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Article

A Note on Invo-Regular Unital Rings

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ABSTRACT: In this paper we provide some important and significant observations on invo-regular rings. This work improves some of the exiting results on invo-regular rings appeared in Ann. Univ. Mariae Curie-Sklodowska Sect. A Mathematica (2018). In addition we provide relations between strongly invo-regular rings and weakly tripotent rings and conclude that there does not exist a noncommutative strongly invo-regular ring.

Keywords: Boolean ring; unit regular ring; invo-regular ring; involution; weakly tripotent ring

MSC 2020: 16U40; 16E50

1. Introduction

In this paper each ring is a unital and associative ring and following [1] we assume that the identity element of a ring is different from the zero element. A ring R is called invo-regular if for each $a \in R$ there exists $b \in \text{Inv}(R)$ such that $a = aba$ [1–3]. Here $\text{Inv}(R)$ is the set of all involutions. One may note that an element b of R satisfying $b^2 = 1$ is called an involution [1–3] and the notion of invo-regular rings is a generalization of the well known notion of unit regular rings [4–6].

It should be emphasized that as per the existing literature ([1], Proposition 2.5) a ring R is invo-regular iff $R \cong R_1 \times R_2$, here R_1 is an invo-regular ring of characteristic two and R_2 is an invo-regular ring of characteristic three.

However we prove that if R is an invo-regular ring and $R \cong R_1 \times R_2$, then the characteristic of R_1 need not be two. In addition we exhibit that if R is an invo-regular ring and $R \cong R_1 \times R_2$, then R_1 need not be Boolean. However it was asserted in [1, Proof of Theorem 2.6] that if R is an invo-regular ring then $R \cong R_1 \times R_2$ and R_1 is a ring of characteristic two which must be a Boolean ring.

One may note that a ring R is called Boolean if for each $a \in R$, we have the identity $a^2 = a$ [7]. A ring R is called tripotent if for each $a \in R$, we have the identity $a^3 = a$ and a ring R is called weakly tripotent if for each $a \in R$, we have the identity $a^3 = a$ or $(1-a)^3 = 1-a$ [7,8]. A ring R is called strongly invo-regular ring if $a^2 = au$ for each $a \in R$ and some $u \in R$ with $u^2 = 1$ [10]. We now provide our observations and results in the next section.

2. Some Important Observations and Results

Proposition 2.1. *If R is an invo-regular ring and $R \cong R_1 \times R_2$, then the characteristic of R_1 need not be two.*

Proof.

Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right\}.$$

Clearly R is a commutative ring of characteristic three under addition and multiplication of matrices modulo three. We have

$$\text{Inv}(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \right\}. \text{ It is easy to check that } R \text{ is an invo-regular}$$

unital ring. Now we have the following cases.

Case I: $R \cong R \times \{0\}$. One may note that R is not a ring of characteristic two.

Case II: $R \cong \{0\} \times R$. It is clear that $\{0\}$ is not a ring of characteristic two.

Case III: $R \cong R_1 \times R_2$. Here $R_1 = Z_3 = R_2$. We note that the characteristic of $R_1 = Z_3$ is not two.

Further we emphasize that if the characteristic of R_1 is two, then the order of R must be even. But the order of R is nine. Thus we see that in the above example the characteristic of R_1 can never be two even though R is an invo-regular ring.

Proposition 2.2. *If R is an invo-regular ring such that $R \cong R_1 \times R_2$, then R_1 need not be a non-zero Boolean ring.*

Proof. Let R is an invo-regular ring and $R \cong R_1 \times R_2$. Clearly the characteristic of R_1 need not be two (we refer Proposition 1). But it is well known that a non-zero Boolean ring must have characteristic two, hence R_1 need not be a non-zero Boolean ring.

Proposition 2.3. *A weakly tripotent ring is a strongly invo-regular ring iff it is a tripotent ring.*

Proof. Let R is a weakly tripotent and strongly invo-regular ring. Then R is a subdirect product of copies of the field of order two and the field of order three [10]. Hence by [9] R is tripotent. Conversely let R is tripotent. Then clearly it is weakly tripotent and by [9] it is a subdirect product of copies of the field of order two and the field of order three. Therefore by [10] it is a strongly invo-regular ring.

Corollary 2.4. *Every strongly invo-regular ring is a tripotent ring. The converse is also true.*

Corollary 2.5. *There does not exist a noncommutative strongly invo-regular ring.*

Proof. Every tripotent ring is commutative [7]. Therefore it follows from Corollary 2.4 that every strongly invo-regular ring is commutative. Hence there does not exist a noncommutative strongly invo-regular ring.

Note 2.6. *Theorem 2.6 [1] states that R is invo-regular ring is equivalent to R is a subdirect product of the field of order two and the field of order three. In order to prove this equivalence it was used that R is an invo-regular ring iff $R \cong R_1 \times R_2$, where R_1 is an invo-regular ring of characteristic two and R_2 is an invo-regular ring of characteristic three and further it was concluded that R_1 is a Boolean ring of characteristic two. However in the view of Propositions 2.1 and 2.2 described above, the proof of this equivalence described in [1] is not necessarily valid. In case Theorem 2.6 [1] is valid then we must have the following results.*

Proposition 2.7. *A weakly tripotent ring is an invo-regular ring iff it is a tripotent ring.*

Corollary 2.8. *Every invo-regular ring is a tripotent ring. The converse is also true.*

Corollary 2.9. *There does not exist a noncommutative invo-regular ring.*

Statement and Declaration: The author declares that there is no competing interest.

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