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Not peer-reviewed version

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Posted Date: 9 October 2023

doi: 10.20944/preprints202310.0452.v1

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A Fluid Perspective of Relativistic Quantum Mechanics

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Abstract: In previous papers we have shown how Schrödinger's equation which includes an electromagnetic field interaction can be deduced from a fluid dynamical Lagrangian of a charged potential flow that interacts with an electromagnetic field. The quantum behaviour was derived from Fisher information terms which were added to the classical Lagrangian. It was thus shown that a quantum mechanical system is driven by information and not only electromagnetic fields. This program was applied also to Pauli's equations by removing the restriction of potential flow and using the Clebsch formalism. Although the analysis was quite successful there were still terms that did not admit interpretation, some of them can be easily traced to the relativistic Dirac theory. Here we repeat the analysis for a relativistic flow, pointing to a new approach for deriving relativistic quantum mechanics.

Keywords: spin; fluid dynamics; electromagnetic interaction

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1. Introduction

A comprehensive introduction to the subject of the variational formalism of non-relativistic fluid dynamics and quantum mechanics and their deep interconnections is given in [1,2] and will not be repeated here.

The original work of Clebsch and all the following publications assume a non-relativistic fluids in which the velocity of the flow is much slower than the speed of light in vacuum c. This is of course to be expected as the work of Clebsch preceded Einstein's work on special relativity by forty eight years. This can also be based on practical basis as relativistic flows are hardly encountered on earth.

The standard approach to relativistic flows is based on the energy-momentum tensor [3–5], however, this approach is not rigorous because the definition of an energy-momentum tensor can only be done if a Lagrangian density is provided [6]. However, no Lagrangian density was known for relativistic flows. In this work we intend to expand Clebsch work to relativistic flow and thus amend this lacuna with a derived Lagrangian density for a relativistic flow from which one can obtain rigorously the energy-momentum tensor of high velocity flows.

We will start by introducing a variational principle for a relativistic charged classical particle with a vector potential interaction and a system of the same. This will be followed by the Eckart [7] Lagrangian variational principles generalized for a relativistic charged fluid. We then introduce an Eulerian-Clebsch variational principle for a relativistic charged fluid. Finally the concept of Fisher information will allow us to suggest a new approach to relativistic quantum fluids.

2. Variational Analysis of Relativistic Trajectories

We study a particle travelling in a constant metric spacetime. The action $\mathcal{A}$ for a relativistic particle is:

$$\mathcal{A} = -mc \int d\tau - e \int A^{\alpha} dx_{\alpha} \tag{1}$$

doi:10.20944/preprints202310.0452.v1

τ is the "length" (interval) along the trajectory:

$$d\tau^2 = \left|\eta^{\alpha\beta} dx_\alpha dx_\beta\right| = |dx_\alpha dx^\alpha| \tag{2}$$

x_α denote the particle's coordinates (the metric raises and lowers indices as is customary), m denotes the particle's mass, e denotes the particle's charge and A^α denotes the four vector potential which is a function of the particle coordinates. A^α transforms under Lorentz transformations as a four dimensional vector. Variational analysis will lead to the equations:

$$m\frac{du^\alpha}{d\tau} = -\frac{e}{c}u^\beta(\partial_\beta A^\alpha - \partial^\alpha A_\beta), \qquad u^\alpha \equiv \frac{dx^\alpha}{d\tau}, \quad \partial^\alpha \equiv \frac{\partial}{\partial x_\alpha}, \quad \partial_\beta \equiv \eta_{\beta\alpha}\partial^\alpha \tag{3}$$

in which the metric $\eta_{\alpha\beta}$ is the Lorentz-Minkowski metric:

$$\eta_{\alpha\beta} = \text{ diag } (1, -1, -1, -1). \tag{4}$$

2.1. Space and Time

For a space-time with a Lorentz-Minkowski metric the partition into spatial and temporal coordinates is easy. The spatial coordinates are $\vec{x} = (x_1, x_2, x_3)$ (in which we introduce a three dimensional vector notation) and the temporal coordinate is x_0. Time is measured in the seconds and space in meters, thus we shall need the conversion coefficient c to convert between the two system of units $x_0 = ct$. We define both a three and a four dimensional velocity as follows:

$$\vec{v} \equiv \frac{d\vec{x}}{dt}, \qquad v = |\vec{v}|, \qquad v_\alpha \equiv \frac{dx_\alpha}{dt} = (\vec{v}, c). \tag{5}$$

It also customary to partition A_α into temporal and spatial pieces:

$$A_\alpha = (A_0, A_1, A_2, A_3) \equiv (A_0, \vec{A}) \equiv (\frac{\phi}{c}, \vec{A}) \tag{6}$$

the factor $\frac{1}{c}$ preceding the scalar potential ϕ allows us to write equations in MKS units, it is not needed for arbitrary types of units. We can now define a magnetic field:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{7}$$

($\vec{\nabla}$ is the nabla operator) and the electric field is defined as:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \tag{8}$$

If $v < c$ we obtain for the differential of the interval:

$$d\tau^2 = c^2 dt^2 (1 - \frac{v^2}{c^2}), \qquad d\tau = cdt\sqrt{1 - \frac{v^2}{c^2}} = \frac{cdt}{\gamma}, \qquad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{9}$$

Thus the spatial part of equation (3) is obtained in the well known form:

$$\frac{d}{dt}(m\gamma\vec{v}) = \frac{d}{dt}\left(m\frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = e\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{10}$$

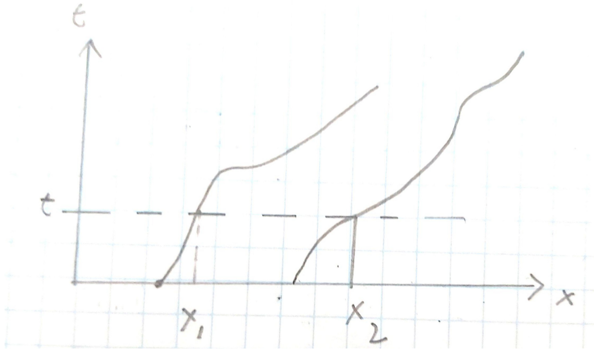

Figure 1. Schematic drawing of two trajectories piercing a time "plane" which is illustrated as a straight line.

2.2. *The Lagrangian*

We may write the action (1) as a temporal integral and thus define a Lagrangian:

$$\begin{aligned} \mathcal{A} &= \int_{t1}^{t2} L dt, \qquad L = L_0 + L_i \\ L_0 &\equiv -mc\frac{d\tau}{dt} = -\frac{mc^2}{\gamma} = -mc^2\sqrt{1-\frac{v^2}{c^2}} \simeq \frac{1}{2}mv^2 - mc^2, \\ L_i &\equiv -eA^\alpha \frac{dx_\alpha}{dt} = e(\vec{A}\cdot\vec{v} - \phi). \end{aligned} \tag{11}$$

in the above the $\simeq$ symbol signifies a classical (low speed) approximation. We notice that the interaction part of the Lagrangian is the same for high and low speeds while the kinetic part takes a different and simpler form for the low speed cases.

2.3. *The Action & Lagrangian for a System of Particles*

Consider N particles indexed by $n \in [1-N]$, with a corresponding mass m_n, charge e_n. Each particle will have a trajectory $x_n^\alpha(\tau_n)$ in which τ_n measures the interval already propagated along the trajectory. Thus:

$$u_n^\alpha \equiv \frac{dx_n^\alpha}{d\tau_n}. \tag{12}$$

We will assume as usual that the particle trajectories pierce through time "planes", and the "plane" t is pierced at position vector $\vec{x}_n(t)$, see figure 1 (actually each "plane" is three dimensional). Thus one can define a velocity $\vec{v}_n \equiv \frac{d\vec{x}_n}{dt}$. The action and Lagrangian for every particle are as before:

$$\begin{aligned} \mathcal{A}_n &= -m_n c\int d\tau_n - e_n \int A^\alpha(x_n^\nu)dx_{\alpha n} = \int_{t1}^{t2} L_n dt, \qquad L_n \equiv L_{0n} + L_{in} \\ L_{0n} &\equiv -\frac{m_n c^2}{\gamma_n} \simeq \frac{1}{2}m_n v_n^2 - m_n c^2, \quad L_{in} \equiv e_n\left(\vec{A}(\vec{x}_n,t)\cdot\vec{v}_n - \phi(\vec{x}_n,t)\right). \end{aligned} \tag{13}$$

Action and Lagrangian for the system of particles are trivially deduced:

$$\mathcal{A}_s = \int_{t1}^{t2} L_s dt, \qquad L_s = \sum_{n=1}^{N} L_n. \tag{14}$$

The variational analysis follows the same route as for a one particle and a set of equations are obtained:

$$m_n \frac{du_n^\alpha}{d\tau_n} = -\frac{e_n}{c} u_n^\beta(\partial_\beta A_n^\alpha - \partial^\alpha A_{\beta n}), \qquad n \in [1-N]. \tag{15}$$

Or the three dimensional form:

$$\frac{d}{dt}(\gamma_n \vec{v}_n) = \frac{e_n}{m_n}\left[\vec{v}_n \times \vec{B}(\vec{x}_n, t) + \vec{E}(\vec{x}_n, t)\right], \qquad n \in [1 - N]. \tag{16}$$

in which we do not sum over repeated Latin indices

3. The Lagrangian Description of a Relativistic Charged Fluid

3.1. Action and Lagrangian

Dynamics of a flow are specified by its composition and the forces which act on it. The fluid is composed of "fluid elements" [7,8]. A "fluid element" is a point particle with an infinitesimal mass $dM_{\vec{\alpha}}$, infinitesimal charge $dQ_{\vec{\alpha}}$, position four vector $x_{\vec{\alpha}\nu}(\tau_{\vec{\alpha}})$ and $u_{\vec{\alpha}\nu}(\tau_{\vec{\alpha}}) \equiv \frac{dx_{\vec{\alpha}\nu}(\tau_{\vec{\alpha}})}{d\tau_{\vec{\alpha}}}$. Here the continuous vector label $\vec{\alpha}$ comes instead of the discrete index n used in the previous section. However, the "fluid element" is not a proper point particle since it has an infinitesimal volume $dV_{\vec{\alpha}}$, infinitesimal entropy $dS_{\vec{\alpha}}$, and an infinitesimal internal energy $dE_{in\ \vec{\alpha}}$. The action for each "fluid element" is following equation (11) of the form:

$$\begin{aligned}
d\mathcal{A}_{\vec{\alpha}} &= -dM_{\vec{\alpha}} c \int d\tau_{\vec{\alpha}} - dQ_{\vec{\alpha}} \int A^{\mu}(x^{\nu}_{\vec{\alpha}}) dx_{\mu\vec{\alpha}} + d\mathcal{A}_{in\ \vec{\alpha}}, \\
d\mathcal{A}_{in\ \vec{\alpha}} &\equiv -\int dE_{in\ \vec{\alpha}} dt.
\end{aligned} \tag{17}$$

The Lagrangian for each "fluid element" can be derived from the above expression as follows:

$$\begin{aligned}
d\mathcal{A}_{\vec{\alpha}} &= \int_{t1}^{t2} dL_{\vec{\alpha}} dt, \qquad dL_{\vec{\alpha}} \equiv dL_{k\vec{\alpha}} + dL_{i\vec{\alpha}} - dE_{in\ \vec{\alpha}} \\
dL_{k\vec{\alpha}} &\equiv -\frac{dM_{\vec{\alpha}} c^2}{\gamma_{\vec{\alpha}}} \simeq \frac{1}{2} dM_{\vec{\alpha}}\, v_{\vec{\alpha}}(t)^2 - dM_{\vec{\alpha}} c^2 \\
dL_{i\vec{\alpha}} &\equiv dQ_{\vec{\alpha}} \left(\vec{A}(\vec{x}_{\vec{\alpha}}(t), t) \cdot \vec{v}_{\vec{\alpha}}(t) - \phi(\vec{x}_{\vec{\alpha}}(t), t)\right).
\end{aligned} \tag{18}$$

The action and Lagrangian of the entire fluid, is integrated over all possible $\vec{\alpha}$'s:

$$\begin{aligned}
L &= \int_{\vec{\alpha}} dL_{\vec{\alpha}} \\
\mathcal{A} &= \int_{\vec{\alpha}} d\mathcal{A}_{\vec{\alpha}} = \int_{\vec{\alpha}} \int_{t1}^{t2} dL_{\vec{\alpha}} dt = \int_{t1}^{t2} \int_{\vec{\alpha}} dL_{\vec{\alpha}} dt = \int_{t1}^{t2} L dt.
\end{aligned} \tag{19}$$

We define a density by dividing a fluid element quantity by its volume. This is done for the Lagrangian, mass, charge and internal energy of every fluid element by introducing the following symbols:

$$\mathcal{L}_{\vec{\alpha}} \equiv \frac{dL_{\vec{\alpha}}}{dV_{\vec{\alpha}}}, \quad \rho_{\vec{\alpha}} \equiv \frac{dM_{\vec{\alpha}}}{dV_{\vec{\alpha}}}, \quad \rho_{c\vec{\alpha}} \equiv \frac{dQ_{\vec{\alpha}}}{dV_{\vec{\alpha}}}, \quad e_{in\ \vec{\alpha}} \equiv \frac{dE_{in\ \vec{\alpha}}}{dV_{\vec{\alpha}}} \tag{20}$$

Every quantity of the density type is a function of $\vec{x}$, in which the "fluid element" labelled $\vec{\alpha}$ is located in time t, for example:

$$\rho(\vec{x}, t) \equiv \rho(\vec{x}_{\vec{\alpha}}(t), t) \equiv \rho_{\vec{\alpha}}(t). \tag{21}$$

We also define "specific" quantities by dividing and attribute of the "fluid element" by its mass, for example a specific internal energy $\varepsilon_{\vec{\alpha}}$ is:

$$\varepsilon_{\vec{\alpha}} \equiv \frac{dE_{in\ \vec{\alpha}}}{dM_{\vec{\alpha}}} \quad \Rightarrow \quad \rho_{\vec{\alpha}} \varepsilon_{\vec{\alpha}} = \frac{dM_{\vec{\alpha}}}{dV_{\vec{\alpha}}} \frac{dE_{in\ \vec{\alpha}}}{dM_{\vec{\alpha}}} = \frac{dE_{in\ \vec{\alpha}}}{dV_{\vec{\alpha}}} = e_{in\ \vec{\alpha}} \tag{22}$$

Thus we can partition the Lagrangian density as follows:

$$\begin{aligned}
\mathcal{L}_{\vec{\alpha}} &= \frac{dL_{\vec{\alpha}}}{dV_{\vec{\alpha}}} = \frac{dL_{k\vec{\alpha}}}{dV_{\vec{\alpha}}} + \frac{dL_{i\vec{\alpha}}}{dV_{\vec{\alpha}}} - \frac{dE_{in\ \vec{\alpha}}}{dV_{\vec{\alpha}}} = \mathcal{L}_{k\vec{\alpha}} + \mathcal{L}_{i\vec{\alpha}} - e_{in\ \vec{\alpha}} \\
\mathcal{L}_{k\vec{\alpha}} &\equiv -\frac{\rho_{\vec{\alpha}} c^2}{\gamma_{\vec{\alpha}}} \simeq \frac{1}{2}\rho_{\vec{\alpha}} v_{\vec{\alpha}}(t)^2 - \rho_{\vec{\alpha}} c^2, \\
\mathcal{L}_{i\vec{\alpha}} &\equiv \rho_{c\vec{\alpha}} \left(\vec{A}(\vec{x}_{\vec{\alpha}}(t), t) \cdot \vec{v}_{\vec{\alpha}}(t) - \varphi(\vec{x}_{\vec{\alpha}}(t), t) \right).
\end{aligned} \tag{23}$$

We can thus write the fluid Lagrangian as a spatial integral:

$$L = \int_{\vec{\alpha}} dL_{\vec{\alpha}} = \int_{\vec{\alpha}} \mathcal{L}_{\vec{\alpha}} dV_{\vec{\alpha}} = \int \mathcal{L}(\vec{x}, t) d^3x \tag{24}$$

which will be used in later section concerned with the Eulerian representation of the fluid.

3.2. Variational Analysis

Let us introduce the symbols $\Delta \vec{x}_{\vec{\alpha}} \equiv \vec{\xi}_{\vec{\alpha}}$ to denote a variation of the trajectory $\vec{x}_{\vec{\alpha}}(t)$. Thus:

$$\Delta \vec{v}_{\vec{\alpha}}(t) = \Delta \frac{d\vec{x}_{\vec{\alpha}}(t)}{dt} = \frac{d\Delta \vec{x}_{\vec{\alpha}}(t)}{dt} = \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}. \tag{25}$$

And thus according to equation (9):

$$\Delta \left(\frac{1}{\gamma_{\vec{\alpha}}} \right) = -\frac{\gamma_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t)}{c^2} \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}, \qquad \Delta \gamma_{\vec{\alpha}} = \frac{\gamma^3_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t)}{c^2} \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}. \tag{26}$$

An "ideal fluid" is defined by the fact that the "fluid element" does exchange mass, nor electric charge, nor heat with other elements, or in a variational form:

$$\Delta dM_{\vec{\alpha}} = \Delta dQ_{\vec{\alpha}} = \Delta dS_{\vec{\alpha}} = 0. \tag{27}$$

According to thermodynamics a change in the internal energy of a "fluid element" satisfies the equation below in the particle's rest frame:

$$\Delta dE_{in\ \vec{\alpha}0} = T_{\vec{\alpha}0} \Delta dS_{\vec{\alpha}0} - P_{\vec{\alpha}0} \Delta dV_{\vec{\alpha}0}. \tag{28}$$

In the above the first term describes the heating of the "fluid element" while the second term is a manifestation the work done by the "fluid element" on neighbouring elements. $T_{\vec{\alpha}0}$ denotes the temperature of the "fluid element" in the rest frame, and $P_{\vec{\alpha}0}$ is the pressure of the same. As the rest mass of the fluid element does not change and does not depend on any specific frame we may divide the above expression by $dM_{\vec{\alpha}}$ to derive the variation of the specific energy:

$$\begin{aligned}
\Delta \varepsilon_{\vec{\alpha}0} &= \Delta \frac{dE_{in\ \vec{\alpha}0}}{dM_{\vec{\alpha}}} = T_{\vec{\alpha}0} \Delta \frac{dS_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} - P_{\vec{\alpha}0} \Delta \frac{dV_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} \\
&= T_{\vec{\alpha}0} \Delta s_{\vec{\alpha}0} - P_{\vec{\alpha}0} \Delta \frac{1}{\rho_{\vec{\alpha}0}} = T_{\vec{\alpha}0} \Delta s_{\vec{\alpha}0} + \frac{P_{\vec{\alpha}0}}{\rho^2_{\vec{\alpha}0}} \Delta \rho_{\vec{\alpha}0}. \quad s_{\vec{\alpha}0} \equiv \frac{dS_{\vec{\alpha}0}}{dM_{\vec{\alpha}}}
\end{aligned} \tag{29}$$

$s_{\vec{\alpha}0}$ is the specific entropy of the fluid element in its rest frame. It follows that (we suppress the indices $\vec{\alpha}$ below):

$$\frac{\partial \varepsilon_0}{\partial s_0} = T_0, \qquad \frac{\partial \varepsilon_0}{\partial \rho_0} = \frac{P_0}{\rho_0^2}. \tag{30}$$

Another useful thermodynamic quantity is the Enthalpy defined for a fluid element in its rest frame as:

$$dW_{\vec{\alpha}0} = dE_{in\ \vec{\alpha}0} + P_{\vec{\alpha}0} dV_{\vec{\alpha}0}. \tag{31}$$

and the specific enthalpy:

$$w_{\vec{\alpha}0} = \frac{dW_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} = \frac{dE_{in\ \vec{\alpha}0}}{dM_{\vec{\alpha}}} + P_{\vec{\alpha}0}\frac{dV_{\vec{\alpha}0}}{dM_{\vec{\alpha}}} = \varepsilon_{\vec{\alpha}0} + \frac{P_{\vec{\alpha}0}}{\rho_{\vec{\alpha}0}}. \tag{32}$$

Combining the above with equation (30) we obtain the useful property:

$$w_0 = \varepsilon_0 + \frac{P_0}{\rho_0} = \varepsilon_0 + \rho_0 \frac{\partial \varepsilon_0}{\partial \rho_0} = \frac{\partial(\rho_0 \varepsilon_0)}{\partial \rho_0}. \tag{33}$$

Moreover:

$$\frac{\partial w_0}{\partial \rho_0} = \frac{\partial(\varepsilon_0 + \frac{P_0}{\rho_0})}{\partial \rho_0} = -\frac{P_0}{\rho_0^2} + \frac{1}{\rho_0}\frac{\partial P_0}{\partial \rho_0} + \frac{\partial \varepsilon_0}{\partial \rho_0} = -\frac{P_0}{\rho_0^2} + \frac{1}{\rho_0}\frac{\partial P_0}{\partial \rho_0} + \frac{P_0}{\rho_0^2} = \frac{1}{\rho_0}\frac{\partial P_0}{\partial \rho_0}. \tag{34}$$

For an ideal fluid, we neglect heat conduction and radiation, and thus only convection is considered. Thus $\Delta dS_{\vec{\alpha}0} = 0$ and it follows that:

$$\Delta dE_{in\ \vec{\alpha}0} = -P_0 \Delta dV_{\vec{\alpha}0}. \tag{35}$$

Let us now establish some relations between the rest frame and any other frame in which the fluid element is in motion (this frame is sometimes denoted the "laboratory" frame). First we notice that at the rest frame there is no velocity (by definition), hence according to equation (9):

$$d\tau = c dt_0 = c dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{c dt}{\gamma} \quad \Rightarrow \quad dt_0 = \frac{dt}{\gamma}. \tag{36}$$

It is well known that the four volume is Lorentz invariant, hence:

$$dV_0 dt_0 = dV dt = dV dt_0 \gamma, \qquad \Rightarrow dV_0 = \gamma dV. \tag{37}$$

Thus:

$$\rho_0 = \frac{dM}{dV_0} = \frac{1}{\gamma}\frac{dM}{dV} = \frac{\rho}{\gamma}, \qquad \Rightarrow \quad \rho = \gamma \rho_0. \tag{38}$$

Moreover, the action given in equation (17) is Lorentz invariant, thus:

$$dE_{in\ \vec{\alpha}0} dt_0 = dE_{in\ \vec{\alpha}} dt = dE_{in\ \vec{\alpha}} dt_0 \gamma \Rightarrow dE_{in\ \vec{\alpha}0} = \gamma dE_{in\ \vec{\alpha}}, dE_{in\ \vec{\alpha}} = \frac{dE_{in\ \vec{\alpha}0}}{\gamma} \tag{39}$$

We can now vary the internal energy of a fluid element:

$$\Delta dE_{in\ \vec{\alpha}} = \Delta\left(\frac{1}{\gamma}\right) dE_{in\ \vec{\alpha}0} + \frac{1}{\gamma}\Delta dE_{in\ \vec{\alpha}0}. \tag{40}$$

Taking into account equation (35) and equation (37) we obtain:

$$\Delta dE_{in\ \vec{\alpha}} = \Delta\left(\frac{1}{\gamma}\right) dE_{in\ \vec{\alpha}0} - \frac{1}{\gamma} P_0 \Delta dV_{\vec{\alpha}0} = \Delta\left(\frac{1}{\gamma}\right) dE_{in\ \vec{\alpha}0} - \frac{1}{\gamma} P_0 \Delta(\gamma dV_{\vec{\alpha}}). \tag{41}$$

Thus using the definition of enthalpy given in equation (31) we may write:

$$\Delta dE_{in\;\vec{\alpha}} = \Delta\left(\frac{1}{\gamma}\right)(dE_{in\;\vec{\alpha}0} + P_0 dV_{\vec{\alpha}0}) - \; P_0 \Delta dV_{\vec{\alpha}} = \Delta\left(\frac{1}{\gamma}\right) dW_{\vec{\alpha}0} - \; P_0 \Delta dV_{\vec{\alpha}}. \tag{42}$$

We shall now vary the volume element. At time t the volume of the fluid element is:

$$dV_{\vec{\alpha},t} = d^3x(\vec{\alpha},t) \tag{43}$$

The Jacobian relates this to the same element at $t = 0$:

$$d^3x(\vec{\alpha},t) = Jd^3x(\vec{\alpha},0), \qquad J \equiv \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) \tag{44}$$

$\vec{\nabla}_0$ is calculated with respect to the $t = 0$ coordinates of the fluid elements:

$$\vec{\nabla}_0 \equiv \left(\frac{\partial}{\partial x(\vec{\alpha},0)_1}, \frac{\partial}{\partial x(\vec{\alpha},0)_2}, \frac{\partial}{\partial x(\vec{\alpha},0)_3}\right). \tag{45}$$

Thus:

$$\begin{aligned} \Delta dV_{\vec{\alpha},t} &= \Delta d^3x(\vec{\alpha},t) = \Delta J\, d^3x(\vec{\alpha},0) = \frac{\Delta J}{J} d^3x(\vec{\alpha},t) = \frac{\Delta J}{J} dV_{\vec{\alpha},t}, \\ (\Delta d^3x(\vec{\alpha},0) &= 0). \end{aligned} \tag{46}$$

The variation of J can thus be derived as:

$$\Delta J = \vec{\nabla}_0 \Delta x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) + \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 \Delta x_2 \times \vec{\nabla}_0 x_3) + \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 \Delta x_3), \tag{47}$$

Now:

$$\begin{aligned} & \vec{\nabla}_0 \Delta x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \vec{\nabla}_0 \xi_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) \\ = \; & \partial_k \xi_1 \vec{\nabla}_0 x_k \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \partial_1 \xi_1 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \partial_1 \xi_1 J. \\ & \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 \Delta x_2 \times \vec{\nabla}_0 x_3) = \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 \xi_2 \times \vec{\nabla}_0 x_3) \\ = \; & \partial_k \xi_2 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_k \times \vec{\nabla}_0 x_3) = \partial_2 \xi_2 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \partial_2 \xi_2 J. \\ & \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 \Delta x_3) = \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 \xi_3) \\ = \; & \partial_k \xi_3 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_k) = \partial_3 \xi_3 \vec{\nabla}_0 x_1 \cdot (\vec{\nabla}_0 x_2 \times \vec{\nabla}_0 x_3) = \partial_3 \xi_3 J. \end{aligned} \tag{48}$$

Thus:

$$\Delta J = \partial_1 \xi_1 J + \partial_2 \xi_2 J + \partial_3 \xi_3 J = \vec{\nabla} \cdot \vec{\xi}\, J, \qquad \Delta dV_{\vec{\alpha},t} = \vec{\nabla} \cdot \vec{\xi}\, dV_{\vec{\alpha},t}. \tag{49}$$

So the variation of the internal energy of equation (42) can be written as:

$$\Delta dE_{in\;\vec{\alpha}} = \Delta\left(\frac{1}{\gamma}\right) dW_{\vec{\alpha}0} - P_0 \vec{\nabla} \cdot \vec{\xi}\, dV_{\vec{\alpha},t}. \tag{50}$$

Taking into account equation (26) this takes the form:

$$\Delta dE_{in\;\vec{\alpha}} = -P_{\vec{\alpha}0} \vec{\nabla} \cdot \vec{\xi}_{\vec{\alpha}}\, dV_{\vec{\alpha},t} - \frac{\gamma_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t)}{c^2} dW_{\vec{\alpha}0} \cdot \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}. \tag{51}$$

The variation of internal energy is the only new calculation with respect to the calculation done for a system of particles described previously, thus the rest of the analysis is trivial. Varying equation (17) we thus obtain:

$$\begin{aligned}
\Delta d\mathcal{A}_{\vec{\alpha}} &= \int_{t1}^{t2} \Delta dL_{\vec{\alpha}} dt, \qquad \Delta dL_{\vec{\alpha}} = \Delta dL_{k\vec{\alpha}} + \Delta dL_{i\vec{\alpha}} - \Delta dE_{in\ \vec{\alpha}} \\
\Delta dL_{k\vec{\alpha}} &= -dM_{\vec{\alpha}} c^2 \Delta \left(\frac{1}{\gamma_{\vec{\alpha}}}\right) = dM_{\vec{\alpha}} \gamma_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t) \cdot \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt}, \\
\Delta dL_{i\vec{\alpha}} &= dQ_{\vec{\alpha}} \left(\Delta \vec{A}(\vec{x}_{\vec{\alpha}}(t), t) \cdot \vec{v}_{\vec{\alpha}}(t) + \vec{A}(\vec{x}_{\vec{\alpha}}(t), t) \cdot \Delta \vec{v}_{\vec{\alpha}}(t)\right. \\
&- \left. \Delta \phi(\vec{x}_{\vec{\alpha}}(t), t)\right). \qquad (52)
\end{aligned}$$

We can now combine the internal and kinetic parts of the varied Lagrangian taking into account the specific enthalpy definition given in equation (32):

$$\Delta dL_{k\vec{\alpha}} - \Delta dE_{in\ \vec{\alpha}} = dM_{\vec{\alpha}} \gamma_{\vec{\alpha}} (\left(1 + \frac{w_0}{c^2}\right) \vec{v}_{\vec{\alpha}}(t) \cdot \frac{d\vec{\xi}_{\vec{\alpha}}(t)}{dt} + P_{\vec{\alpha}0} \vec{\nabla} \cdot \vec{\xi}_{\vec{\alpha}}\ dV_{\vec{\alpha},t}. \qquad (53)$$

The electromagnetic interaction variation terms are not different than in the low speed (non-relativistic) case, see for example equations A47 and A48 of [1], and their derivation will not be repeated here:

$$d\vec{F}_{L\vec{\alpha}} \equiv dQ_{\vec{\alpha}} \left[\vec{v}_{\vec{\alpha}} \times \vec{B}(\vec{x}_{\vec{\alpha}}(t), t) + \vec{E}(\vec{x}_{\vec{\alpha}}(t), t)\right] \qquad (54)$$

and:

$$\Delta dL_{i\vec{\alpha}} = \frac{d(dQ_{\vec{\alpha}} \vec{A}(\vec{x}_{\vec{\alpha}}(t), t) \cdot \vec{\xi}_{\vec{\alpha}})}{dt} + d\vec{F}_{L\vec{\alpha}} \cdot \vec{\xi}_{\vec{\alpha}}. \qquad (55)$$

Introducing the shorthand notation:

$$\bar{\lambda} \equiv 1 + \frac{w_0}{c^2}, \qquad \lambda \equiv \gamma \bar{\lambda} = \gamma \left(1 + \frac{w_0}{c^2}\right). \qquad (56)$$

The variation of the action of a relativistic fluid element is:

$$\begin{aligned}
\Delta d\mathcal{A}_{\vec{\alpha}} &= \int_{t1}^{t2} \Delta dL_{\vec{\alpha}} dt = \left(dM_{\vec{\alpha}} \lambda_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t) + dQ_{\vec{\alpha}} \vec{A}(\vec{x}_{\vec{\alpha}}(t), t)\right) \cdot \vec{\xi}_{\vec{\alpha}} \Big|_{t1}^{t2} \\
&- \int_{t1}^{t2} (dM_{\vec{\alpha}} \frac{d(\lambda_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t))}{dt} \cdot \vec{\xi}_{\vec{\alpha}} - d\vec{F}_{L\vec{\alpha}} \cdot \vec{\xi}_{\vec{\alpha}} - P_{\vec{\alpha}0} \vec{\nabla} \cdot \vec{\xi}_{\vec{\alpha}}\ dV_{\vec{\alpha},t}) dt. \qquad (57)
\end{aligned}$$

The variation of the relativistic fluid action is thus:

$$\begin{aligned}
\Delta \mathcal{A} &= \int_{\vec{\alpha}} d\mathcal{A}_{\vec{\alpha}} = \int_{\vec{\alpha}} (dM_{\vec{\alpha}} \lambda_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t) + dQ_{\vec{\alpha}} \vec{A}(\vec{x}(\vec{\alpha}, t), t)) \cdot \vec{\xi}_{\vec{\alpha}} \Big|_{t1}^{t2} \\
&- \int_{t1}^{t2} \int_{\vec{\alpha}} (dM_{\vec{\alpha}} \frac{d(\lambda_{\vec{\alpha}} \vec{v}_{\vec{\alpha}}(t))}{dt} \cdot \vec{\xi}_{\vec{\alpha}} - d\vec{F}_{L\vec{\alpha}} \cdot \vec{\xi}_{\vec{\alpha}} - P_{\vec{\alpha}0} \vec{\nabla} \cdot \vec{\xi}_{\vec{\alpha}}\ dV_{\vec{\alpha}}) dt. \qquad (58)
\end{aligned}$$

Now according to equation (20) we may write:

$$dM_{\vec{\alpha}} = \rho_{\vec{\alpha}}\ dV_{\vec{\alpha}}, \qquad dQ_{\vec{\alpha}} = \rho_{c\vec{\alpha}}\ dV_{\vec{\alpha}} \qquad (59)$$

using the above relations we may turn the $\vec{\alpha}$ integral into a volume integral and thus write the variation of the fluid action in which we suppress the $\vec{\alpha}$ labels:

$$\Delta \mathcal{A} = \int (\rho \lambda \vec{v} + \rho_c \vec{A}) \cdot \vec{\xi} dV \Big|_{t1}^{t2} - \int_{t1}^{t2} \int (\rho \frac{d(\lambda \vec{v})}{dt} \cdot \vec{\xi} - \vec{f}_L \cdot \vec{\xi} - P_0 \vec{\nabla} \cdot \vec{\xi}) dV dt. \qquad (60)$$

doi:10.20944/preprints202310.0452.v1

in the above we introduced the Lorentz force density:

$$\vec{f}_{L\vec{\alpha}} \equiv \frac{d\vec{F}_{L\vec{\alpha}}}{dV_{\vec{\alpha}}} = \rho_{c\vec{\alpha}} \left[\vec{v}_{\vec{\alpha}} \times \vec{B}(\vec{x}_{\vec{\alpha}}(t), t) + \vec{E}(\vec{x}_{\vec{\alpha}}(t), t)\right]. \tag{61}$$

Now, since:

$$P_0 \vec{\nabla} \cdot \vec{\xi} = \vec{\nabla} \cdot (P_0 \vec{\xi}) - \vec{\xi} \cdot \vec{\nabla} P_0, \tag{62}$$

and using Gauss theorem the variation of the action can be written as:

$$\begin{aligned} \Delta \mathcal{A} &= \int (\rho \lambda \vec{v} + \rho_c \vec{A}) \cdot \vec{\xi} dV \Big|_{t1}^{t2} \\ &- \int_{t1}^{t2} \left[\int (\rho \frac{d(\lambda \vec{v})}{dt} - \vec{f}_L + \vec{\nabla} P_0) \cdot \vec{\xi} dV - \oint P_0 \vec{\xi} \cdot d\vec{\Sigma}\right] dt. \end{aligned} \tag{63}$$

It follows that the variation of the action will vanish for a $\vec{\xi}$ such that $\vec{\xi}(t1) = \vec{\xi}(t2) = 0$ and vanishing on a surface encapsulating the fluid, but other than that arbitrary only if the Euler equation for a relativistic charged fluid is satisfied, that is:

$$\frac{d(\lambda \vec{v})}{dt} = -\frac{\vec{\nabla} P_0}{\rho} + \frac{\vec{f}_L}{\rho} \tag{64}$$

for the particular case that the fluid element is made of identical microscopic particles each with a mass m and a charge e, it follows that the mass and charge densities are proportional to the number density n:

$$\rho = m\, n, \quad \rho_c = e\, n \Rightarrow \frac{\vec{f}_L}{\rho} = k \left[\vec{v} \times \vec{B} + \vec{E}\right], k \equiv \frac{e}{m} \tag{65}$$

thus except from the terms related to the internal energy the equation is similar to that of a point particle. For a neutral fluid one obtains the form:

$$\frac{d(\lambda \vec{v})}{dt} = -\frac{\vec{\nabla} P_0}{\rho}. \tag{66}$$

Some authors prefer to write the above equation in terms of the energy per element of the fluid per unit volume in the rest frame which is the sum of the internal energy contribution and the rest mass contribution:

$$e_0 \equiv \rho_0 c^2 + \rho_0 \varepsilon_0. \tag{67}$$

It is easy to show that:

$$\bar{\lambda} = 1 + \frac{w_0}{c^2} = \frac{e_0 + P_0}{\rho_0 c^2}. \tag{68}$$

And using the above equality and some manipulations we may write equation (66) in a form which is preferable by some authors:

$$(e_0 + P_0) \frac{\gamma}{c^2} \frac{d(\gamma \vec{v})}{dt} = -\vec{\nabla} P_0 - \frac{\gamma^2}{c^2} \frac{dP_0}{dt} \vec{v}. \tag{69}$$

In practical fluid dynamics a fluid is described in terms of localized quantities, instead of quantities related to unseen infinitesimal "fluid elements". This is the Eulerian description of fluid dynamics in which one uses flow fields rather than "fluid elements" as will be discussed below.

4. The Clebsch Approach to the Relativistic Charged Eulerian Fluid

Here we follow the derivation of [1,2,9] but now taking into account the relativistic nature of the flow, this implies taking into account an action which is invariant under Lorentz transformations. Let us consider the action:

$$\begin{aligned}
\mathcal{A} &\equiv \int \mathcal{L} d^3x dt, \qquad \mathcal{L} \equiv \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_i \\
\mathcal{L}_0 &\equiv -\rho(\frac{c^2}{\gamma} + \varepsilon) = -\rho_0(c^2 + \varepsilon_0) = -e_0, \qquad \mathcal{L}_2 \equiv \nu \partial^\nu (\rho_0 u_\nu) - \rho_0 \alpha u_\nu \partial^\nu \beta, \\
\mathcal{L}_i &\equiv -\rho_c A^\nu v_\nu, \qquad v_\nu \equiv \frac{dx_\nu}{dt}.
\end{aligned} \tag{70}$$

In the non relativistic limit we may write:

$$\mathcal{L}_0 \simeq \rho(\frac{1}{2}v^2 - \varepsilon - c^2) \tag{71}$$

Taking into account that:

$$u_\mu = \gamma(c, \vec{v}) \tag{72}$$

and also that $\rho = \gamma\rho_0$ according to equation (38), it is easy to write the above Lagrangian densities in a space-time formalism:

$$\mathcal{L}_2 = \nu[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v})] - \rho \alpha \frac{d\beta}{dt} \qquad \mathcal{L}_i = \rho_c \left(\vec{A} \cdot \vec{v} - \phi\right) \tag{73}$$

Here we consider the variational variables to be functions of space and time (fields). Those include a vector velocity field $\vec{v}(\vec{x}, t)$ and density scalar field $\rho(\vec{x}, t)$. The way to include in the above formalism conservation of quantities which is easy in the Lagrange approach such as: label, mass, charge and entropy is by using Lagrange multipliers ν, α that enforce the equations:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \\
&\frac{d\beta}{dt} = 0
\end{aligned} \tag{74}$$

Provided $\rho \neq 0$ those are the continuity equation which ensures mass conservation and the conditions that β is a label (comoving function). Combining variation with respect to β with the continuity equation (74) will lead to the equation (similar to the derivation of equations (67-69) of [2]):

$$\frac{d\alpha}{dt} = 0 \tag{75}$$

Hence for $\rho \neq 0$ both α and β are labels. The specific internal energy ε_0 defined in equation (22) is dependent on the thermodynamic properties of the fluid. This is formulated through an equation of state as a function of the density and specific entropy. Here we shall assume a barotropic fluid, that is $\varepsilon_0(\rho_0)$ is a function of the density ρ_0 alone. The electromagnetic potentials $\vec{A}, \phi$ are given functions of coordinates and thus are not varied. Another assumption in our analysis is that each fluid element is composed of microscopic particles of mass m and charge e, thus it follows from equation (65) that:

$$\rho_c = k\rho. \tag{76}$$

Let us now take the variational derivative with respect to the density ρ, we obtain:

$$\begin{aligned}\delta_\rho A &= \int d^3x dt \delta\rho[-\frac{c^2}{\gamma} - w_0\frac{\delta\rho_0}{\delta\rho} - \frac{\partial\nu}{\partial t} - \vec{v}\cdot\vec{\nabla}\nu + k(\vec{A}\cdot\vec{v} - \phi)] \\ &+ \oint d\vec{S}\cdot\vec{v}\delta\rho\nu + \int d\vec{\Sigma}\cdot\vec{v}\delta\rho[\nu] + \int d^3x\nu\delta\rho|_{t_0}^{t_1} \end{aligned} \tag{77}$$

Or as:

$$\begin{aligned}\delta_\rho A &= \int d^3x dt \delta\rho[-\frac{c^2 + w_0}{\gamma} - \frac{\partial\nu}{\partial t} - \vec{v}\cdot\vec{\nabla}\nu + k(\vec{A}\cdot\vec{v} - \phi)] \\ &+ \oint d\vec{S}\cdot\vec{v}\delta\rho\nu + \int d\vec{\Sigma}\cdot\vec{v}\delta\rho[\nu] + \int d^3x\nu\delta\rho|_{t_0}^{t_1} \end{aligned} \tag{78}$$

Hence if $\delta\rho$ disappears on the boundary and cut, and in initial and final times we obtain:

$$\frac{d\nu}{dt} = \frac{\partial\nu}{\partial t} + \vec{v}\cdot\vec{\nabla}\nu = -\frac{c^2 + w_0}{\gamma} + k(\vec{A}\cdot\vec{v} - \phi) \tag{79}$$

In the above we use the material derivative defined by the prevalent form:

$$\frac{dg(\vec{\alpha},t)}{dt} = \frac{dg(\vec{x}(\vec{\alpha},t),t)}{dt} = \frac{\partial g}{\partial t} + \frac{d\vec{x}}{dt}\cdot\vec{\nabla}g = \frac{\partial g}{\partial t} + \vec{v}\cdot\vec{\nabla}g \tag{80}$$

once g is taken to be dependent on $\vec{x}, t$.

Finally we vary the action with respect to $\vec{v}$, taking into account that:

$$\delta_{\vec{v}}\frac{1}{\gamma} = -\gamma\frac{\vec{v}\cdot\delta\vec{v}}{c^2} \tag{81}$$

This will result in:

$$\begin{aligned}\delta_{\vec{v}}A &= \int d^3x dt \rho\delta\vec{v}\cdot[\gamma\vec{v} - \frac{w_0}{\rho}\frac{\delta\rho_0}{\delta\vec{v}} - \vec{\nabla}\nu - \alpha\vec{\nabla}\beta + k\vec{A}] \\ &+ \oint d\vec{S}\cdot\delta\vec{v}\rho\nu + \int d\vec{\Sigma}\cdot\delta\vec{v}\rho[\nu]. \end{aligned} \tag{82}$$

However:

$$\frac{\delta\rho_0}{\delta\vec{v}} = \rho\frac{\delta\frac{1}{\gamma}}{\delta\vec{v}} = -\rho\gamma\frac{\vec{v}}{c^2} \tag{83}$$

Taking in account the definition of λ (see equation (56)), we thus have:

$$\begin{aligned}\delta_{\vec{v}}A &= \int d^3x dt \rho\delta\vec{v}\cdot[\lambda\vec{v} - \vec{\nabla}\nu - \alpha\vec{\nabla}\beta + k\vec{A}] \\ &+ \oint d\vec{S}\cdot\delta\vec{v}\rho\nu + \int d\vec{\Sigma}\cdot\delta\vec{v}\rho[\nu]. \end{aligned} \tag{84}$$

the above boundary terms contain integration over the external boundary $\oint d\vec{S}$ and an integral over the cut $\int d\vec{\Sigma}$ that must be introduced in case that ν is not single valued, more on this case in later sections. The external boundary term vanishes; in the case of astrophysical flows for which $\rho = 0$ on the free flow boundary, or the case in which the fluid is contained in a vessel which induces a no flux boundary condition $\delta\vec{v}\cdot\hat{n} = 0$ ($\hat{n}$ is a unit vector normal to the boundary). The cut "boundary" term vanish when the velocity field varies only parallel to the cut that is it satisfies a Kutta type condition. If the boundary terms vanish $\vec{v}$ must have the following form:

$$\lambda\vec{v} = \alpha\vec{\nabla}\beta + \vec{\nabla}\nu - k\vec{A} \tag{85}$$

this is a generalization of Clebsch representation of the flow field (see for example [7], [10, page 248]) for a relativistic charged flow.

4.1. Euler's equations

We shall now show that a velocity field given by equation (85), such that the functions α, β, ν satisfy the corresponding equations (74,79,75) must satisfy Euler's equations. Let us calculate the material derivative of $\lambda\vec{v}$:

$$\frac{d(\lambda\vec{v})}{dt} = \frac{d\vec{\nabla}\nu}{dt} + \frac{d\alpha}{dt}\vec{\nabla}\beta + \alpha\frac{d\vec{\nabla}\beta}{dt} - k\frac{d\vec{A}}{dt} \tag{86}$$

It can be easily shown that:

$$\begin{aligned}
\frac{d\vec{\nabla}\nu}{dt} &= \vec{\nabla}\frac{d\nu}{dt} - \vec{\nabla}v_n\frac{\partial\nu}{\partial x_n} = \vec{\nabla}\left(-\frac{c^2+w_0}{\gamma} + k\vec{A}\cdot\vec{v} - k\phi\right) - \vec{\nabla}v_n\frac{\partial\nu}{\partial x_n} \\
\frac{d\vec{\nabla}\beta}{dt} &= \vec{\nabla}\frac{d\beta}{dt} - \vec{\nabla}v_n\frac{\partial\beta}{\partial x_n} = -\vec{\nabla}v_n\frac{\partial\beta}{\partial x_n}
\end{aligned} \tag{87}$$

In which x_n is a Cartesian coordinate and a summation convention is assumed. Inserting the result from equations (87) into equation (86) yields:

$$\begin{aligned}
\frac{d(\lambda\vec{v})}{dt} &= -\vec{\nabla}v_n\left(\frac{\partial\nu}{\partial x_n} + \alpha\frac{\partial\beta}{\partial x_n}\right) + \vec{\nabla}\left(-\frac{c^2+w_0}{\gamma} + k\vec{A}\cdot\vec{v} - k\phi\right) - k\frac{d\vec{A}}{dt} \\
&= -\vec{\nabla}v_n(\lambda v_n + kA_n) + \vec{\nabla}\left(-\frac{c^2+w_0}{\gamma} + k\vec{A}\cdot\vec{v} - k\phi\right) - k\partial_t\vec{A} - k(\vec{v}\cdot\vec{\nabla})\vec{A} \\
&= -\frac{1}{\gamma}\vec{\nabla}w_0 + k\vec{E} + k(v_n\vec{\nabla}A_n - v_n\partial_n\vec{A}),
\end{aligned} \tag{88}$$

the electric field is defined in equation (8). Also according to equation (7):

$$(v_n\vec{\nabla}A_n - v_n\partial_n\vec{A})_l = v_n(\partial_l A_n - \partial_n A_l) = \epsilon_{lnj}v_n B_j = (\vec{v}\times\vec{B})_l, \tag{89}$$

Hence we obtain the Euler equation of a charged relativistic fluid in the form:

$$\frac{d(\lambda\vec{v})}{dt} = -\frac{1}{\gamma}\vec{\nabla}w_0 + k\left[\vec{v}\times\vec{B} + \vec{E}\right] = -\frac{1}{\rho}\vec{\nabla}P_0 + k\left[\vec{v}\times\vec{B} + \vec{E}\right], \tag{90}$$

since (see equation (34)):

$$\vec{\nabla}w_0 = \frac{\partial w_0}{\partial\rho_0}\vec{\nabla}\rho_0 = \frac{1}{\rho_0}\frac{\partial P_0}{\partial\rho_0}\vec{\nabla}\rho_0 = \frac{1}{\rho_0}\vec{\nabla}P_0. \tag{91}$$

Equation (90) is identical to equation (64) and thus it is proven that the Euler equations can be derived from the action (73) thus all the equations of relativistic charged fluid dynamics can be derived from the action (73) using arbitrary and unrestricted variations.

4.2. Simplified action

It may be claimed the previous approach introduced unnecessary complications to the theory of relativistic fluid dynamics by adding additional three more scalar fields α, β, ν to the physical set $\vec{v}, \rho$. We will show that this is just a superficial impression and equation (70) given in a pedagogical form can be simplified. It is easy to show that defining a four dimensional Clebsch four vector:

$$v^\mu_C \equiv \alpha\partial^\mu\beta + \partial^\mu\nu = \left(\frac{1}{c}(\alpha\partial_t\beta + \partial_t\nu), \alpha\vec{\nabla}\beta + \vec{\nabla}\nu\right) = \left(\frac{1}{c}(\alpha\partial_t\beta + \partial_t\nu), \vec{v}_C\right) \tag{92}$$

and a four dimensional electromagnetic Clebsch four vector:

$$v^{\mu}_{E} \equiv v^{\mu}_{C} + kA^{\mu} = (\frac{1}{c}(\alpha\partial_t\beta + \partial_t\nu + k\phi), \vec{v}_C - k\vec{A}). \tag{93}$$

It follows from equation (79) and equation (85) that:

$$v_{\mu} = -\frac{v_{E\mu}}{\lambda} \Rightarrow \vec{v} = \frac{\vec{v}_E}{\lambda}. \tag{94}$$

Eliminating $\vec{v}$ the Lagrangian density appearing in equation (73) can be written (up to surface terms) in the compact form:

$$\mathcal{L}[\rho_0, \alpha, \beta, \nu] = \rho_0 \left[c\sqrt{v_{E\mu}v^{\mu}_{E}} - \varepsilon_0 - c^2 \right] \tag{95}$$

This Lagrangian density will yield the four equations (74,75,79), after those equations are solved we can substitute the scalar fields α, β, ν into equation (85) to obtain $\vec{v}$. Hence, the general charged relativistic barotropic fluid dynamics problem is modified such that instead of solving the Euler and continuity equations we need to solve an alternative equivalent set which can be derived from the Lagrangian density $\hat{\mathcal{L}}$.

5. Conclusion

While the analogies between spin fluid dynamics classic Clebsch fluid dynamics are quite convincing [1,2] still there are terms in spin fluid dynamics that lack classical interpretation. It was thus suggested that those term originate from a relativistic Clebsch theory which was the main motivation to the current paper. Indeed following the footsteps of the pervious papers [11,12] we may replace the internal energy in equation (95) with a Lorentz invariant Fisher information term to obtain a new Lagrangian density of relativistic quantum mechanics of a particle with spin:

$$\mathcal{L}[\rho_0, \alpha, \beta, \nu] = \rho_0 \left[c\sqrt{v_{E\mu}v^{\mu}_{E}} - c^2 \right] - \frac{\hbar^2}{2m}\partial^{\mu}a_0\partial_{\mu}a_0, \qquad a_0 \equiv \sqrt{\frac{\rho_0}{m}}. \tag{96}$$

in the above m is the particle's mass and $\hbar$ is Planck's constant divided by 2π.

A side benefit of the above work is the ability to canonically derive the momentum energy tensor of a relativistic fluid.

As the current paper is of limited scope, we were not able to compare the above lagrangian with its low speed limit and derive the relevant quantum equation, hopefully this will be done in a following more expanded paper.

Not less important is the comparison between the fluid route to relativistic quantum mechanics and the more established route of the Dirac equation, this certainly deserve and additional paper which I hope to compose in the near future.

References

1. Yahalom, A. Fisher Information Perspective of Pauli's Electron. Entropy 2022, 24, 1721. https://doi.org/10.3390/e24121721.
2. Yahalom, A. (2023). Fisher Information Perspective of Pauli's Electron. In: Skiadas, C.H., Dimotikalis, Y. (eds) 15th Chaotic Modeling and Simulation International Conference. CHAOS 2022. Springer Proceedings in Complexity. Springer, Cham. https://doi.org/10.1007/978-3-031-27082-6_26
3. Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 1972.
4. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; W.H. Freeman & Company: New York, NY, USA, 1973.
5. Padmanabhan, T. *Gravitation: Foundations and Frontiers*; Cambridge University Press: Cambridge, UK, 2010.
6. H. Goldstein , C. P. Poole Jr. & J. L. Safko, Classical Mechanics, Pearson; 3 edition (2001).

7. C. Eckart, "Variation Principles of Hydrodynamics ," *Phys. Fluids*, vol. 3, 421, 1960.
8. F.P. Bretherton "A note on Hamilton's principle for perfect fluids," Journal of Fluid Mechanics / Volume 44 / Issue 01 / October 1970, pp 19 31 DOI: 10.1017/S0022112070001660, Published online: 29 March 2006.
9. A. Yahalom "The Fluid Dynamics of Spin". Molecular Physics, Published online: 13 Apr 2018. http://dx.doi.org/10.1080/00268976.2018.1457808 (arXiv:1802.09331 [physics.flu-dyn]).
10. Lamb, H., *Hydrodynamics*, 1945 (New York: Dover Publications).
11. A. Yahalom "The Fluid Dynamics of Spin - a Fisher Information Perspective" arXiv:1802.09331v2 [cond-mat.] 6 Jul 2018. Proceedings of the Seventeenth Israeli - Russian Bi-National Workshop 2018 "The optimization of composition, structure and properties of metals, oxides, composites, nano and amorphous materials".
12. Asher Yahalom "The Fluid Dynamics of Spin - a Fisher Information Perspective and Comoving Scalars" Chaotic Modeling and Simulation (CMSIM) 1: 17-30, 2020.