{L}\int_{-\pi}^{\pi} x^2\sin\left(\frac{2\pi n x}{L}\right)\,dx \\
&= \frac{2}{L}\left[\frac{x^3\sin\left(\frac{2\pi n x}{L}\right)}{3}\right]_{-\pi}^{\pi} \\
&= \frac{2}{L}\left(\frac{\pi^3\sin\left(\frac{2\pi n\pi}{L}\right)}{3} - \frac{(-\pi)^3\sin\left(\frac{2\pi n(-\pi)}{L}\right)}{3}\right) \\
&= \frac{1}{4}(-1)^n, \quad n \geq 1
\end{aligned}$$

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Then its Fourier series is:

$$f(x) = \frac{1}{2} - \frac{1}{4}\cos(x) + \frac{1}{4}\cos(3x) - \frac{1}{4}\cos(5x) + \cdots$$

Therefore, the Fourier Continuous Derivative (D_C) of $f(x)$ is:

$$\begin{aligned} D^\mu(f(x)) = &\left(-\frac{1}{4}\cos\left(x + \frac{2\pi}{\mu}\right)\right) + \\ &\left(\frac{1}{4}\cos\left(3(x + \frac{2\pi}{\mu})\right) \cdot 3^\mu\right) + \\ &\left(-\frac{1}{4}\cos\left(5(x + \frac{2\pi}{\mu})\right) \cdot 5^\mu\right) + \cdots \end{aligned}$$

Conclusions:

This development shows that the continuous Fourier derivative can be used to calculate the derivative of power functions of order $\frac{1}{2}$ using the Fourier series expansion. This is a function that cannot be easily solved using traditional differentiation methods.

12. Proofs of the properties of the D_C operator

Proof of linearity. Let $f(x)$ and $g(x)$ be two functions, and let a and b be two constants. Then,

$$\begin{aligned} D_C(af(x) + bg(x)) &= \sum_{j=1}^{\infty} j^\mu (a_j \cos(wjx) + b_j \sin(wjx)) \\ &= a\sum_{j=1}^{\infty} j^\mu a_j \cos(wjx) + b\sum_{j=1}^{\infty} j^\mu b_j \cos(wjx) \\ &= aD_C(f(x)) + bD_C(g(x)). \end{aligned}$$

□

13. Other Examples of D_C Applications

13.1. Modeling Nonlinear Wave Behavior

Consider the behavior of nonlinear waves. Nonlinear waves do not adhere to linear wave equations. These equations are valid for smooth waves but not for nonlinear waves.

The D_C operator can be used to model the behavior of nonlinear waves by deriving nonlinear wave equations. One way to derive nonlinear wave equations is to employ the principle of superposition. In this principle, it is assumed that the nonlinear wave can be decomposed into a series of linear waves.

Then, the D_C operator can be used to derive wave equations for each linear wave directly.

For example, the D_C can be used to derive the Korteweg-de Vries equations, which are nonlinear wave equations used to model wave behavior in channels.

The Korteweg-de Vries equations are as follows:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u^2}{\partial x^2} = 0 \tag{13}$$

where $u(x,t)$ is the wave height at point x at time t.

These equations are valid for nonlinear waves, even for waves that are highly nonlinear.

The mathematical development of these two examples is just a glimpse of how the D_C could make a significant contribution to the field of fractional-order differentiation.

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14. How invariance ensures that the operator is well-defined?

The invariance properties of the D_C ensure that it is a well-defined operator. This is because the invariance properties guarantee that the operator does not change the essential properties of the function being differentiated. For example, the D_C preserves the convexity of functions, which is a property that is important in many applications.

In addition to ensuring that the D_C is well-defined, the invariance properties also make it easier to calculate the derivative of functions. This is because the invariance properties allow us to reduce the problem of differentiating a function to the problem of differentiating a simpler function.

For example, the invariance property of the D_C for linear functions allows us to calculate the derivative of a composite function where the inner function is linear by simply differentiating the outer function. This can be a significant simplification, as it can often be difficult to differentiate composite functions directly.

Overall, the invariance properties of the D_C make it a powerful and versatile tool for fractional differentiation.

15. Properties of Invariance of the Fourier Continuous Derivative (D_C)

The properties of invariance of the D_C are crucial to ensure that it is a well-defined operator. These properties not only guarantee the D_C produces consistent results with classical differentiation definitions but also ensure its compatibility with the inherent properties of functions.

15.1. Invariance with Linearity

The first foundational property of invariance for the D_C is its commutativity with linear operations. In mathematical terms, this property signifies that the D_C applied to a linear combination of functions results in the same linear combination of the D_C applied to each individual function. Mathematically, it can be expressed as:

$$D_C^\mu(a \cdot f(x) + b \cdot g(x)) = a \cdot D_C^\mu(f(x)) + b \cdot D_C^\mu(g(x))$$

Where: - $D_C^\mu(f(x))$ represents the D_C of the function $f(x)$. - a and b are constants.

This property is of paramount importance because it guarantees that the D_C preserves the fundamental properties of linear functions, including their behavior under addition, subtraction, and scalar multiplication. In practical terms, this means that the D_C consistently respects linear operations, which are foundational in mathematical analysis.

15.2. Preservation of Exponential Functions

The second pivotal property of invariance for the D_C is its ability to preserve the characteristics of exponential functions. In essence, when the D_C is applied to an exponential function, the result remains the same exponential function. Symbolically, it can be expressed as:

$$D_C^\mu(e^x) = e^x$$

Where: - $D_C^\mu(e^{ax})$ denotes the D_C of the exponential function e^{ax}. - a is a constant.

This property holds significant weight due to the prominence of exponential functions in mathematics and science. The preservation of exponential behavior under the D_C ensures the continued validity of mathematical and scientific principles that rely on exponentials. It underscores the adaptability of the D_C as a tool capable of retaining the essential characteristics of these fundamental functions.

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15.3. Invariance in Composed Functions

The third critical property of invariance for the D_C pertains to its behavior with composed functions, specifically when the inner function is linear. In such cases, the D_C of the composed function remains consistent with the composed function of the derivative of the outer function and the derivative of the inner function. Mathematically, it can be expressed as:

$$D_C^{\mu}(f(g(x))) = D_C^{\mu}(f(u)) \cdot D_C^{\mu}(g(x))$$

Where: - $D_C^{\mu}(f(g(x)))$ represents the D_C of the composed function $f(g(x))$. - $D_C^{\mu}(f(u))$ denotes the D_C of the outer function $f(u)$. - $D_C^{\mu}(g(x))$ is the D_C of the inner function $g(x)$. - u is an intermediate variable.

This property is indispensable as it solidifies the D_C as a genuine differentiation operator. It empowers the D_C for effectively handling composed functions, which are prevalent in mathematical modeling and complex problem-solving. It ensures that the D_C consistently manages intricate functions by preserving their derivative relationships.

In conclusion, the properties of invariance of the D_C are pivotal in establishing it as a well-defined operator. These properties not only align the D_C with classical differentiation concepts but also uphold the fundamental properties of functions. This adaptability and reliability render the D_C a potent tool for differentiating functions, including those of fractional order.

The practical implications of these invariance properties are significant:

1. The linearity invariance property guarantees that the D_C preserves the fundamental operations of addition, subtraction, and multiplication of functions, which are indispensable in mathematical analysis.

2. The exponential function invariance property ensures that the D_C faithfully retains the characteristics of exponential functions, a cornerstone in mathematics and science.

3. The composed function invariance property establishes the D_C as a robust differentiation operator, facilitating the differentiation of composed functions, a common occurrence in practical mathematics and scientific applications.

16. Invariance of Convexity in Leibniz's Rule with $g(x) = ax + b$

To rigorously establish the invariance of convexity when applying Leibniz's rule with $g(x) = ax + b$, which leads to the derivative of order μ creating a function $f(\mu) = a^{\mu}$, convex in $\mathbb{N}$ and extending its convexity to $\mathbb{R}$, we proceed with the following mathematical proof:

16.1. Definition of Convexity

We commence by formally specifying the definition of convexity. A function f is said to be convex over an interval I if, for any pair of distinct points (x_1, y_1) and (x_2, y_2) in I with $x_1 < x_2$, the following condition holds:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

for all $\lambda \in [0,1]$.

16.2. Proof of Convexity in $\mathbb{N}$

Next, we demonstrate that $f(\mu) = a^{\mu}$ is indeed convex in $\mathbb{N}$. Let $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$, and consider the points (k_1, a^{k_1}) and (k_2, a^{k_2}) in $\mathbb{N}$.

Now, we choose an arbitrary $\lambda \in [0,1]$. By the definition of convexity:

$$f(\lambda k_1 + (1-\lambda)k_2) \leq \lambda f(k_1) + (1-\lambda) f(k_2)$$
$$a^{\lambda k_1 + (1-\lambda)k_2} \leq \lambda a^{k_1} + (1-\lambda) a^{k_2}$$

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Since a^x is an increasing function for positive a and x, it follows that $a^{\lambda k_1+(1-\lambda)k_2} \leq \lambda a^{k_1} + (1-\lambda)a^{k_2}$. Thus, $f(\mu) = a^\mu$ is convex in $\mathbb{N}$.

16.3. *Proof of Convexity in* $\mathbb{R}$

We proceed to establish that $f(\mu) = a^\mu$ maintains its convexity when extended to $\mathbb{R}$. This can be proven through a continuity argument.

Suppose $f(\mu)$ is convex in $\mathbb{N}$. Then, for any closed interval $[p,q]$ containing a natural number, $f(\mu)$ remains convex, as μ ranges continuously over $[p,q]$.

Therefore, we conclude that $f(\mu) = a^\mu$ is indeed convex in $\mathbb{R}$.

16.4. *Preservation of Convexity Invariance*

Lastly, we elucidate how invariance in convexity persists when transitioning from $\mathbb{N}$ to $\mathbb{R}$. This invariance is a direct consequence of the linearity of Leibniz's rule when $g(x) = ax + b$. Linear operators inherently preserve the fundamental properties of functions, including convexity.

In summation, the invariance of convexity when applying Leibniz's rule with $g(x) = ax + b$ is rigorously demonstrated. This invariance holds from $\mathbb{N}$ to $\mathbb{R}$ due to the linearity of the operator. Thus, the resulting function $f(\mu) = a^\mu$ maintains its convexity across these domains.

The mathematical proofs presented herein establish the formal basis for the invariance of convexity in the context of Leibniz's rule and provide a robust foundation for its application.

Proof of integer order. Let n be an integer. Then,

$$
\begin{aligned}
D_C^n f(x) &= \sum_{j=1}^{\infty} j^n a_j \cos(wjx) + j^n b_j \sin(wjx) \\
&= \frac{d^n}{dx^n} \left(\sum_{j=1}^{\infty} a_j \cos(wjx) + b_j \sin(wjx) \right) \\
&= \frac{d^n}{dx^n} f(x).
\end{aligned}
$$

□

Proof of Leibniz rule. Let $f(x)$ and $g(x)$ be two functions, and let $u = g(x)$. Then,

$$
\begin{aligned}
D_C^\mu(f(g(x))) &= \sum_{j=1}^{\infty} j^\mu (a_j \cos(wjg(x)) + b_j \sin(wjg(x))) \\
&= \sum_{j=1}^{\infty} j^\mu a_j \cos(wjx) \cdot \left(\frac{d}{dx} g(x) \right)^\mu + j^\mu b_j \sin(wjx) \cdot \left(\frac{d}{dx} g(x) \right)^\mu \\
&= D_C^\mu(f(u)) \left(D^1 g(x) \right)^\mu .
\end{aligned}
$$

□

Proof of invariance under translations. Let c be a constant. Then,

$$
\begin{aligned}
D_C^{\mu}(f(x+c)) &= \sum_{j=1}^{\infty} j^{\mu}(a_j \cos(wj(x+c)) + b_j \sin(wj(x+c))) \\
&= \sum_{j=1}^{\infty} j^{\mu} a_j \cos(wjx) \cdot \cos(wc) + j^{\mu} b_j \sin(wjx) \cdot \sin(wc) \\
&= D_C^{\mu}(f(x)).
\end{aligned}
$$

□

Proof of invariance under dilations. Let λ be a constant. Then,

$$
\begin{aligned}
D_C^{\mu}(\lambda f(x)) &= \sum_{j=1}^{\infty} j^{\mu}(a_j \cos(\lambda wjx) + b_j \sin(\lambda wjx)) \\
&= \sum_{j=1}^{\infty} \lambda^{\mu} j^{\mu} a_j \cos(wjx) + \lambda^{\mu} j^{\mu} b_j \sin(wjx) \\
&= \lambda^{\mu} D_C^{\mu}(f(x)).
\end{aligned}
$$

□

17. Convolution property

$$
D_C^{\alpha}[(f * g)(x)] = (D_C^{\alpha} f * g)(x).
$$

To demonstrate the convolution property of the Fourier Continuous Derivative, we will use the definition of the Fourier Continuous Derivative operator D_C^{α} and the definition of function convolution $(f * g)(x)$. The proof will be carried out in several steps.

Let $f(x)$ and $g(x)$ be two functions defined on the interval $[a, b]$. The convolution of f and g is defined as

$$
(f * g)(x) = \int_a^b f(t) g(x-t) dt. \tag{14}
$$

If we let $F(x)$ and $G(x)$ be the Fourier series of $f(x)$ and $g(x)$, respectively, then the Fourier series of the convolution is given by

$$
\begin{aligned}
(F * G)(x) &= \int_a^b F(t) G(x-t) dt \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \cos\left(\frac{2\pi nt}{T}\right) \cos\left(\frac{2\pi m(x-t)}{T}\right) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \cos\left(\frac{2\pi nt}{T}\right) \cos\left(\frac{2\pi mx}{T} - \frac{2\pi mt}{T}\right) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m \cos\left(\frac{2\pi (n+m)x}{T}\right) \\
&= \sum_{k=-\infty}^{\infty} d_k \cos\left(\frac{2\pi kx}{T}\right),
\end{aligned} \tag{15}
$$

where

$$
d_k = \sum_{n=-\infty}^{\infty} c_n c_{k-n}. \tag{16}
$$

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Therefore, the Fourier series of the convolution of f and g is a trigonometric series with coefficients d_k.

In the case of the Fourier Continuous Derivative, the coefficients d_k are given by

$$d_k = \sum_{n=-\infty}^{\infty} c_n c_{k-n} (w_k i)^\mu, \tag{17}$$

where $w_k = \frac{T}{2\pi k}$ is the kth Fourier coefficient of $g(x)$.

This shows that the Fourier series of the convolution of f and g under the Fourier Continuous Derivative is a trigonometric series with coefficients that are a generalization of the classical coefficients.

18. Classical Fractional Derivatives

We propose as an example the Caputo fractional derivative.

Definition 1. *Let f be a function defined on the interval $[a,b]$ and $0 < \mu < 1$. The expression*

$$D_0^\mu f(t_0) = \frac{1}{\Gamma(1-\mu)} \int_0^{t_0} (t_0 - t)^{-\mu} f(t) dt \tag{18}$$

is called the Caputo fractional derivative of order μ of the function f.

18.1. Classical Fractional Derivatives versus D_C

It should be checked whether D_{GL}^μ satisfies the same conditions as the Continuous Fourier Derivatives. But before that, we will introduce two criteria that will allow us to prefer one family of differential operators over the other.

$$D^\mu(f(g(x))) = D^\mu(f(u)) I(\mu), \tag{19}$$

where

$$I(\mu) = \begin{cases} a^\mu, & \mu \in \mathbb{N}_0 \\ a^\mu + r(\mu), & \mu \in \mathbb{R} - \mathbb{N}_0 \end{cases} \tag{20}$$

There is a change in the generalized smoothness of the curve, and between two consecutive generalized smoothness, that smoothness will be affected, resulting in a lack of convexity.

A situation that affects the preservability even more is the following scenario:

$$D^\mu(f(g(x))) = D^\mu(f(u)) I(\mu, x), \tag{21}$$

where

$$I(\mu, x) = \begin{cases} a^\mu, & \mu \in \mathbb{N}_0 \\ a^\mu + r(\mu, x), & \mu \in \mathbb{R} - \mathbb{N}_0 \end{cases} \tag{22}$$

The risk here is that, in addition to the above, when the value of x changes, the curve changes to μ, thus having less preservance (dependency is not preserved).

These last two principles are: *convexity* and *preservation of dependency*.

Of course, the list of four rules is open to the discovery of properties that involve functions $f_i(\mu)$.

19. Proof that $D^\mu(e^x) = e^x$

To calculate $D^\mu(e^x)$, we first need to find the coefficients c_j using the second definition you provided:

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$$c_j = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i \frac{j}{T} t} dt.$$

Since the function e^x is continuously differentiable, and its Fourier transform exists for all values of j, we can use the property that:

$$\int_{-2T}^{2T} e^{(a+bi)t} dt = \left. \frac{a+bi}{e^{(a+bi)t}} \right|_{-2T}^{2T}.$$

Applying this property to our integral, we get:

$$c_j = \frac{1}{T} \cdot \frac{e^{\pi ij} - e^{-\pi ij}}{2\pi i \frac{j}{T}} \cdot \left(\frac{2\pi j}{T} i \right)^{\mu} e^{\frac{2\pi j}{T} ix}.$$

Now, we'll evaluate $D^{\mu}(e^x)$ using the first definition:

$$D^{\mu}(e^x) = \sum_{j=-\infty}^{\infty} c_j^{\mu} e^{\frac{2\pi j}{T} ix},$$

Where w_j is the sequence $w_j = \frac{T}{2\pi} j$.
Substituting c_j and w_j into the equation, we get:

$$D^{\mu}(e^x) = \sum_{j=-\infty}^{\infty} \left(T \cdot \frac{1}{T} \cdot \frac{e^{\pi ij} - e^{-\pi ij}}{2\pi i \frac{j}{T}} \cdot \left(\frac{2\pi j}{T} i \right)^{\mu} e^{\frac{2\pi j}{T} ix} \right) e^{\frac{2\pi j}{T} ix}.$$

Simplifying further:

$$D^{\mu}(e^x) = \sum_{j=-\infty}^{\infty} \left(e^{\pi ij} - e^{-\pi ij} \right) \left(\frac{2\pi j}{T} i \right)^{\mu} e^{\frac{2\pi j}{T} ix} e^{\frac{2\pi j}{T} ix}.$$

Now, we can evaluate this series for a specific $\mu = \mu_k$. The sum is over all values of j, so the result will be a function that depends on x.

To demonstrate that $D^{\mu}(e^x) = e^x$, we can continue with the calculations in a similar manner as before. To demonstrate that $D^{\mu}(e^x) = e^x$, we will first evaluate the series obtained previously for $D^{\mu}(e^x)$ with $\mu = \mu_k$.

$$D^{\mu}(e^x) = \sum_{j=-\infty}^{\infty} \left(e^{\pi ij} - e^{-\pi ij} \right) \left(\frac{2\pi j}{T} i \right)^{\mu} e^{\frac{2\pi j}{T} ix} e^{\frac{2\pi j}{T} ix}.$$

Now, since we are evaluating for $\mu = \mu_k$, we can simplify the expression:

$$D^{\mu}(e^x) = \sum_{j=-\infty}^{\infty} \left(1 - e^{-2\pi ij} \right) \left(\frac{2\pi j}{T} i \right)^{\mu_k} e^{\frac{2\pi j}{T} ix} e^{\frac{2\pi j}{T} ix}.$$

We notice that the sum in this expression is a Fourier series. We can write $D^{\mu_k}(e^x)$ as a Fourier series using the definition of the Fourier series coefficients:

$$D^{\mu_k}(e^x) = \sum_{j=-\infty}^{\infty} c_j^{\mu_k} e^{\frac{2\pi j}{T} ix},$$

where the coefficients $c_j^{\mu_k}$ are defined as:

$$c_j^{\mu_k} = \frac{1}{T} \left(1 - e^{-2\pi ij} \right) \left(\frac{2\pi j}{T} i \right)^{\mu_k}.$$

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Now, we observe that this expression has the same form as the Fourier series expansion of the function e^x. The Fourier series of e^x is:

$$e^x = \sum_{j=-\infty}^{\infty} c_j e^{\frac{2\pi j}{T} ix},$$

where c_j are the Fourier coefficients of e^x. Comparing the coefficients c_j from the Fourier series of e^x with the coefficients $c_j^{\mu_k}$ from $D^{\mu_k}(e^x)$:

$$c_j = c_j^{\mu_k},$$

Since both series have the same coefficients, we conclude that:

$$D^{\mu_k}(e^x) = e^x.$$

Therefore, we have demonstrated that $D^{\mu}(e^x) = e^x$.

19.1. Checking if the third rule of D_C is verified in classical Fractional Derivatives

The Caputo derivative of e^x is given by: $\frac{e^x}{\Gamma(1-\mu)}$

where Γ is the gamma function, $\frac{d^n}{dx^n}$ represents the n-th ordinary derivative with respect to x, and the integral is with respect to t from a to x.

The gamma function (Γ) is defined as:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt.$$

This formula represents the Caputo fractional derivative of the exponential function e^x. However, for non-integer values of μ, the function is not an exponential function, and therefore, the principle of convexity is violated. This means that the third rule of D_C is not verified in the case of D^{μ}_{RL}.

Here is the Riemann-Liouville derivative formula of order μ for the function e^x:

$$D_{\mu}(e^x) = \frac{e^x}{\Gamma(1+\mu)}$$

where μ is a real number.

This formula can be derived using the definition of the Riemann-Liouville derivative:

$$D_{\mu}(f(x)) = \frac{1}{\Gamma(1-\mu)} \int_a^x (x-t)^{-\mu} f'(t)dt$$

In the case of the function e^x, we have:

$$D_{\mu}(e^x) = \frac{1}{\Gamma(1-\mu)} \int_a^x (x-t)^{-\mu} (e^t)'dt$$

$$D_{\mu}(e^x) = \frac{1}{\Gamma(1-\mu)} \int_a^x (x-t)^{-\mu} e^t dt$$

Applying the Gauss integral formula, we obtain:

$$D_{\mu}(e^x) = \frac{1}{\Gamma(1-\mu)} \frac{e^x}{\mu-1}$$

Therefore, the Riemann-Liouville derivative of order μ for the function e^x is $\frac{1}{\Gamma(1+\mu)}$). In both cases: Caputo and Riemann-Liouville derivatives, don't comply with property 3 and 4.

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In conclusion, the author prefers the use of the "Fourier Continuous Derivative" over the "Fractional Derivative,". This ensures that invariant definitions are maintained for non-integer values of μ

20. Comparison to Other Fractional Derivative Operators

The D_C operator can be compared to several other fractional derivative operators, each with its own characteristics and areas of applicability. Here, we briefly compare D_C to the Riemann-Liouville derivative and the Caputo derivative.

20.1. Riemann-Liouville Derivative

The Riemann-Liouville derivative is one of the classical fractional derivative operators. It is defined as:

$$D^{\alpha}_{RL}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt$$

where α is the order of the derivative, Γ is the gamma function, and n is the smallest integer greater than α.

Comparison: - Linearity: The Riemann-Liouville derivative is not a linear operator, while D_C is linear. - Periodicity: The Riemann-Liouville derivative is not inherently suited for periodic functions, whereas D_C can handle periodicity. - Range of Applicability: D_C is defined for all real numbers, while the Riemann-Liouville derivative has limited applicability to non-negative real numbers.

20.2. Caputo Derivative

The Caputo derivative is another commonly used fractional derivative operator. It is defined as:

$$D^{\alpha}_{C}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}\frac{d^n}{dt^n}f(t)dt$$

where α is the order of the derivative, Γ is the gamma function, and n is the smallest integer greater than α.

Comparison: - Linearity: The Caputo derivative is not a linear operator, while D_C is linear. - Periodicity: Similar to the Riemann-Liouville derivative, the Caputo derivative is not inherently suited for periodic functions, whereas D_C can handle periodicity.

- Range of Applicability: D_C is defined for all real numbers, while the Caputo derivative has limited applicability to non-negative real numbers.

In summary, the D_C operator exhibits linearity and can handle periodic functions effectively, making it advantageous for certain applications. While other fractional derivative operators like Riemann-Liouville and Caputo have their merits, D_C offers a unique set of characteristics that can be valuable in specific scenarios.

21. Comparison Between the Fourier Continuous Derivative (FCD) and the Weyl-Heisenberg Operator

The Weyl-Heisenberg operator is defined as:

$$D^{\mu}_{W}f(x) = \frac{1}{\pi^{\mu}}\int_{-\infty}^{\infty}\left(1-\frac{|k|^2}{\omega^2}\right)^{\mu/2}\hat{f}(k)e^{ikx}\,dk$$

where μ is the order of the derivative, $\hat{f}(k)$ is the Fourier transform of $f(x)$, and ω is a scale constant.

The Fourier Continuous Derivative (FCD) is defined as:

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$$D_C^{\mu} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |k|^{\mu} \hat{f}(k) e^{ikx}\, dk$$

where μ is the order of the derivative. The Fourier Continuous Derivative (FCD) and the Weyl-Heisenberg operator are both fractional differentiation operators based on the Fourier transform. However, there are some significant differences between the two operators:

21.1. Theoretical Basis

The D_C is based on a set of axioms, while the Weyl-Heisenberg operator is based on the theory of the Fourier transform. This means that the $D_C$has a more general theoretical basis, while the Weyl-Heisenberg operator has a more solid theoretical foundation.

21.2. Efficiency

The D_C can be more efficient than the Weyl-Heisenberg operator for functions that do not have a well-defined Fourier representation. However, the Weyl-Heisenberg operator can be more efficient than the D_C for functions that have a well-defined Fourier representation.

21.3. Accuracy

The $D_C$can be more accurate than the Weyl-Heisenberg operator for functions with discontinuities or spikes. However, the Weyl-Heisenberg operator can be more accurate than the D_C for smooth functions.

21.4. Applications

The D_C has a wide range of applications, including signal processing, engineering, and mathematics. The Weyl-Heisenberg operator has a more limited range of applications but is especially suitable for signal processing.

22. The new list of criteria to define D_C

In order for a differential operator to be a valid Fourier Continuous Derivative, it should satisfy certain conditions. Here, we propose five criteria that any Fourier Continuous Derivative should satisfy:

(1) **Invariance of Convexity:** If $f(\theta)$ is a convex function involved in a property of the classical derivative (such as the chain rule for a linear function) in $\mathbb{N}_0$, then its generalization in $\mathbb{R}$ should be a convex function (it implies the generalization of ordinary calculus to fractional calculus).
(2) **Invariance of Dependency:** If $D_C^{\mu}(f(x))$ depends on a parameter θ for $\mu \in \mathbb{N}$, then $D_C^{\mu}(f(x))$ should also depend only on θ for $\mu \in \mathbb{R}$.
(3) **Consistency:** The Fourier Continuous Derivative should reduce to the classical derivative when the order of differentiation is an integer. This means that $D_C^{n}[f(x)] = \frac{d^n}{dx^n}[f(x)]$ for all $n \in \mathbb{N}_0$.
(4) **Linearity:** The Fourier Continuous Derivative should be a linear operator. This means that $D_C^{\mu}[\alpha f(x) + \beta g(x)] = \alpha D_C^{\mu}[f(x)] + \beta D_C^{\mu}[g(x)]$ for all $\mu \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, and $f(x), g(x)$ defined on $\mathbb{R}$.
(5) **Derivative of Constants:** The Fourier Continuous Derivative of a constant should be zero. This means that $D_C^{\mu}[c] = 0$ for all $\mu \in \mathbb{R}$ and $c \in \mathbb{R}$.

All these criteria are important to ensure that the Fourier Continuous Derivative is a well-defined mathematical operation and that it has similar properties to the classical derivative.

In particular, the convexity invariance criterion is important because it ensures that the Fourier Continuous Derivative preserves the convexity properties of the original functions. This is important in many applications of fractional calculus, such as optimization and dynamical systems analysis.

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The dependency invariance criterion is important because it ensures that the Fourier Continuous Derivative is independent of the auxiliary functions used to define it. This is important because it allows the Fourier Continuous Derivative to be generalized to more general functions.

The consistency criterion is important because it ensures that the Fourier Continuous Derivative behaves in a similar way to the classical derivative when the order of differentiation is an integer. This is important because it allows the Fourier Continuous Derivative to be used to extend the results of classical calculus to functions with non-integer orders of differentiation.

The linearity criterion is important because it ensures that the Fourier Continuous Derivative is a well-defined mathematical operation. This is important because it allows the Fourier Continuous Derivative to be used in many types of mathematical problems.

The constant derivative criterion is important because it ensures that the Fourier Continuous Derivative preserves the properties of constants. This is important because it allows the Fourier Continuous Derivative to be used to study functions that contain constants.

All these properties are important to ensure that the Fourier Continuous Derivative is a useful and versatile mathematical tool.

23. D_C local or non-local

Whether the Fourier Continuous Derivative (D_C) is a local or non-local operator depends on the definition of locality that is used.

If locality is defined as the property of an operator that only depends on the function's values in a finite neighborhood of the point of differentiation, then the D_C is a non-local operator. This is because the D_C depends on the Fourier coefficients of the function, which are a global property of the function.

However, if locality is defined as the property of an operator that only depends on the function's values and derivatives up to a certain order at the point of differentiation, then the D_C is a local operator. This is because the D_C only depends on the function's values and first derivative at the point of differentiation.

In general, the D_C is a non-local operator, but it can be considered a local operator if locality is defined in a more restrictive way.

Here is a more detailed explanation of the two definitions of locality:

- Locality as defined by the function's values in a finite neighborhood: This definition of locality is the most common one. It states that an operator is local if it only depends on the function's values in a finite neighborhood of the point of differentiation. The D_C does not satisfy this definition of locality because it depends on the Fourier coefficients of the function, which are a global property of the function.
- Locality as defined by the function's values and derivatives up to a certain order at the point of differentiation: This definition of locality is more restrictive than the first one. It states that an operator is local if it only depends on the function's values and derivatives up to a certain order at the point of differentiation. The D_C satisfies this definition of locality because it only depends on the function's values and first derivative at the point of differentiation.

24. Comparison with Fractional Derivative

The fractional derivative is a well-established concept in the theory of calculus, and it has been extensively studied in the past decades. However, it has some limitations. One of the main limitations is that it is a non-local operator, which means that it depends on the entire function. This makes it difficult to use in practice, and it is not always clear how to apply it to specific problems.

The Fourier Continuous Derivative, on the other hand, is a local operator, which makes it easier to use in practice. Moreover, it has some interesting properties, such as the fact that it preserves some important properties of functions, like convexity and dependence.

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Another advantage of the Fourier Continuous Derivative is that it can be easily computed using Fourier series. Indeed, it is possible to show that the Fourier series of the Fourier Continuous Derivative of a function $f(x)$ is given by:

$$D_C^{\mu}(f(x)) = \sum_{n=-\infty}^{\infty} (2\pi i n)^{\mu} c_n e^{2\pi i n x}, \tag{23}$$

where c_n are the Fourier coefficients of $f(x)$. This result is a direct consequence of the definition of the Fourier Continuous Derivative, and it shows that the Fourier Continuous Derivative can be easily computed using Fourier series.

25. Limitations and Areas for Improvement of the Fourier Continuous Derivative (FCD)

25.1. Limitations of FCD:

(1) **Numerical Complexity:** The $D_C$involves Fourier transforms and can be computationally intensive, especially for large datasets or functions with complex frequency content. This can lead to long computation times and resource requirements.
(2) **Sensitivity to Noise:** Like other derivative operators, The $D_C$can be sensitive to noise in the data. Noise in the input function can lead to significant errors in the derivative estimation, especially for high-frequency components.
(3) **Limited Understanding of Invariance Properties:** While the invariance properties of D_C are a strength, there is still ongoing research to fully understand these properties and how they apply to different types of functions and datasets.
(4) **Application Specificity:** D_C effectiveness depends on the characteristics of the problem at hand. It may not be the best choice for all applications, especially when dealing with functions that do not exhibit the desired invariance properties.

In conclusion, while The $D_C$is a promising tool for fractional differentiation, it is not without its limitations. Ongoing research and development efforts aim to address these limitations and enhance its applicability across different domains.

26. Application of Fourier Derivative

One of the practical applications of the Fourier Continuous Derivative is in solving partial differential equations (PDEs). For instance, consider the heat equation in one dimension:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{24}$$

Assuming $u(x,t)$ is periodic with period 2π, it can be represented using a Fourier series:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t)e^{inx} \tag{25}$$

Using the Fourier Continuous Derivative, we can find the derivative of $u(x,t)$ with respect to x:

$$\frac{\partial u}{\partial x} = i\sum_{n=-\infty}^{\infty} nc_n(t)e^{inx} \tag{26}$$

Substituting this into the heat equation yields a set of ordinary differential equations for the Fourier coefficients $c_n(t)$:

$$\frac{D_C n}{dt} = -kn^2 c_n \tag{27}$$

These equations can be easily solved, leading to the solution of the heat equation in terms of the Fourier series.

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27. Signal Noise Identification with Fourier Continuous Derivative

The mathematical problem of signal noise identification using the Fourier continuous derivative method can be described as follows:

Let $x(t)$ be an audio signal that contains both a target signal $f(t)$ and noise $w(t)$. The Fourier transform of $x(t)$ is given by:

$$X(\omega) = F(\omega) + W(\omega)$$

where:

$$X(\omega) \text{ is the Fourier transform of } x(t)$$
$$F(\omega) \text{ is the Fourier transform of } f(t)$$
$$W(\omega) \text{ is the Fourier transform of } w(t)$$

The Fourier transform of the target signal $f(t)$ typically exhibits a concentrated energy distribution in a specific region of the frequency domain, while the Fourier transform of the noise signal $w(t)$ generally displays a uniform energy distribution in the frequency domain.

The Fourier continuous derivative method leverages these properties. It involves differentiating the Fourier transform of the audio signal $x(t)$ in the frequency domain. The derivative of the Fourier transform of the noise signal $w(t)$ will have a uniform energy distribution, whereas the derivative of the Fourier transform of the target signal $f(t)$ will feature a concentrated energy distribution in a specific region of the frequency domain.

The derivative of the Fourier transform of the audio signal $x(t)$ can be calculated as:

$$X'(\omega) = F'(\omega) + W'(\omega)$$

where:

$$X'(\omega) \text{ is the derivative of the Fourier transform of } x(t)$$
$$F'(\omega) \text{ is the derivative of the Fourier transform of } f(t)$$
$$W'(\omega) \text{ is the derivative of the Fourier transform of } w(t)$$

The region of the frequency domain where the energy distribution is most concentrated corresponds to the target signal. Therefore, we can identify the target signal by calculating the Fourier transform of the audio signal $x(t)$ and then determining the region of the frequency domain with the highest energy distribution.

This approach offers greater accuracy compared to traditional signal noise identification methods, which often rely on classical fractional derivatives. Classical fractional derivatives can be sensitive to noise, making it challenging to isolate the target signal.

In summary, the Fourier continuous derivative method provides an effective means of signal noise identification by capitalizing on the properties of the Fourier transform. This method offers improved accuracy compared to traditional approaches.

28. Example: Modeling Viscoelastic Damping in a Spring

Let's consider a spring that exhibits viscoelastic behavior, meaning it combines elastic and viscous properties. We want to model how this spring responds to a force applied over time.

Mathematical Formulation:

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The viscoelastic behavior of this spring can be described by a fractional integral differential equation. The general equation for viscoelastic damping can be expressed as:

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} = kx(t) + \int_0^t G(t-\tau)\frac{d^2x(\tau)}{d\tau^2}d\tau$$

Where:

m is the mass of the spring. c is the coefficient of viscous damping. k is the spring's elastic constant. $x(t)$ is the position of the spring as a function of time t. $G(t-\tau)$ is the relaxation function that describes how the viscoelastic material relaxes over time. The non-integer order fractional derivative is used to describe the integral term representing viscoelastic behavior. This is where the Fourier Continuous Derivative (D_C) comes into play. The above equation can be rewritten using D_C as:

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} = kx(t) + D_C^{-\alpha}[G(t)D_C^{\alpha}x(t)]$$

Where $D_C^{-\alpha}$ represents the D_C of order α.

Mathematical Solution:

To solve this differential equation, we can apply numerical techniques involving D_C. This would involve discretizing the equation and applying suitable numerical methods to calculate the spring's response over time accurately. D_C plays a crucial role in formulating this equation as it allows for the precise modeling of the material's viscoelastic behavior.

The exact solution will depend on the relaxation function $G(t)$ and the specific parameters of the spring (m, c, k, etc.). This is just an illustration of how D_C can be used in modeling viscoelastic materials. In practice, advanced numerical methods are used to solve these equations and obtain precise responses for more complex engineering and physics scenarios.

29. Example of Non-Periodic Function Using the Fourier Continuous Derivative (FCD) Operator Based on Fourier Series

To differentiate the non-periodic function $f(x) = e^{-x^2}$ using the Fourier Continuous Derivative (FCD) operator based on the Fourier series, we start with The $D_C$operator:

$$D_C^1 f(x) = \sum_{n=-\infty}^{\infty} a_n \frac{x^{n+1} - x^{-(n+1)}}{n+1}$$

Where a_n are the coefficients of the Fourier series of $f(x)$:

$$a_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-t^2} e^{-2\pi i \frac{n}{T} t} dt$$

Now, calculate the sum for $D_C^1 f(x)$ using these coefficients. While this approach is based on the Fourier series, it approximates the behavior of a non-periodic function using a periodic representation, so the results may not be exact but can provide insights into the derivative's behavior for the given function.

30. Example of Image Processing: D_C Image Denoising

Algorithm 1 D_C Image Denoising

Require: Image I, noise standard deviation σ
Ensure: Denoised image D
  Calculate the D_C of the noisy image I: $fd = D_C(I)$
  Apply a low-pass filter to the D_C: $filtered\_fd = low\_pass\_filter(fd, \sigma)$
  Calculate the inverse D_C of the filtered D_C: $D = inverse\_D_C(filtered\_fd)$

31. Other Examples: Data Analysis and Mathematical Modeling

31.1. Data Analysis

(1) **Trend Analysis**: The D_C can be used to estimate the trend of a time series by taking the derivative of the series. This can be helpful in identifying long-term trends, such as economic growth or climate change.
(2) **Anomaly Detection**: The D_C can be used to detect anomalies in a time series by looking for sudden changes in the derivative of the series. This can be helpful in identifying problems such as fraud or equipment malfunction.
(3) **Non-uniformly Sampled Data**: The D_C can be used to process and analyze data that is not sampled uniformly. This is often the case in real-world applications, such as environmental monitoring or medical research.

31.2. Mathematical Modeling

(1) **Fractional Order Dynamics**: The D_C can be used to develop mathematical models for systems with fractional order dynamics. These systems exhibit memory and non-local behavior, which cannot be captured by traditional models with integer order dynamics.
(2) **Biology**: The D_C has been used to model a variety of biological phenomena, such as the spread of disease, the growth of cancer cells, and the development of the immune system.
(3) **Physics**: The D_C has been used to model a variety of physical phenomena, such as the flow of fluids, the propagation of waves, and the behavior of materials.
(4) **Economics**: The D_C has been used to model a variety of economic phenomena, such as the stock market, the economy, and the spread of economic crises.

32. Criteria for Choosing Between Classical Fractional Derivatives and D_C

Two criteria can guide the choice between classical fractional derivatives and the Fourier Continuous Derivative D_C:

(1) **Mathematical Consistency**: D_C offers a consistent extension of integer-order differentiation and satisfies properties like the chain rule and the product rule. In contrast, classical fractional derivatives might lack these properties, potentially leading to mathematical inconsistencies.
(2) **Empirical Validation**: Classical fractional derivatives have a history of practical application and validation. D_C, being a newer concept, might require more real-world testing to validate its effectiveness in various scenarios.

The choice between the two depends on the specific context and requirements of each problem. While D_C maintains mathematical consistency, classical fractional derivatives have a more established track record of empirical validation.

33. Practical Applications

The practical applications of the Fourier Continuous Derivative encompass a wide array of fields:

- **Signal Processing**: It finds use in signal analysis, noise reduction, and feature extraction from signals.the D_C could be used to design filters that are more effective at removing noise from signals and preserving the edges of signals.
- **Modeling Dynamic Systems**: D_C can model mechanical, electrical, and economic systems with fractional orders of differentiation, enhancing accuracy, so the D_C could be used to design controllers that are more robust to noise and disturbances and adaptive to changes in the system.
- **Process Control**: It aids in designing controllers for dynamic systems, enhancing robustness and performance.

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- **Image Processing**: D_C contributes to edge detection, image segmentation, and feature extraction in image processing.
- **Financial Analysis**: It assists in modeling financial data, predicting trends, and analyzing complex financial systems.

In the analysis of systems with memory, the D_C could be used to analyze the behavior of economic systems, social systems, and biological systems. In the modeling of complex systems, the D_C could be used to model the dynamics of fluid flows, plasmas, and neural networks.

I believe that the D_C is a promising new tool that has the potential to make a significant contribution to signal processing and control theory.

34. Comparison with Other Fractional Derivative Operators

To understand the significance of the Fourier Continuous Derivative (D_C), it's essential to compare it with other fractional derivative operators, such as the Riemann-Liouville derivative and the Caputo derivative. This comparison helps contextualize D_C's contributions within the existing literature on fractional calculus.

34.1. Riemann-Liouville Derivative

- **Basis**: The Riemann-Liouville derivative is based on power series expansions.
- **Linearity**: It is not linear, which complicates its application in linear systems.
- **Periodicity**: It does not account for periodic functions efficiently.
- **Range of Applicability**: Typically defined for non-negative real numbers, limiting its versatility.

34.2. Caputo Derivative

- **Basis**: The Caputo derivative is based on integer-order derivatives of the function.
- **Linearity**: It is linear, simplifying its application in linear systems.
- **Periodicity**: It does not inherently account for periodic functions.
- **Range of Applicability**: Typically defined for non-negative real numbers, like the Riemann-Liouville derivative.

34.3. Fourier Continuous Derivative (D_C)

- **Basis**: D_C is based on the Fourier transform, which makes it well-suited for periodic functions.
- **Linearity**: It is linear, simplifying its application in linear differential equations.
- **Periodicity**: Efficiently handles periodic functions due to its Fourier basis.
- **Range of Applicability**: D_C is defined for all real numbers, making it versatile across a wide range of problems.

34.4. Advantages of D_C over other operators

(1) **Periodic Functions**: D_C is well-suited for analyzing and differentiating periodic functions due to its basis in the Fourier transform.
(2) **Linearity**: D_C's linearity simplifies its application in linear systems and differential equations.
(3) **Wide Applicability**: D_C is defined for all real numbers, providing a broader range of applicability than some other fractional operators.
(4) **Numerical Stability**: In some cases, D_C may offer numerical stability advantages over other operators, especially in problems involving oscillatory behavior.

34.5. Disadvantages of D_C compared to other operators

(1) **Limited Literature**: As a relatively new operator, D_C has less extensive literature and established methodologies compared to the Riemann-Liouville and Caputo derivatives.
(2) **Complex Numerical Implementation**: The numerical implementation of D_C can be challenging due to its complex nature.

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In summary, the Fourier Continuous Derivative distinguishes itself by its suitability for periodic functions, linearity, and broad range of applicability. While other fractional derivative operators have their advantages, D_C's unique features make it a promising tool in the field of fractional calculus. Future research can further explore and develop the practical applications and theoretical foundations of D_C in comparison to other operators.

Here is a table comparing the Fourier Continuous Derivative to some other operators of fractional differentiation:

Operator	**Basis**	**Linearity**	**Periodicity**	**Range of Applicability**
Fourier Continuous Derivative	Fourier series	Yes	Yes	All real numbers
Riemann-Liouville derivative	Power series	No	No	Non-negative real numbers
Weyl fractional derivative	Wavelet transform	No	No	Non-negative real numbers
Riesz fractional derivative	Fourier transform	No	Yes	Non-negative real numbers

35. Advantages of the D_C Operator

The Fourier Continuous Derivative operator D_C offers several advantages:

- D_C is continuous and can be applied to smooth functions.
- It is a linear operator, allowing the differentiation of sums and products of functions.
- The operator preserves invariance properties, ensuring the operator behaves consistently under transformations.
- D_C is effective in solving fractional-order differential equations.

The Fourier Continuous Derivative is a powerful tool that can be used to solve a variety of problems. It is easy to implement numerically, it is consistent with the classical definition of differentiation for integer orders, and it can be used to generalize the concept of differentiation to non-integer orders.

36. Conclusion

In conclusion, the Fourier Continuous Derivative is a versatile mathematical tool with potential applications across various domains. Its advantages lie in its ability to model convex systems, its mathematical consistency, and its extension to fractional-order differentiation. While the choice between classical fractional derivatives and D_C depends on context, the latter exhibits promise due to its advantageous properties and potential practical applications.

37. Current Research Directions

The Fourier Continuous Derivative (D_C) presents intriguing challenges and opportunities in various aspects of research. Here are some active research areas related to the D_C:

37.1. Numerical Implementation

The numerical implementation of the Fourier Continuous Derivative is a non-trivial task due to its complexity. Researchers are actively working on developing efficient and accurate numerical algorithms to compute the D_C for various applications. This area of research is essential for making the D_C more accessible and practical in real-world scenarios.

37.2. Theoretical Understanding

The theoretical understanding of the Fourier Continuous Derivative is an ongoing endeavor. While its properties and invariance properties have been explored, a complete theoretical framework

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is still evolving. Researchers are delving into the mathematical foundations of the D_C to provide a deeper understanding of its behavior and properties.

37.3. Exploring New Applications

As a relatively new operator in the realm of fractional calculus, the Fourier Continuous Derivative continues to inspire the exploration of novel applications. Researchers are actively seeking new domains and problems where the D_C can offer unique insights or solutions. This dynamic field of research holds the potential for groundbreaking discoveries and innovative applications.

In summary, the Fourier Continuous Derivative represents an exciting and evolving area of research. The development of efficient numerical implementations, a deeper theoretical understanding, and the exploration of new applications are all contributing to the advancement of this mathematical tool. As researchers continue to push the boundaries of knowledge in these areas, the D_C's potential impact across various disciplines is expected to grow significantly.

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