Article

Some Remarks about Regular Relations

Chongyun Zhao and Guanghao Jiang *

Posted Date: 10 October 2023

doi: 10.20944/preprints202310.0646.v1

Keywords: Regular relations；Galois adjunction； Dually order generating；Category theory

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 October 2023 doi:10.20944/preprints202310.0646.v1

Article

Some Remarks about Regular Relations †

Chongyun Zhao and Guanghao Jiang *

Department of Mathematics, Huaibei Normal University, Huaibei 235000, P. R. China
* Correspondence: guanghaoj@126.com

† Research supported by the NNSF of China (12071188).

Abstract: In this paper, we give the Galois adjunction form of the regular relation, and by using residuated mapping, we find the connection between it and order-generating set. Furthermore, we obtain the equivalent conclusion of "regular". In addition, by using the specialization order of topological space and the lower approximate operator in the rough set, we find a continuous homeomorphic mapping from a regular space to a *Alexandrov* topological space.

Keywords: regular relations; Galois adjunction; dually order generating; category theory

MSC: 06A11; 06B35; 54H10

1. Introduction

The regular relation was first characterized by Zareckiǐ[1], further research is proposed by Markowsky[2] and Xu Xiaoquan[3]. In this article, the regular relation is explained by Galois adjunction, if we see Hyting Algebra as a special Galois adjunction, we can also get that it is a regular form. In the book[4], Gierz G gave the definition of order-generating set, using the concept of residuated mapping of Blyth[5], we get the order-generating set is equivalent with the Galois adjunction, and it also can be used to link it with dually order generating, and we induce that several equivalent conditions of the $\nleq$ which is a regular relation. The knowledge of category theory and basic topology knowlege refer to [6] of Xu Luoshan. And if we let the regular set as a tool, we construct the relationship between regular relation and regular space.

Because rough sets and topology are based on set theory, there is a very natural and close relationship between them. The rough set theory was founded by Pawlak [10], and the core of rough set theory is the upper and lower approximate operator. Considering that the relationship on a non-empty set can be a general binary relationship, then we get a rough set, Li Yanyan [13] was devoted to the study of topological properties of generalized rough approximation operators induced by arbitrary binary relations, and the view that every generalized approximation space can be produced by one special topological spaces was held by Sun Tao[14], Zhan Wanrong[11] gave the sufficient condition of two different binary relations inducing the same topology and studied the structure of the equivalence class. In this paper, from the lower approximate operator on the rough set and the special order-induced topology, we get a continuous map from the regular topological space to the *Alexandorv* topology which is consisted by the $R-open$ set in the rough set.

2. The basic notations and definitions

Firstly, there are some notations and definitions which need to be mentioned. For a set X, we call ρ a *binary relation* on X, written $\rho : X \to X$, $\rho \subseteq X \times X = X^2$. Let $\mathscr{B}(X)$ denote the set of all binary relations on X. For $\rho, \alpha \in \mathscr{B}$(X), we define $\alpha \circ \rho = \{(x,z) \in$ X^2:$\exists y \in$ X,with$(x,y) \in \rho \wedge (y,z) \in \alpha\}$. We called it the composition of ρ and α by denoting $\alpha \circ \rho$.

doi:10.20944/preprints202310.0646.v1

Definition 2.1. For a binary relation ρ on a set X, define

$$\begin{aligned}\rho^{-1} &= \{(y,x) \in X^2 : (x,y) \in \rho\};\\ \rho^c &= X^2 \setminus \rho;\\ (\rho^{-1})^c &= (\rho^c)^{-1};\\ \rho(A) &= \{y \in X : \exists a \in A,\ (a,y) \in \rho\}\ (A \subseteq X);\\ \Phi_\rho(X) &= \{\rho(A) : A \subseteq X\}.\end{aligned}$$

Definition 2.2. If taking X and Y as objects of the small category of set, let $\rho \in \mathscr{B}(X)$ and $\rho^2 = \rho \circ \rho = \rho$, we call ρ is *idempotent*; if $\exists \sigma : Y \to X$ such that $\rho \circ \sigma \circ \rho = \rho$, we call ρ is *regular*.

Example 2.3. An arbitrary mapping f from X to Y:$f : X \to Y$ as a binary relation is *regular*, that is because if we define a relation σ from Y to X: $(y,x) \in \sigma \Leftrightarrow y = f(x)$, then we have$\rho \circ \sigma \circ \rho = \rho$, and by the definition 2.2, we have the mapping is a special *regular*.

3. The Galois adjunction representation of regular relations

Definition 3.1. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors, for $\forall A \in ob(\mathscr{C}), \forall B \in ob(\mathscr{D})$, there exists a bijection:

$$\varphi = \varphi_{A,B} : Hom_{\mathscr{D}}(F(A), B) \to Hom_{\mathscr{C}}(A, G(B))$$

is a natural transformation, then we call (F, G, φ) is an adjunction from $\mathscr{C}$ to $\mathscr{D}$, and F is the $left$ $adjunction$ of G, conversely, G is the *right adjunction* of F, we denote: $F \dashv G$.

Lemma 3.2. *$F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors, if (F, G, φ) is an adjunction from $\mathscr{C}$ to $\mathscr{D}$, then $\exists \eta$ is a natural transformation: $id_{\mathscr{C}} \to G \circ F$ such that $\forall A \in ob(\mathscr{C})$, $\eta(A) : A \to G(F(A))$; and $\exists \varepsilon$ is a natural transformation: $F \circ G \to id_{\mathscr{D}}$ such that $\forall B \in ob(\mathscr{D})$, $\varepsilon(B) : G(F(B)) \to B$.*

Definition 3.3. Let $f : P \to Q$ and $g : Q \to P$ are order preserving mapping, if for $\forall a \in P, \forall b \in Q$ satisfy:

$$f(a) \le b \Leftrightarrow a \le g(b).$$

Then we call the order pair (f, g) is a Galois adjunction between P and Q.

By the definition 3.1, if we restrict category on poset, restrict functor on mapping, then we have:
$(1) f(a) \le b \to a \le g(b);$
$(1') a \le g(b) \to f(a) \le b;$
$(2) id_P \le g \circ f;$
$(2') g \circ f \le id_P.$
If we put (1) and (1′) together, put (2) and (2′) together, then we get the following theorem:

Theorem 3.4. *If $f : P \to Q$ and $g : Q \to P$ are order preserving mapping, then the following conditions are equivalent:*
$(1) f \dashv g : P \rightharpoonup Q;$
$(2) id_P \le g \circ f, f \circ g \le id_Q.$

Proof. (1) ⇒ (2) By definition 3.3, $\forall x \in P$, we have $f(x) \le f(x)$, we see the right $f(x)$ as b and the left x as a, then we get $x \le gf(x)$. $\forall y \in Q$, we have $g(y) \le g(y)$, we see the right y as b and the left $g(y)$ as a, then we get $fg(y) \le y$. Thus, we get (2).

(2) ⇒ (1) If $f(x) \le y$, by (2), we have $x \le gf(x) \le g(y)$; conversely, if $x \le g(y)$, we have $f(x) \le fg(y) \le y$. Therefore, we get $f \dashv g : P \rightharpoonup Q$.
□

doi:10.20944/preprints202310.0646.v1

And if we use f and g in both sides at the same time in the Theorem3.4 (2), then there exists the following consequence:

Proposition 3.5. *If $f \dashv g : P \rightharpoonup Q$, then $fgf = f, gfg = g$.*

Proof. Using (2) in Theorem 3.4, $id_P \le g \circ f \Leftrightarrow id_P \circ g \le g \circ f \circ g, f \circ g \le id_Q \Leftrightarrow g \circ f \circ g \le g \circ id_Q$, from this, we get $gfg = g$. Similarly, we can also get $fgf = f$.
□

By the form of Galois adjunction in Proposition3.5, we find that it is a regular relation. If we externalize Galois adjunction to Heyting Algebra, let's see what we can get.

Definition 3.6. Suppose H is a lattice, and there exist binary operation: $\wedge$ and $\to$: $H \times H \to H$, such that

$$a \wedge b \le c \Leftrightarrow b \le a \to c,$$

then we call H is a Heyting Algebra.

If we take the two binary operation as f and g, that means: $f : a \wedge (-)$, $g : a \to (-)$, it is easy to get f and g can constitute the Galois adjunction, now, we verify whether it is a regular relation.

Theorem 3.7. *H is a Heyting Algebra, $a \in H$, $f : a \wedge (-)$, $g : a \to (-)$, we have*
(1)$fgf = f$;
(2)$gfg = g$.

Proof. By the previous introduction, we know that

$$\forall b \in H, f(b) = a \wedge b, g(b) = a \to b.$$

(1) That is to prove $a \wedge (a \to (a \wedge b)) = a \wedge b$.

(i) First to prove $a \wedge (a \to (a \wedge b)) \le a \wedge b$. Because $a \to (a \wedge b) \le a \to (a \wedge b)$, then by the Definition 3.6, we get $a \wedge (a \to (a \wedge b)) \le a \wedge b$.

(ii) Secondly, we show that $a \wedge b \le a \wedge (a \to (a \wedge b))$. Because $a \wedge b \le a \wedge b$, then we have $b \le a \to (a \wedge b)$, both sides act with $a \wedge (-)$, we know that it is not transfer the direction of the inequality sign, thus we obtain that $a \wedge b \le a \wedge (a \to (a \wedge b))$, then we get (1).

(2) That is to prove $a \to (a \wedge (a \to b)) = a \to b$.

(i) First to prove $a \to (a \wedge (a \to b)) \le a \to b$. By the Definition 3.6, we have $a \to (a \wedge (a \to b)) \le a \to b \Leftrightarrow a \wedge (a \to (a \wedge (a \to b))) \le b$, we just need to prove $a \to (a \wedge (a \to b)) \le b$, using the Definition 3.6 again, we just need to satisfy $a \wedge (a \to b) \le a \to b$, however, it is obviously;.

(ii) Secondly, we show $a \to b \le a \to (a \wedge (a \to b))$. It is apparently that $a \wedge (a \to b) \le a \wedge (a \to b)$, from the Definition3.6, we get $a \to b \le a \to (a \wedge (a \to b))$, then we get (2). However, if we see the Heyting Algebra as a special Galois adjunction, the above conditions can easily get.
□

Definition 3.8. If L is a dcpo, $\forall x \in P$, the set $\Downarrow x$ is direct and $x = \Downarrow x$, then we call P is a continuous poset.

Example 3.9. P is a poset, $Idl(P)$ is the idea complement lattice of P,$a \in H$, $\vee$ and $\downarrow$ are order-preserving mapping,

$$\vee : Idl(P) \to P; \downarrow : P \to Idl(P).$$

we have the following questions:
(1)$\vee \downarrow \vee = \vee$?
(2) $\downarrow \vee \downarrow = \downarrow$?

In the other hand, if P is a continuous poset, under a order-preserving mapping, the image of P is a continuous poset or not? $\vee$ or $\downarrow$ is *regular*? we just prove $\vee$ is *regular*.

Proof. For an arbitrary direct set $D \in P$, then we have $\downarrow D \in Idl(P)$, for (1), we just need to verify $\vee(\downarrow(\vee(\downarrow D))) = \vee(\downarrow D)$. By the definition3.8, because $\Downarrow x$ is obviously a down-set, we just replace x with D, we can obtain (1). Likely, we can also get $\downarrow$ is *regular*,or we can say $\vee$ and $\downarrow$ are residuated.
$\square$

The Proposition 3.5 giving the regular form of Galois adjunction, next we will show that it is indeed a regular relation.

Theorem 3.10. *If $f : P \to Q$ and $g : Q \to P$ are order preserving mapping, and $g^* = (f^{-1} \circ g \circ f^{-1})^c$, then $g^* = \bigvee\{g : Q \to P | f \circ g \circ f \subseteq f\}$.*

Proof. Firstly to show $g \subseteq g^*$. Suppose $p \in P, q \in Q$, using counter-certification method, if $g \nsubseteq g^*$, which means $(q, p) \in g \wedge (q, p) \notin g^*$, then $(q, p) \in f^{-1} \circ g \circ f^{-1}$, therefore, there exists $s \in P, t \in Q$ such that $(q, s) \in f^{-1}, (s, t) \in g, (t, p) \in f^{-1}$, and by $f \circ g \circ f \subseteq f$, we get $(s, t) \in f$, it is contradiction, so that $g \subseteq g^*$.

Secondly, we show that g^* is the maximum of g, we just need to prove $f \circ (f^{-1} \circ g \circ f^{-1})^c \circ f \subseteq f$. we make a notation $\triangle$ for the left formula, if $(x, y) \in \triangle$, then $\exists u, v$ such that $(x, u) \in f, (u, v) \in g^*, (v, y) \in f$, in that way $(u, v) \in (f^{-1} \circ g \circ f^{-1})^c$, $(u, v) \notin f^{-1} \circ g \circ f^{-1}$, and we know that $(u, x) \in f^{-1}, (y, v) \in f^{-1}$, it can be seen from the composition of the relation that $(x, y) \notin g$. Because there only have two mappings, and if there is a "relationship" between g and f we can get $(x, y) \in f$.
$\square$

4. Order characterization for regular relations

The vital thing is "relationship" between g and f, now we introduce the concept of residual mapping. And by residual mapping, we can have the order characterization for the regular relation.

Definition 4.1. If E, F are ordered set, and a mapping $f : E \to F$ that satisfies either of the following equivalent conditions is said to be *residuated*:
(1) the inverse image under f of every principal down-set of F is a down-set of E;
(2) f is isotone and there is an isotone mapping $g : F \to E$ such that $g \circ f \geq id_E$ and $f \circ g \leq id_F$.

If we strengthen the condition in f and g, we might as well set they are bijection, then we can get the next proposition.

Proposition 4.2. *If E, F are ordered set, $f : E \to F, g : F \to E$, the following conclusions are equivalent:*
(1) f is injective and surjective;
(2) $g \circ f = id_E, f \circ g = id_F$;
(3) g is injective and surjective.

By this proposition, and connect it with the Definition 4.1, if we denote f^{-1} as the inverse image under f, we get the wanted conclusion is:

If we still use $f : P \to Q$ and $g : Q \to P$ to denote the order-preserving mapping, P and Q are order sets, we get $f(P) \cong g(Q)$, and we have two order generating sets X and Y. According to this isomorphism relation between P and Q, we can also get the *regular* relation.

Theorem 4.3. *If $f : P \to Q$ and $g : Q \to P$ are order preserving mapping in the posets P and Q, X is a subset of P, Y is a subset of Q, then the following conditions are equivalent:*
(1)$f \dashv g : P \rightharpoonup Q$;
(2)$\forall y \in Q, g(y) = \bigvee\{X \cap f^{-1}(\downarrow y)\}$;
(3)$\forall x \in P, f(x) = \bigwedge\{Y \cap g^{-1}(\uparrow x)\}$.

We call X and Y are order generating, X is the join-generated by P, Y is the meet-generated by Q; and we call X is a dually order generating with Y, or we say X is the residual by Y.

Proof. (1) $\Rightarrow$ (2) Because f is an order preserving mapping, suppose $y \in Q$, if we see f and g as binary relation on the power set of P and Q, we give the notation

$$f^{\rightarrow}(X) = \{f(x)|x \in X\}; f^{\leftarrow}(Y) = \{x \in X|f(x) \in Y\}.$$

For convenience, we let $f^{\rightarrow} = f, f^{\leftarrow} = f^{-1}$, by these notations, let $X = f^{-1}(\downarrow y)$, $X \neq, \forall x \in X$, we can find a unique y such that $g(y) = x$, because $g(y) \leq g(y)$, according to (1), we have $fg(y) \leq y$ which means

$$f^{-1}fg(y) \leq f^{-1}(y) \Leftrightarrow g(y) \in f^{-1}(\downarrow y),$$

next, verify its maximumity. For $\forall x' \in f^{-1}(\downarrow y)$, $f(x') \leq y$, by(1), we have $x' \leq g(y)$, then we get (2).

(2) $\Rightarrow$ (1) For $\forall x \in P, y \in Q$, if $x \leq g(y)$, by $g(y) = \bigvee\{X \cap f^{-1}(\downarrow y)\}$ and f is an order preserving mapping, we have $f(x) \leq fg(y) \leq y$; conversely, if $f(x) \leq y$, then we get $x \in f^{-1}(\downarrow y)$, that means $x \leq gf(x) \leq y$.

(1) $\Leftrightarrow$ (3) Like (2), (3) is the dually form of (2).

□

By the Domain theory, we know that the order generating means if there are $x, y \in P$, P is a poset, and $y \not\leq x$, then there is an element p such that $x \leq p \wedge y \not\leq p$. Now we can obtain the equivalent conditions of order generating.

Theorem 4.4. *For a poset Q, and Y is a subset of Q, if we consider f as a binary relation which is regular, and if we denote $y \not\leq x$ means $(x, y) \in f$,the following statements are equivalent:*

(1)*the relation $\not\leq$ is regular;*

$(2)\forall x, y \in Q$, *if $y \not\leq x$, $\exists u, v \in Q$, such that $u \not\leq x, y \not\leq v$;*

(3)*then there is an element p such that: $x \leq p \wedge y \not\leq p$;*

(4) *Y is order generating.*

Proof. (1) $\Rightarrow$ (2) If the relation $\not\leq$ is regular, that means f is a regular relation, then $\exists \sigma \in \mathscr{B}(Y)$ with $f\sigma f = f$. If $y \not\leq x$, we have $(x, y) \in f$, and $u, v \in Q$, suppose $(u, v) \in \sigma$, then we have $(x, u) \in f, (v, y) \in f$, then we can obtain (2).

(2) $\Rightarrow$ (3) If this poset Q is a chain, then the condition (3) is easy to get, we consider Q is a general poset. If we let $\sigma = Id$, we have $p \not\leq x, y \not\leq p$, and from Definition 4.1, we know that there is an only g is residuated with f, and by Theorem3.4, we get $g = f^{-1}$, that means $p \not\leq x \Leftrightarrow x \leq p$, we get (3).

(3) $\Leftrightarrow$ (4) By the definition of order generating.

(3) $\Rightarrow$ (1) By Theorem4.3, we know order generating is equivalent with Galois adjunction, and the Galois adjunction is a *regular* relation, then we get (1). □

5. Topology characterization for regular spaces

By the *regular* of the relation, we consider a *regular* space in topology.

Definition 5.1. Suppose X is a topology space, $\forall x \in X$, and a closed set $A \subseteq X, x \notin A$, then exists neighborhood U of x and V of A, such that $U \cap V = \emptyset$, then we call X is a *regular* space.

Definition 5.2. Topology space$(X, \mathcal{T})$, and $A \subseteq X$, if $A = \overline{A}^{\circ}$, we call A is a *regular* set.

Theorem 5.3. *A regular space can induce a regular relation, a regular relation can induce a regular set, and a regular set can induce a regular space.*

Proof. (1)Suppose a topology space X, a single set $\{x\}$ and a closed set A, $x \in X, x \notin A \subseteq X$, there are two binary relation ρ and σ:

doi:10.20944/preprints202310.0646.v1

$$\rho : \{x\} \to A; \qquad \sigma : A \to \{x\}.$$

Now we try to find if there is a relation in a *regular* space. If we see ρ as a injective mapping, σ as a surjective mapping, and by the set-including order in the *regular* space, we can easily get:

$$\rho(\{x\}) \subseteq A; \qquad \{x\} \subseteq \sigma(A). \quad (*)$$

By the formula $(*)$, we know that $\rho \dashv \sigma$, then by the Galois adjunction, we can get ρ and σ are *regular* relation, in the other hand, σ is surjective, $\exists a \in A$ such that $\sigma(a) = x$; ρ is injective, therefore, $\exists b \in A, b = \rho(x) \in A$, $\{x\}$ is single, so that $\sigma(b) = x \in \{x\}$, from this, we get $\sigma \circ \rho \circ \sigma \subseteq \sigma$. The opposite is also true.

(2) we have $\rho(A) = \rho \circ \sigma \circ \rho(A)$, we know $\rho^{-1} \circ \rho(A) = \rho^{-1} \circ \rho \circ \sigma \circ \rho(A)$, that is: $A = \sigma \circ \rho(A)$, we see ρ as a closure operator in A, σ as an interior operator in A, we get: $A = \overline{A}^{\circ}$, and from Proposition 3.5 and the idempotent of the closure operator, we have $\rho = \rho \circ \sigma \circ \rho, \rho \circ \rho = \rho$, so that $\sigma \circ \rho = \rho$, which means $\overline{A} = \overline{A}^{\circ}$, then we get $A = \overline{A}$, A is a closed set.

(3)$\forall x \in X, A \subseteq X$, by(2), we know A is a closed set, obviously, $U = X - A$ is an open neighbor of x, because $x \in \{x\} \subseteq \overline{\{x\}} \subseteq U$, we suppose $V = X - \overline{\{x\}}$, then V is an open neighbor of A, and $U \cap V = \emptyset$, which means X is a *regular* space. ☐

Definition 5.4. Set $(X, \mathcal{T})$ as a topological space, for any $x, y \in X$, specify $x \leq_s y \Leftrightarrow x \in \overline{\{y\}}$. It is the pre-order on the set X, which is called the specialized order of the topological space X, also known as the special order induced by the topology.

Remark 5.5. If $\leq_s$ is the specialized order of $(X, \mathcal{T})$, for an arbitrary $x \in X$, we have $\overline{\{x\}} = \downarrow x$, from this, every close set of X is a down-set, and every open set of X is an up-set.

Definition 5.6. Suppose U is a non-empty set, ρ is the binary relation in U, and the ordered pair (U, ρ) is called generalized approximate space, or rough set.

Definition 5.7. Let (U, ρ) is a rough set, $x \in U, A \subseteq U$. $\rho_s(x) = \{y \in U | x\rho y\}$ is the successor neighborhood of x, and the preceding neighborhood is $\rho_p(x) = \{y \in U | y\rho x\}$. And the operators $\underline{\rho}(A), \overline{\rho}(A)$ are called lower operator and upper operator.

$$\underline{\rho}(A) = \{x \in U | \rho_s(x) \subseteq A\}, \ \overline{\rho}(A) = \{x \in U | \rho_p(x) \subseteq A\}.$$

Definition 5.8. Let (U, ρ) is a rough set, $A, B \subseteq U$. If $\overline{\rho}(A) \subseteq A$, then A is a $R - close$ set; if $B \subseteq \underline{\rho}(B)$, then B is a $R - open$ set.

Corollary 5.9. *(U, ρ) is a rough set, thus all $R - open$ set in (U, ρ) constitute a topology $\mathcal{T}_{\subset} = \{A \subseteq U | A \subseteq \underline{\rho}(A)\}$ is the Alexandrov topology.*

Corollary 5.10. *(U, ρ) is a rough set, the operator $\underline{\rho}(\overline{\rho})$ is the interior(closure) operator of a certain topology on U if and only if the relation ρ is a pre-order.*

Corollary 5.11. *Let U is a non-empty set, ρ is the regular relation, we have ρ is a partial order in U.*

Proof. By the form of *regular*, exists a relation $\sigma \in \mathscr{B}(U)$, such that $\rho = \rho \circ \sigma \circ \rho$, if we take ρ as the including relation, and

(1) $\forall A \subseteq U, A\rho A$;
(2)$\forall A, B, C \subseteq U, A\rho B, B\rho C$, we get $A \subseteq C$;
(3)$\forall A, B \subseteq U, A \subseteq B, B \subseteq A$, we get $A = B$.
From the three conditions above, we know ρ is the partial order. ☐

Theorem 5.12. *Suppose $(X, \mathcal{T})$ is a regular topology space, ρ is the specialization order relation on $\mathcal{T}$, if we see (X, ρ) as a rough set, and we denote the $Alexandrov$ topology which induced by ρ as $\mathcal{T}_{\mathcal{A}}$:*

A function $f : (X, \mathcal{T}) \to (X, \mathcal{T}_{\mathcal{A}}), f(x) = \underline{\rho}(\{x\})$ is continuous and f is a homeomorphic mapping.

Proof. Firstly, we see (X, ρ) as a poset, by the specialization order of topological space, $x, y \in X, x\rho y \Leftrightarrow x \in \overline{\{y\}}$, and from Remark5.5, we know that $\forall x \in X, \overline{\{x\}} = \downarrow x$, from this, we have every close set is a down set, and every open set is an up set. Because every open set in $\mathcal{T}_{\mathcal{A}}$ is an up set, to prove its continuity, we just need to prove

$$f^{-1}(\underline{\rho}(\{x\})) \in \mathcal{T}.$$

By Theorem5.3, we have ρ is regular, and by Corollary5.10, 5.11, $\underline{\rho}$ is a interior operator, so that A is an open set that means $\underline{\rho}(A) = A$. By the definition of $R - open$ set, it is clearly $\rho(A)$ is a $R - open$ set. From the specialization order, suppose $u \in X$, if $u \in f^{-1}(\underline{\rho}(\{x\})) = f^{-1}(\uparrow x) = f^{-1}(X - \overline{\{x\}}) = f^{-1}(X - \downarrow x) = \{u \in X | f(u) = \underline{\rho}(\{u\}) \nsubseteq \downarrow x\}$, this shows that $\underline{\rho}(\{x\}) \nsubseteq \downarrow x$. Therefore, exists $u \in \mathcal{T}_{\mathcal{A}}$ such that $u \in \underline{\rho}(\{x\})$ but $u \notin \downarrow x$, from the regularity of topological space, it can be concluded that $\exists V, W \in \mathcal{T}$ such that $u \in V, \downarrow x \subseteq W$ and $V \cap W = \emptyset$, then we get $u \in V \in \mathcal{T}$, then we have $f^{-1}(\underline{\rho}(\{x\})) \in \mathcal{T}$, which means f is continuous. Similarly, we reverse the proof process, we have f^{-1} is continuous, too.

Actually, if $x_1 = x_2$, we obtain $f(x_1) = f(x_2)$, then f is a injective mapping, and from the proof above, to get the surjection of f is easy. Thus, we have the conclusion that f is a homeomorphic mapping. ☐

References

1. A.Zareckii *The semigroup of binaryrelations*[J], Mat. Sbornik, 1963, 61: 291-305.
2. Markowsky G., *Idempotents and product representations with applications to the semigroup of binary relations*[J], Semigroup Forum, 1972, 5: 95–119.
3. Xu Xiaoquan and Liu Yingming,*Relational representations of hypercontinuous lattices*[J], Domain Theory,Logic, and Computation, Kluwer Academic Publisher, 2003, 65–74.
4. Gierz G, et al. Continuous Lattices and Domains[M]. Cambrige University Press, 2003.
5. T.S.Blyth, Lattices and Ordered Algebraic Structures[M]. Springer Press, 2004.
6. Xu Luoshan, Mao Xuxin and He Qingyu, Applied Topology[M]. Scince Press, 2022.
7. Schein B. M., *Regular elements of the semigroup of all binary relations*[J], Semigroup Forum, 1976, 13: 95–102.
8. Bandelt H. J., *On regularity classes of binary relations*[J], Universal Algebra and Applications, Banach Center Publications, PWN-Polish Scientific Publishers, 1982, 9: 329–333.
9. Yang.J.C.,*A theorem on the semigroup of all binary relations*[J],Proc. of Amer.Math.Soc., 1969,22:134-135.
10. Pawlak Z.,*Rough sets*[J],International Journal of Computer and information sciences,1982,11:341-356.
11. Zhan Wanrong, Yu Hai and Zhang Ruiling,*Topological Properties of Generalized Approximate Spaces*[J],Fuzzy Systems and Mathematics,2016,30(5):174-178.
12. Xu Xiaoquan and Luo Maokang,*Regular Relations and Normality of Topologies*[J],ACTA MATHEMATICA SINICA,Chinese Serises,2009,52(2):393-402.
13. Li Yanyan and Qin Keyun,*On Topological Properties of Generalized Rough Approximation Operators*[J],Computer Science,2022,49(3):263-268.
14. Sun Tao, Chen Haiyan, Fan Nianbai and Huang Shuqi,*A Kind of Topological Spaces Produced by Generalized Approximation Spaces*[J], Journal of Hainan Normal University(Natural Science),2012,25(3):237-240.
15. Li Zhaowen and Zhang Ke,*The Topological Structure on Approximation Spaces*[J],ACTA MATHEMATICAE APPLICATE SINICA,2010,33(4):750-757.