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Fuzzy Normed Algebras Generated by Homomorphisms

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Abstract: As a new approach, for a nonzero normed algebra A, we will define some different classes of algebra fuzzy norms on A generated by homomorphisms and continuous homomorphisms. Also as a source of examples and counterexamples in the field of fuzzy normed algebras, separate continuity of the elements of each class are investigated.

Keywords: fuzzy normed algebra; homomorphism; separate continuity

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1. Introduction and Preliminaries

The idea of fuzzy norm on a linear space introduced by Katsaras [11] in 1984. Also a type of fuzzy metric introduced by O. Kaleva and S. Seikkala [10] in 1984. Felbin [7] in 1992 introduced an idea of fuzzy norm on a linear space, such that its corresponding fuzzy metric is of type of introduced by O. Kaleva and S. Seikkala. Another idea of a fuzzy norm on a linear space was introduced by Cheng and Mordeson [6] in 1994. Following Cheng and Mordeson, a definition of a fuzzy norm whose associated fuzzy metric is similar to Kramosil and Michalek type [12], was introduced by T. Bag and S. K. Samanta [1] in 2003. A large number of papers concerning fuzzy norms have been published by different authors such as papers [2–5,8,9].

The concept of fuzzy normed algebra is different from the notion of fuzzy normed linear space in one step that had to be done.

In this paper we will consider A as a normed algebra over the field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Also let $Hom(A, \mathbb{F})$ be the set of all algebra homomorphisms from A into $\mathbb{F}$.

Given a normed algebra A, we will introduce some different classes of fuzzy normed algebras. Also separate continuity of the elements of each class are investigated. The results of this paper can be applied as a source of examples and counterexamples in the field of fuzzy normed algebras.

2. Algebra Fuzzy Norms Generated by Homomorphisms

Definition 2.1. [1] Let X be a linear space and let $N : X \times \mathbb{R} \longrightarrow [0,1]$ be a function such that for all $x, y \in X$ and for all $s, t \in \mathbb{R}$,

1. $N(x,t) = 0$ for all $t \leq 0$,
2. $N(x,t) = 1$ for all $t > 0$ if and only if $x = 0$,
3. $N(cx,t) = N(x, \frac{t}{|c|})$ for all $c \neq 0$,
4. $N(x+y, s+t) \geq \min(N(x,s), N(y,t))$,
5. For each fixed $x \in X$, $N(x,.) : \mathbb{R} \longrightarrow [0,1]$ is an increasing function, and $\lim\limits_{t \longrightarrow \infty} N(x,t) = 1$.

   Then (X, N) is called a fuzzy normed linear space.

Definition 2.2. [13] Let A be an algebra and let $N : A \times \mathbb{R} \longrightarrow [0,1]$ be a function such that for all $a, b \in A$ and for all $s, t \in \mathbb{R}$,

1. $N(a,t) = 0$ for all $t \leq 0$,
2. $N(a,t) = 1$ for all $t > 0$ if and only if $a = 0$,
3. $N(\alpha a, t) = N(a, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$,
4. $N(a+b, s+t) \geq \min(N(a,s), N(b,t))$,
5. For each fixed $a \in A, N(a,.) : \mathbb{R} \longrightarrow [0,1]$ is an increasing function, and $\lim\limits_{t \longrightarrow \infty} N(a,t) = 1$,
6. $N(ab, st) \geq N(a,s)N(b,t)$.
   Then (A, N) is called a fuzzy normed algebra.

Example 2.3. If A is a normed algebra, then $N : A \times \mathbb{R} \longrightarrow [0,1]$ defined by

$$N(a,t) = \begin{cases} 0 & t \leq \|a\| \\ 1 & t > \|a\| \end{cases}$$

is an algebra fuzzy norm on A.

Proposition 2.4. *Let A be a normed algebra over the field $\mathbb{F}$ and $\varphi \in Hom(A, \mathbb{F})$. Define $N_\varphi : A \times \mathbb{R} \longrightarrow [0,1]$ by*

$$N_\varphi(a,t) = \begin{cases} 0 & t \leq \|a\| + |\varphi(a)| \\ \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|} & t > \|a\| + |\varphi(a)|. \end{cases}$$

Then N_φ is an algebra fuzzy norm on A.

Proof. We only prove parts (2), (4), (5), and (6) of Definition 2.2.
(2): If $a = 0$, then clearly $N_\varphi(a,t) = 1$ for all $t > 0$. For the converse let $N_\varphi(a,t) = 1$ for all $t > 0$ and suppose by contradiction that $a \neq 0$.
If $t = \|a\| + |\varphi(a)|$, then $N_\varphi(a,t) = 0$ that is a contradiction.
(4): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $s + t \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_\varphi(a,s) = 0$ or $N_\varphi(b,t) = 0$.
Hence $0 = N_\varphi(a+b, s+t) \geq \min(N_\varphi(a,s), N_\varphi(b,t)) = 0$.
If $s + t > 0$ and $s + t \leq \|a+b\| + |\varphi(a+b)|$, then $s \leq \|a\| + |\varphi(a)|$ or $t \leq \|b\| + |\varphi(b)|$. Therefore

$$0 = N_\varphi(a+b, s+t) \geq \min(N_\varphi(a,s), N_\varphi(b,t)) = 0.$$

If $s + t > \|a+b\| + |\varphi(a+b)|$, then $N_\varphi(a+b, s+t) = \frac{s+t-\|a+b\|-|\varphi(a)+\varphi(b)|}{s+t+\|a+b\|+|\varphi(a)+\varphi(b)|}$.
In this case if $s \leq \|a\| + |\varphi(a)|$ or $t \leq \|b\| + |\varphi(b)|$, then clearly

$$\frac{s+t-\|a+b\|-|\varphi(a)+\varphi(b)|}{s+t+\|a+b\|+|\varphi(a)+\varphi(b)|} \geq \min(N_\varphi(a,s), N_\varphi(b,t)) = 0.$$

If $s > \|a\| + |\varphi(a)|$ and $t > \|b\| + |\varphi(b)|$, then $N_\varphi(a,s) = \frac{s-\|a\|-|\varphi(a)|}{s+\|a\|+|\varphi(a)|}$ and $N_\varphi(b,t) = \frac{t-\|b\|-|\varphi(b)|}{t+\|b\|+|\varphi(b)|}$.
One can easily verify that if $N_\varphi(a,s) \leq N_\varphi(b,t)$, then

$$s(\|b\| + |\varphi(b)|) \leq t(\|a\| + |\varphi(a)|).$$

Also if $N_\varphi(b,t) \leq N_\varphi(a,s)$, then $t(\|a\| + |\varphi(a)|) \leq s(\|b\| + |\varphi(b)|)$.
A straightforward calculation reveals that
if $s(\|b\| + |\varphi(b)|) \leq t(\|a\| + |\varphi(a)|)$, then

$$\begin{aligned} N_\varphi(a+b,s+t) &= \frac{s+t-\|a+b\|-|\varphi(a)+\varphi(b)|}{s+t+\|a+b\|+|\varphi(a)+\varphi(b)|} \\ &\geq \frac{s-\|a\|-|\varphi(a)|}{s+\|a\|+|\varphi(a)|} \\ &= \min(N_\varphi(a,s), N_\varphi(b,t)). \end{aligned}$$

Also if $t(\|a\| + |\varphi(a)|) \leq s(\|b\| + |\varphi(b)|)$, then

$$\begin{aligned} N_\varphi(a+b,s+t) &= \frac{s+t-\|a+b\|-|\varphi(a)+\varphi(b)|}{s+t+\|a+b\|+|\varphi(a)+\varphi(b)|} \\ &\geq \frac{t-\|b\|-|\varphi(b)|}{t+\|b\|+|\varphi(b)|} \\ &= \min(N_\varphi(a,s), N_\varphi(b,t)). \end{aligned}$$

Therefore $N_\varphi(a+b,s+t) \geq \min(N_\varphi(a,s), N_\varphi(b,t))$ for all $a,b \in A$ and $s,t \in \mathbb{R}$.
(5): Let $a \in A$ and $s \leq t$. If $N_\varphi(a,s) = 0$, then clearly $N_\varphi(a,s) \leq N_\varphi(a,t)$. If $N_\varphi(a,s) \neq 0$, then $N_\varphi(a,s) = \frac{s-\|a\|-|\varphi(a)|}{s+\|a\|+|\varphi(a)|}$ and $s > \|a\| + |\varphi(a)|$. Hence $t > \|a\| + |\varphi(a)|$ and $N_\varphi(a,t) = \frac{t-\|a\|-|\varphi(a)|}{t+\|a\|+|\varphi(a)|}$. Since $s \leq t$,

$$s(\|a\| + |\varphi(a)|) \leq t(\|a\| + |\varphi(a)|).$$

Therefore a straightforward calculation reveals that

$$N_\varphi(a,s) = \frac{s-\|a\|-|\varphi(a)|}{s+\|a\|+|\varphi(a)|} \leq \frac{t-\|a\|-|\varphi(a)|}{t+\|a\|+|\varphi(a)|} = N_\varphi(a,t).$$

This shows that $N_\varphi(a,.)$ is an increasing function for all $a \in A$.
Also $\lim\limits_{t\to\infty} N_\varphi(a,t) = \lim\limits_{t\to\infty} \frac{t-\|a\|-|\varphi(a)|}{t+\|a\|+|\varphi(a)|} = 1$.
(6): Let $a,b \in A$ and $s,t \in \mathbb{R}$. If $st \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_\varphi(a,s) = 0$ or $N_\varphi(b,t) = 0$.
Hence $0 = N_\varphi(ab,st) \geq N_\varphi(a,s)N_\varphi(b,t) = 0$.
If $st > 0$ and $st \leq \|ab\| + |\varphi(ab)|$, then $s \leq \|a\| + |\varphi(a)|$ or $t \leq \|b\| + |\varphi(b)|$. Therefore

$$0 = N_\varphi(ab,st) \geq N_\varphi(a,s)N_\varphi(b,t) = 0.$$

Let $st > \|ab\| + |\varphi(ab)|$, $s > \|a\| + |\varphi(a)|$ and $t > \|b\| + |\varphi(b)|$. It is easy but tedious to show that the inequality

$$N_\varphi(ab,st) \geq N_\varphi(a,s)N_\varphi(b,t) \tag{2.1}$$

is equivalent to

$$\begin{aligned} st\Big(s(\|b\| + |\varphi(b)|) + t(\|a\| + |\varphi(a)|\Big) \geq\ & st|\varphi(a)||\varphi(b)| + st\|ab\| \\ & + \|ab\|\|a\|\|b\| + \|ab\|\|a\||\varphi(b)| \\ & + \|ab\|\|b\||\varphi(a)| + \|ab\||\varphi(a)||\varphi(b)| \\ & + \|a\|\|b\||\varphi(a)||\varphi(b)| + \|a\||\varphi(a)||\varphi(b)|^2 \\ & + \|b\||\varphi(b)||\varphi(a)|^2 + |\varphi(a)|^2|\varphi(b)|^2. \end{aligned}$$

So by hypothesis, we have

$$\begin{aligned}
& st\Big(s(\|b\| + |\varphi(b)|) + t(\|a\| + |\varphi(a)|\Big) \\
& \geq st\Big((\|a\| + |\varphi(a)|)(\|b\| + |\varphi(b)|) + (\|b\| + |\varphi(b)|)(\|a\| + |\varphi(a)|)\Big) \\
& = 2st\Big((\|a\| + |\varphi(a)|)(\|b\| + |\varphi(b)|)\Big) \\
& = (st + st)\Big(\|a\|\|b\| + \|a\||\varphi(b)| + \|b\||\varphi(a)| + |\varphi(a)||\varphi(b)|\Big) \\
& = st\Big(\|a\|\|b\| + \|a\||\varphi(b)| + \|b\||\varphi(a)| + |\varphi(a)||\varphi(b)|\Big) \\
& + st\Big(\|a\|\|b\| + \|a\||\varphi(b)| + \|b\||\varphi(a)| + |\varphi(a)||\varphi(b)|\Big) \\
& \geq st\Big(\|a\|\|b\| + |\varphi(a)||\varphi(b)|\Big) \\
& + \Big(\|ab\| + |\varphi(a)\varphi(b)|\Big)\Big(\|a\|\|b\| + \|a\||\varphi(b)| + \|b\||\varphi(a)| + |\varphi(a)||\varphi(b)|\Big) \\
& \geq st\Big(\|ab\| + |\varphi(a)||\varphi(b)|\Big) \\
& + \Big(\|ab\| + |\varphi(a)\varphi(b)|\Big)\Big(\|a\|\|b\| + \|a\||\varphi(b)| + \|b\||\varphi(a)| + |\varphi(a)||\varphi(b)|\Big) \\
& = st|\varphi(a)||\varphi(b)| + st\|ab\| + \|ab\|\|a\|\|b\| + \|ab\|\|a\||\varphi(b)| \\
& + \|ab\|\|b\||\varphi(a)| + \|ab\||\varphi(a)||\varphi(b)| + \|a\|\|b\||\varphi(a)||\varphi(b)| \\
& + \|a\||\varphi(a)||\varphi(b)|^2 + \|b\||\varphi(b)||\varphi(a)|^2 + |\varphi(a)|^2|\varphi(b)|^2.
\end{aligned}$$

This shows that inequality 2.1 holds for all $a, b \in A$ and $s, t \in \mathbb{R}$. Hence N_φ is an algebra fuzzy norm on A. ☐

Proposition 2.5. *Let A be a normed algebra, $\varphi : A \longrightarrow \mathbb{F}$ be a continuous homomorphism, and $\epsilon > 0$ be an element such that $(\epsilon + \|\varphi\|) \geq 1$. Define $N_{\varphi,\epsilon} : A \times \mathbb{R} \longrightarrow [0,1]$ by*

$$N_{\varphi,\epsilon}(a,t) = \begin{cases} 0 & t \leq \|a\|(\epsilon + \|\varphi\|) \\ \frac{t-|\varphi(a)|}{t+|\varphi(a)|} & t > \|a\|(\epsilon + \|\varphi\|). \end{cases}$$

Then $N_{\varphi,\epsilon}$ is an algebra fuzzy norm on A.

Proof. We only prove parts (2), (4), (5), and (6) of Definition 2.2.
(2) : If $a = 0$, then clearly $N_{\varphi,\epsilon}(a,t) = 1$ for all $t > 0$. For the converse let $N_{\varphi,\epsilon}(a,t) = 1$ for all $t > 0$ and suppose by contradiction that $a \neq 0$.
If $t = \|a\|(\epsilon + \|\varphi\|)$, then $N_{\varphi,\epsilon}(a,t) = 0$ that is a contradiction.
(4) : Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $s + t \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_{\varphi,\epsilon}(a,s) = 0$ or $N_{\varphi,\epsilon}(b,t) = 0$.
Hence $0 = N_{\varphi,\epsilon}(a+b, s+t) \geq \min(N_{\varphi,\epsilon}(a,s), N_{\varphi,\epsilon}(b,t)) = 0$.
If $s + t > 0$ and $s + t \leq \|a+b\|(\epsilon + \|\varphi\|)$, then $s \leq \|a\|(\epsilon + \|\varphi\|)$ or $t \leq \|b\|(\epsilon + \|\varphi\|)$. Therefore

$$0 = N_{\varphi,\epsilon}(a+b, s+t) \geq \min(N_{\varphi,\epsilon}(a,s), N_{\varphi,\epsilon}(b,t)) = 0.$$

If $s+t >\|a+b\|(\epsilon+\|\varphi\|)$, then $N_{\varphi,\epsilon}(a+b,s+t) = \frac{s+t-|\varphi(a)+\varphi(b)|}{s+t+|\varphi(a)+\varphi(b)|}$.
In this case if $s \leq\|a\|(\epsilon+\|\varphi\|)$ or $t \leq\|b\|(\epsilon+\|\varphi\|)$, then clearly

$$\frac{s+t-|\varphi(a)+\varphi(b)|}{s+t+|\varphi(a)+\varphi(b)|} \geq \min(N_{\varphi,\epsilon}(a,s), N_{\varphi,\epsilon}(b,t)) = 0.$$

If $s >\|a\|(\epsilon+\|\varphi\|)$ and $t >\|b\|(\epsilon+\|\varphi\|)$, then $N_{\varphi,\epsilon}(a,s) = \frac{s-|\varphi(a)|}{s+|\varphi(a)|}$ and $N_{\varphi,\epsilon}(b,t) = \frac{t-|\varphi(b)|}{t+|\varphi(b)|}$.
One can easily verify that if $N_{\varphi,\epsilon}(a,s) \leq N_{\varphi,\epsilon}(b,t)$, then $2s|\varphi(b)| \leq 2t|\varphi(a)|$. Also if $N_{\varphi,\epsilon}(b,t) \leq N_{\varphi,\epsilon}(a,s)$, then $2t|\varphi(a)| \leq 2s|\varphi(b)|$.
A straightforward calculation reveals that if $2s|\varphi(b)| \leq 2t|\varphi(a)|$, then

$$\begin{aligned} N_{\varphi,\epsilon}(a+b,s+t) &= \frac{s+t-|\varphi(a)+\varphi(b)|}{s+t+|\varphi(a)+\varphi(b)|} \\ &\geq \frac{s-|\varphi(a)|}{s+|\varphi(a)|} \\ &= \min(N_{\varphi,\epsilon}(a,s), N_{\varphi,\epsilon}(b,t)). \end{aligned}$$

Also if $2t|\varphi(a)| \leq 2s|\varphi(b)|$, then

$$\begin{aligned} N_{\varphi,\epsilon}(a+b,s+t) &= \frac{s+t-|\varphi(a)+\varphi(b)|}{s+t+|\varphi(a)+\varphi(b)|} \\ &\geq \frac{t-|\varphi(b)|}{t+|\varphi(b)|} \\ &= \min(N_{\varphi,\epsilon}(a,s), N_{\varphi,\epsilon}(b,t)). \end{aligned}$$

Therefore $N_{\varphi,\epsilon}(a+b,s+t) \geq \min(N_{\varphi,\epsilon}(a,s), N_{\varphi,\epsilon}(b,t))$ for all $a,b \in A$ and $s,t \in \mathbb{R}$.
(5) : Let $a \in A$ and $s \leq t$. If $N_{\varphi,\epsilon}(a,s) = 0$, then clearly $N_{\varphi,\epsilon}(a,s) \leq N_{\varphi,\epsilon}(a,t)$. If $N_{\varphi,\epsilon}(a,s) \neq 0$, then $N_{\varphi,\epsilon}(a,s) = \frac{s-|\varphi(a)|}{s+|\varphi(a)|}$ and $s >\|a\|(\epsilon+\|\varphi\|)$. Hence $t >\|a\|(\epsilon+\|\varphi\|)$ and $N_{\varphi,\epsilon}(a,t) = \frac{t-|\varphi(a)|}{t+|\varphi(a)|}$.
Since $s \leq t$, $2s|\varphi(a)| \leq 2t|\varphi(a)|$. Therefore a straightforward calculation reveals that $N_{\varphi,\epsilon}(a,s) = \frac{s-|\varphi(a)|}{s+|\varphi(a)|} \leq \frac{t-|\varphi(a)|}{t+|\varphi(a)|} = N_{\varphi,\epsilon}(a,t)$. This shows that $N_{\varphi,\epsilon}(a,.)$ is an increasing function for all $a \in A$. Also

$$\lim_{t\to\infty} N_{\varphi,\epsilon}(a,t) = \lim_{t\to\infty} \frac{t-|\varphi(a)|}{t+|\varphi(a)|} = 1.$$

(6): Let $a,b \in A$ and $s,t \in \mathbb{R}$. If $st \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_{\varphi,\epsilon}(a,s) = 0$ or $N_{\varphi,\epsilon}(b,t) = 0$.
Hence $0 = N_{\varphi,\epsilon}(ab,st) \geq N_{\varphi,\epsilon}(a,s)N_{\varphi,\epsilon}(b,t) = 0$.
Let $st > 0$ and $st \leq \|ab\|(\epsilon+\|\varphi\|)$. The condition $\epsilon+\|\varphi\| \geq 1$ implies that
$s \leq \|a\|(\epsilon+\|\varphi\|)$ or $t \leq \|b\|(\epsilon+\|\varphi\|)$. Therefore

$$0 = N_{\varphi,\epsilon}(ab,st) \geq N_{\varphi,\epsilon}(a,s)N_{\varphi,\epsilon}(b,t) = 0.$$

Finally let $st > \|ab\|(\epsilon+\|\varphi\|)$, $s > \|a\|(\epsilon+\|\varphi\|)$ and $t > \|b\|(\epsilon+\|\varphi\|)$. It is easy to see that the inequality

$$N_{\varphi,\epsilon}(ab,st) \geq N_{\varphi,\epsilon}(a,s)N_{\varphi,\epsilon}(b,t) \tag{2.2}$$

is equivalent to $st|\varphi(a)||\varphi(b)| + |\varphi(a)|^2|\varphi(b)|^2 \leq s^2t|\varphi(b)| + st^2|\varphi(a)|$.
Since $s > \|a\|(\epsilon + \|\varphi\|) \geq \|a\|\|\varphi\| \geq |\varphi(a)|$ and $t > \|b\|(\epsilon + \|\varphi\|) \geq \|b\|\|\varphi\| \geq |\varphi(b)|$, we have

$$\begin{aligned} st|\varphi(a)||\varphi(b)| + |\varphi(a)|^2|\varphi(b)|^2 &= st|\varphi(a)||\varphi(b)| + |\varphi(b)|^2|\varphi(a)||\varphi(a)| \\ &\leq st(s)|\varphi(b)| + t^2s|\varphi(a)| \\ &= s^2t|\varphi(b)| + st^2|\varphi(a)|. \end{aligned}$$

This shows that inequality 2.2 holds for all $a, b \in A$ and $s, t \in \mathbb{R}$. $\square$

Remark 2.6. Note that the condition $\epsilon + \|\varphi\| \geq 1$ in Proposition 2.5 is necessary. Indeed, for $A = \mathbb{R}$, $\varphi = 0$, and $0 < \epsilon < 1$ we have $\epsilon + \|\varphi\| = \epsilon < 1$. Also $N_{0,\epsilon}(1 \cdot 1, \sqrt{\epsilon} \cdot \sqrt{\epsilon}) = N_{0,\epsilon}(1, \epsilon) = 0$ and $N_{0,\epsilon}(1, \sqrt{\epsilon}) = 1$. Since $\epsilon \leq |1|(\epsilon + 0)$ and $\sqrt{\epsilon} > |1|(\epsilon + 0)$. So the inequality $N_{0,\epsilon}(1 \cdot 1, \sqrt{\epsilon} \cdot \sqrt{\epsilon}) \geq N_{0,\epsilon}(1, \sqrt{\epsilon})N_{0,\epsilon}(1, \sqrt{\epsilon})$ dose not hold. Hence $N_{0,\epsilon}$ is not an algebra fuzzy norm .

Proposition 2.7. *Let A be a normed algebra and $\psi \in Hom(A, \mathbb{F})$. Then the maps*

$$N_\psi^{(1)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_\psi^{(1)}(a, t) = \begin{cases} 0 & t \leq \|a\| + |\psi(a)| \\ \frac{t}{t + \|a\| + |\psi(a)|} & t > \|a\| + |\psi(a)|, \end{cases}$$

and

$$N_\psi^{(2)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_\psi^{(2)}(a, t) = \begin{cases} 0 & t \leq |\psi(a)| \\ \frac{t}{t + \|a\| + |\psi(a)|} & t > |\psi(a)|, \end{cases}$$

are algebra fuzzy norms on A.
Also if $\ker \psi = \{0\}$*, then the map*

$$N_\psi^{(3)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_\psi^{(3)}(a, t) = \begin{cases} 0 & t \leq |\psi(a)| \\ \frac{t}{t + |\psi(a)|} & t > |\psi(a)|, \end{cases}$$

is an algebra fuzzy norm on A.

Proof. At the first we will prove that $N_\psi^{(2)}$ is an algebra fuzzy norm. At the end we will only prove part (6) of Definition 2.2 for $N_\psi^{(1)}$ and $N_\psi^{(3)}$. The proof of the other parts are similar. Note that for investigation of the fuzzy norm properties in the case $N_\psi^{(3)}$, the condition $\ker \psi = \{0\}$ will be used in part (2) of Definition 2.2, since $|\psi(a)| = 0$ implies that $a = 0$.
(1): If $t \leq 0$, then clearly $N_\psi^{(2)}(a, t) = 0$ for all $a \in A$.
(2): If $a = 0$, then $\psi(a) = 0$ and consequently for all $t > 0 = |\psi(a)|$, $N_\psi^{(2)}(a, t) = \frac{t}{t+0+0} = 1$. For the converse let $N_\psi^{(2)}(a, t) = 1$ for all $t > 0$. At the first we will show that $\psi(a) = 0$. Suppose by contradiction that $\psi(a) \neq 0$.
If $t = |\psi(a)|$, then $N_\psi^{(2)}(a, t) = 0$ that is a contradiction. This shows that $\psi(a) = 0$. So by hypothesis,

for all $t > 0 = |\psi(a)|$, $\frac{t}{t+\|a\|+0} = N_\psi^{(2)}(a,t) = 1$. It follows that $\|a\| = 0$ that implies $a = 0$.
(3): If $\alpha \neq 0$, then

$$\begin{aligned} N_\psi^{(2)}(\alpha a, t) &= \begin{cases} 0 & t \leq |\psi(\alpha a)| \\ \frac{t}{t+\|\alpha a\|+|\psi(\alpha a)|} & t > |\psi(\alpha a)| \end{cases} \\ &= \begin{cases} 0 & \frac{t}{|\alpha|} \leq |\psi(a)| \\ \frac{\frac{t}{|\alpha|}}{\frac{t}{|\alpha|}+\|a\|+|\psi(a)|} & \frac{t}{|\alpha|} > |\psi(a)| \end{cases} \\ &= N_\psi^{(2)}(a, \frac{t}{|\alpha|}), a \in A, t \in \mathbb{R}. \end{aligned}$$

(4): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $s + t \leq |\psi(a+b)| = |\psi(a) + \psi(b)|$, then $s \leq |\psi(a)|$ or $t \leq |\psi(b)|$. So

$$0 = N_\psi^{(2)}(a+b, s+t) \geq \min(N_\psi^{(2)}(a,s), N_\psi^{(2)}(b,t)) = 0.$$

Let $s + t > |\psi(a) + \psi(b)|$ and $s \leq |\psi(a)|$ or $t \leq |\psi(b)|$. Then clearly,

$$\begin{aligned} N_\psi^{(2)}(a+b, s+t) &= \frac{s+t}{s+t+\|a+b\|+|\psi(a)+\psi(b)|} \\ &> 0 \\ &= \min(N_\psi^{(2)}(a,s), N_\psi^{(2)}(b,t)). \end{aligned}$$

Let $s + t > |\psi(a) + \psi(b)|$ and $s > |\psi(a)|$ and $t > |\psi(b)|$. Then $N_\psi^{(2)}(a+b, s+t) = \frac{s+t}{s+t+\|a+b\|+|\psi(a)+\psi(b)|}$ and
$N_\psi^{(2)}(a,s) = \frac{s}{s+\|a\|+|\psi(a)|}$ and $N_\psi^{(2)}(b,t) = \frac{t}{t+\|b\|+|\psi(b)|}$.
If $\frac{s}{s+\|a\|+|\psi(a)|} \leq \frac{t}{t+\|b\|+|\psi(b)|}$, then

$$s(\|b\| + |\psi(b)|) \leq t(\|a\| + |\psi(a)|). \tag{2.3}$$

If $\frac{t}{t+\|b\|+|\psi(b)|} \leq \frac{s}{s+\|a\|+|\psi(a)|}$, then

$$t(\|a\| + |\psi(a)|) \leq s(\|b\| + |\psi(b)|). \tag{2.4}$$

In the case where $N_\psi^{(2)}(a,s) \leq N_\psi^{(2)}(b,t)$, inequality 2.3 implies that

$$\begin{aligned} \frac{s+t}{s+t+\|a+b\|+|\psi(a)+\psi(b)|} &= N_\psi^{(2)}(a+b, s+t) \\ &\geq \frac{s}{s+\|a\|+|\psi(a)|} \\ &= \min(N_\psi^{(2)}(a,s), N_\psi^{(2)}(b,t)). \end{aligned}$$

Also in the case where $N_\psi^{(2)}(b,t) \leq N_\psi^{(2)}(a,s)$, inequality 2.4 implies that
$N_\psi^{(2)}(a+b, s+t) \geq \min(N_\psi^{(2)}(a,s), N_\psi^{(2)}(b,t))$.
(5): Let $s \leq t$. If $N_\psi^{(2)}(a,s) = 0$, then clearly $N_\psi^{(2)}(a,s) \leq N_\psi^{(2)}(a,t)$. Let $N_\psi^{(2)}(a,s) \neq 0$. Then $s > |\psi(a)|$. It follows that $t > |\psi(a)|$.
Since $s(\|a\| + |\psi(a)|) \leq t(\|a\| + |\psi(a)|)$, $N_\psi^{(2)}(a,s) \leq N_\psi^{(2)}(a,t)$. So $N_\psi^{(2)}(a,.)$ is an increasing function and also

$$\lim_{t\to\infty} N_\psi^{(2)}(a,t) = \lim_{t\to\infty} \frac{t}{t+\|a\|+|\psi(a)|} = 1.$$

(6): Let $a,b \in A$ and $s,t \in \mathbb{R}$. If $st \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_{\psi}^{(2)}(a,s) = 0$ or $N_{\psi}^{(2)}(b,t) = 0$.
Hence $0 = N_{\psi}^{(2)}(ab,st) \geq N_{\psi}^{(2)}(a,s)N_{\psi}^{(2)}(b,t) = 0$.
If $st > 0$ and $st \leq |\psi(ab)|$, then $s \leq |\psi(a)|$ or $t \leq |\psi(b)|$. Therefore

$$0 = N_{\psi}^{(2)}(ab,st) \geq N_{\psi}^{(2)}(a,s)N_{\psi}^{(2)}(b,t) = 0.$$

Let $st > |\psi(ab)|$, $s > |\psi(a)|$, and $t > |\psi(b)|$. Clearly

$$st + \|ab\| + |\psi(a)||\psi(b)| \leq (s + \|a\| + |\psi(a)|)(t + \|b\| + |\psi(b)|).$$

So $\frac{st}{st+\|ab\|+|\psi(a)||\psi(b)|} \geq \frac{st}{(s+\|a\|+|\psi(a)|)(t+\|b\|+|\psi(b)|)}$. Hence

$$N_{\psi}^{(2)}(ab,st) \geq N_{\psi}^{(2)}(a,s)N_{\psi}^{(2)}(b,t). \tag{2.5}$$

Therefore inequality 2.5 holds for all $a,b \in A$ and $s,t \in \mathbb{R}$.
Now we will investigate part (6) of Definition 2.2 for $N_{\psi}^{(1)}$ and $N_{\psi}^{(3)}$.

$$N_{\psi}^{(1)} : A \times \mathbb{R} \longrightarrow [0,1]$$
$$N_{\psi}^{(1)}(a,t) = \begin{cases} 0 & t \leq \|a\| + |\psi(a)| \\ \frac{t}{t+\|a\|+|\psi(a)|} & t > \|a\| + |\psi(a)|, \end{cases}$$

(6): Let $a,b \in A$ and $s,t \in \mathbb{R}$. If $st \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_{\psi}^{(1)}(a,s) = 0$ or $N_{\psi}^{(1)}(b,t) = 0$.
Hence $0 = N_{\psi}^{(1)}(ab,st) \geq N_{\psi}^{(1)}(a,s)N_{\psi}^{(1)}(b,t) = 0$.
If $st > 0$ and $st \leq \|ab\| + |\psi(ab)|$, then $s \leq \|a\| + |\psi(a)|$ or $t \leq \|b\| + |\psi(b)|$. Therefore

$$0 = N_{\psi}^{(1)}(ab,st) \geq N_{\psi}^{(1)}(a,s)N_{\psi}^{(1)}(b,t) = 0.$$

Finally in the case where $st > \|ab\| + |\psi(a)\psi(b)|$, $s > \|a\| + |\psi(a)|$, and $t > \|b\| + |\psi(b)|$ clearly $st + \|ab\| + |\psi(a)||\psi(b)| \leq (s + \|a\| + |\psi(a)|)(t + \|b\| + |\psi(b)|)$. So $\frac{st}{st+\|ab\|+|\psi(a)||\psi(b)|} \geq \frac{st}{(s+\|a\|+|\psi(a)|)(t+\|b\|+|\psi(b)|)}$. Hence

$$N_{\psi}^{(1)}(ab,st) \geq N_{\psi}^{(1)}(a,s)N_{\psi}^{(1)}(b,t). \tag{2.6}$$

Therefore inequality 2.6 holds for all $a,b \in A$ and $s,t \in \mathbb{R}$.

$$N_{\psi}^{(3)} : A \times \mathbb{R} \longrightarrow [0,1]$$
$$N_{\psi}^{(3)}(a,t) = \begin{cases} 0 & t \leq |\psi(a)| \\ \frac{t}{t+|\psi(a)|} & t > |\psi(a)|, \end{cases}$$

(6): Let $a,b \in A$ and $s,t \in \mathbb{R}$. If $st \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_{\psi}^{(3)}(a,s) = 0$ or $N_{\psi}^{(3)}(b,t) = 0$.
Hence $0 = N_{\psi}^{(3)}(ab,st) \geq N_{\psi}^{(3)}(a,s)N_{\psi}^{(3)}(b,t) = 0$.
If $st > 0$ and $st \leq |\psi(ab)|$, then $s \leq |\psi(a)|$ or $t \leq |\psi(b)|$. Therefore

$$0 = N_{\psi}^{(3)}(ab,st) \geq N_{\psi}^{(3)}(a,s)N_{\psi}^{(3)}(b,t) = 0.$$

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Finally in the case $st > |\psi(ab)|$, $s > |\psi(a)|$, and $t > |\psi(b)|$ clearly $st + |\psi(a)||\psi(b)| \leq (s + |\psi(a)|)(t + |\psi(b)|)$. So $\frac{st}{st+|\psi(a)||\psi(b)|} \geq \frac{st}{(s+|\psi(a)|)(t+|\psi(b)|)}$ and consequently

$$N_{\psi}^{(3)}(ab, st) \geq N_{\psi}^{(3)}(a, s) N_{\psi}^{(3)}(b, t). \tag{2.7}$$

This shows that for all $a, b \in A$ and $s, t \in \mathbb{R}$ inequality 2.7 holds. ☐

Proposition 2.8. *Let A be a normed algebra and $\eta \in Hom(A, \mathbb{F})$. Then the map*

$$N_{\eta}^{(1)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_{\eta}^{(1)}(a, t) = \begin{cases} 0 & t \leq \|a\| + |\eta(a)| \\ \frac{t - \|a\| - |\eta(a)|}{t} & t > \|a\| + |\eta(a)|, \end{cases}$$

is an algebra fuzzy norm on A.
Also if $\ker \eta = \{0\}$, then the map

$$N_{\eta}^{(2)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_{\eta}^{(2)}(a, t) = \begin{cases} 0 & t \leq |\eta(a)| \\ \frac{t - |\eta(a)|}{t} & t > |\eta(a)|, \end{cases}$$

is an algebra fuzzy norm on A.

Proof. We only prove part (6) of Definition 2.2 for $N_{\eta}^{(1)}$ and $N_{\eta}^{(2)}$.

$$N_{\eta}^{(1)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_{\eta}^{(1)}(a, t) = \begin{cases} 0 & t \leq \|a\| + |\eta(a)| \\ \frac{t - \|a\| - |\eta(a)|}{t} & t > \|a\| + |\eta(a)|, \end{cases}$$

(6): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $st \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_{\eta}^{(1)}(a, s) = 0$ or $N_{\eta}^{(1)}(b, t) = 0$. Hence $0 = N_{\eta}^{(1)}(ab, st) \geq N_{\eta}^{(1)}(a, s) N_{\eta}^{(1)}(b, t) = 0$.
If $st > 0$ and $st \leq \|ab\| + |\eta(a)\eta(b)|$, then $s \leq \|a\| + |\eta(a)|$ or $t \leq \|b\| + |\eta(b)|$. Therefore

$$0 = N_{\eta}^{(1)}(ab, st) \geq N_{\eta}^{(1)}(a, s) N_{\eta}^{(1)}(b, t) = 0.$$

Finally in the case where $st > \|ab\| + |\eta(ab)|$, $s > \|a\| + |\eta(a)|$, and $t > \|b\| + |\eta(b)|$ we will show that the inequality

$$N_{\eta}^{(1)}(ab, st) \geq N_{\eta}^{(1)}(a, s) N_{\eta}^{(1)}(b, t) \tag{2.8}$$

holds. It is easy to see that inequality 2.8 is equivalent to

$$s(\|b\| + |\eta(b)|) + t(\|a\| + |\eta(a)|) \geq \|a\|\|b\| + \|ab\| + \|a\||\eta(b)| + \|b\||\eta(a)| + 2|\eta(a)||\eta(b)|.$$

In this case

$$s(\|b\| + |\eta(b)|) \geq (\|a\| + |\eta(a)|)(\|b\| + |\eta(b)|)$$

and

$$t(\|a\| + |\eta(a)|) \geq (\|b\| + |\eta(b)|)(\|a\| + |\eta(a)|).$$

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Hence

$$s(\|b\|+|\eta(b)|)+t(\|a\|+|\eta(a)|)\geq 2(\|a\|\|b\|+\|a\||\eta(b)|+\|b\||\eta(a)|+|\eta(a)||\eta(b)|).$$

Therefore

$$\begin{aligned}s(\|b\|+|\eta(b)|)+t(\|a\|+|\eta(a)|)&\geq(\|a\|\|b\|+\|a\||\eta(b)|+\|b\||\eta(a)|+2|\eta(a)||\eta(b)|)\\&+(\|a\|\|b\|)+(\|a\||\eta(b)|+\|b\||\eta(a)|)\\&\geq(\|a\|\|b\|+\|ab\|+\|a\||\eta(b)|+\|b\||\eta(a)|+2|\eta(a)||\eta(b)|)\\&+(\|a\||\eta(b)|+\|b\||\eta(a)|)\\&\geq\|a\|\|b\|+\|ab\|+\|a\||\eta(b)|+\|b\||\eta(a)|+2|\eta(a)||\eta(b)|.\end{aligned}$$

So inequality 2.8 holds for all $a,b\in A$ and $s,t\in\mathbb{R}$.

$$N_\eta^{(2)}:A\times\mathbb{R}\longrightarrow[0,1]$$

$$N_\eta^{(2)}(a,t)=\begin{cases}0 & t\leq|\eta(a)|\\ \frac{t-|\eta(a)|}{t} & t>|\eta(a)|,\end{cases}$$

(6): Let $a,b\in A$ and $s,t\in\mathbb{R}$. If $st\leq 0$, then $s\leq 0$ or $t\leq 0$. So $N_\eta^{(2)}(a,s)=0$ or $N_\eta^{(2)}(b,t)=0$.
Hence $0=N_\eta^{(2)}(ab,st)\geq N_\eta^{(2)}(a,s)N_\eta^{(2)}(b,t)=0$.
If $st>0$ and $st\leq|\eta(a)\eta(b)|$, then $s\leq|\eta(a)|$ or $t\leq|\eta(b)|$. Therefore

$$0=N_\eta^{(2)}(ab,st)\geq N_\eta^{(2)}(a,s)N_\eta^{(2)}(b,t)=0.$$

Let $st>|\eta(ab)|$, $s>|\eta(a)|$, and $t>|\eta(b)|$. It is easy to see that the inequality

$$N_\eta^{(2)}(ab,st)\geq N_\eta^{(2)}(a,s)N_\eta^{(2)}(b,t)\tag{2.9}$$

is equivalent to $s|\eta(b)|+t|\eta(a)|\geq 2|\eta(a)||\eta(b)|$. So in this case we have,

$$s|\eta(b)|\geq|\eta(a)||\eta(b)|$$

and

$$t|\eta(a)|\geq|\eta(b)||\eta(a)|.$$

Hence

$$s|\eta(b)|+t|\eta(a)|\geq 2|\eta(a)||\eta(b)|.$$

This shows that for all $a,b\in A$ and $s,t\in\mathbb{R}$ inequality 2.9 holds. □

3. The separate continuity of N_φ, $N_{\varphi,\epsilon}$, $N_\psi^{(i)}$, $N_\eta^{(j)}$, $1\leq i\leq 3$, $1\leq j\leq 2$

In this section we characterize separate continuity of the algebra fuzzy norms N_φ, $N_{\varphi,\epsilon}$, $N_\psi^{(i)}$, $N_\eta^{(j)}$, $1\leq i\leq 3$, $1\leq j\leq 2$, where $\epsilon>0$ and φ,ψ,η are continuous homomorphisms from A into $\mathbb{F}$.

Theorem 3.1. *Let A be a normed algebra and $\varphi:A\longrightarrow\mathbb{F}$ be a continuous homomorphism. If $N_\varphi:A\times\mathbb{R}\longrightarrow[0,1]$ is defined by*

$$N_\varphi(a,t)=\begin{cases}0 & t\leq\|a\|+|\varphi(a)|\\ \frac{t-\|a\|-|\varphi(a)|}{t+\|a\|+|\varphi(a)|} & t>\|a\|+|\varphi(a)|,\end{cases}$$

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then the map

$$\begin{aligned} N_\varphi(.,t) : A &\longrightarrow [0,1] \\ a &\longrightarrow N_\varphi(a,t), \end{aligned}$$

is continuous for all $t \in \mathbb{R}$.
Also the map

$$\begin{aligned} N_\varphi(a,.) : \mathbb{R} &\longrightarrow [0,1] \\ t &\longrightarrow N_\varphi(a,t), \end{aligned}$$

is continuous for all $a \in A$ *except* $a = 0$.

Proof. If $t \leq 0$, then $N_\varphi(a,t) = 0$ for all $a \in A$. So $N_\varphi(.,t) : A \longrightarrow [0,1]$ is a constant function and consequently is continuous on A for all $t \leq 0$.
For a fixed $t > 0$, let $a \in A$ and $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_n \longrightarrow a$ as $n \longrightarrow \infty$. If $t = \|a\| + |\varphi(a)|$, then for all subsequences $\{a_{n_k}\}_{k=1}^{\infty} \subseteq \{a_n\}_{n=1}^{\infty}$ satisfying $t \leq \|a_{n_k}\| + |\varphi(a_{n_k})|$, $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} N_\varphi(x_{n_k},t) = \lim_{k\to\infty} 0 = 0 = N_\varphi(a,t).$$

Also for all subsequences $\{a_{n_k}\}_{k=1}^{\infty} \subseteq \{a_n\}_{n=1}^{\infty}$ satisfying $t > \|a_{n_k}\| + |\varphi(a_{n_k})|, k \in \mathbb{N}$, we have

$$\begin{aligned} \lim_{k\to\infty} N_\varphi(a_{n_k},t) &= \lim_{k\to\infty} \frac{t - \|a_{n_k}\| - |\varphi(a_{n_k})|}{t + \|a_{n_k}\| + |\varphi(a_{n_k})|} \\ &= \frac{t-t}{t+t} \\ &= 0 \\ &= N_\varphi(a,t). \end{aligned}$$

If $t < \|a\| + |\varphi(a)|$, then there exists an $N \in \mathbb{N}$ such that $t < \|a_n\| + |\varphi(a_n)|$ for all $n \geq N$. So $\lim\limits_{n\to\infty} N_\varphi(a_n,t) = \lim\limits_{n\to\infty} 0 = 0 = N_\varphi(a,t)$.
If $t > \|a\| + |\varphi(a)|$, then there exists an $N \in \mathbb{N}$ such that $t > \|a_n\| + |\varphi(a_n)|$ for all $n \geq N$. So

$$\begin{aligned} \lim_{n\to\infty} N_\varphi(a_n,t) &= \lim_{n\to\infty} \frac{t - \|a_n\| - |\varphi(a_n)|}{t + \|a_n\| + |\varphi(a_n)|} \\ &= \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|} \\ &= N_\varphi(a,t). \end{aligned}$$

Hence for each $t > 0$, $N_\varphi(.,t)$ is continuous at every $a \in A$. So $N_\varphi(.,t)$ is continuous on A for all $t \in \mathbb{R}$.
For any fixed $a \neq 0$, let $t \in \mathbb{R}$ and $t_n \longrightarrow t$ as $n \longrightarrow \infty$. If $t = \|a\| + |\varphi(a)|$, then for all subsequences $\{t_{n_k}\}_{k=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ satisfying $t_{n_k} \leq \|a\| + |\varphi(a)|$ we have

$$\lim_{k\to\infty} N_\varphi(a,t_{n_k}) = \lim_{k\to\infty} 0 = 0 = N_\varphi(a,t).$$

Also for all subsequences $\{t_{n_k}\}_{k=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ satisfying $t_{n_k} > \|a\| + |\varphi(a)|$ we have

$$\begin{aligned}\lim_{k\to\infty} N_\varphi(a, t_{n_k}) &= \lim_{k\to\infty} \frac{t_{n_k} - \|a\| - |\varphi(a)|}{t_{n_k} + \|a\| + |\varphi(a)|} \\ &= \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|} \\ &= \frac{t-t}{t+t} \\ &= 0 \\ &= N_\varphi(a,t).\end{aligned}$$

If $t < \|a\| + |\varphi(a)|$, then there exists an $N \in \mathbb{N}$ such that $t_n < \|a\| + |\varphi(a)|$ for all $n \geq N$. So $\lim_{n\to\infty} N_\varphi(a, t_n) = \lim_{n\to\infty} 0 = 0 = N_\varphi(a,t)$.
If $t > \|a\| + |\varphi(a)|$, then there exists an $N \in \mathbb{N}$ such that $t_n > \|a\| + |\varphi(a)|$ for all $n \geq N$. So

$$\begin{aligned}\lim_{n\to\infty} N_\varphi(a, t_n) &= \lim_{n\to\infty} \frac{t_n - \|a\| - |\varphi(a)|}{t_n + \|a\| + |\varphi(a)|} \\ &= \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|} \\ &= N_\varphi(a,t).\end{aligned}$$

It follows that for any fixed $a \neq 0$, $t_n \longrightarrow t$ as $n \longrightarrow \infty$, implies that $\lim_{n\longrightarrow\infty} N_\varphi(a, t_n) = N_\varphi(a,t)$.
Hence $N_\varphi(a, .) : \mathbb{R} \longrightarrow [0,1]$ is continuous for all $a \neq 0$.
We shall show that $N_\varphi(0, .) : \mathbb{R} \longrightarrow [0,1]$ is not continuous at $t = 0$.
Since

$$N_\varphi(0,t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0, \end{cases}$$

$\lim_{t\longrightarrow 0^+} N_\varphi(0,t) = 1 \neq N_\varphi(0,0) = 0$. This shows that $N_\varphi(0, .) : \mathbb{R} \longrightarrow [0,1]$ is not continuous on $\mathbb{R}$. □

Theorem 3.2. *Let A be a normed algebra, $\varphi : A \longrightarrow \mathbb{F}$ be a continuous homomorphism, and $\epsilon > 0$ be an element such that $(\epsilon + \|\varphi\|) \geq 1$. If*
$N_{\varphi,\epsilon} : A \times \mathbb{R} \longrightarrow [0,1]$ is defined by

$$N_{\varphi,\epsilon}(a,t) = \begin{cases} 0 & t \leq \|a\|(\epsilon + \|\varphi\|) \\ \frac{t - |\varphi(a)|}{t + |\varphi(a)|} & t > \|a\|(\epsilon + \|\varphi\|), \end{cases}$$

then

1. *the map*

$$\begin{aligned} N_{\varphi,\epsilon}(.,t) : A &\longrightarrow [0,1] \\ a &\longrightarrow N_{\varphi,\epsilon}(a,t), \end{aligned}$$

is continuous for all $t \leq 0$.

2. *if $t > 0$, then $N_{\varphi,\epsilon}(.,t)$ is continuous at every $a \in A \setminus S$, where $S = \{a \in A \mid \|a\| = \frac{t}{\epsilon + \|\varphi\|}\}$.*

3. *the map $N_{\varphi,\epsilon}(a, .)$ is continuous at every $t \in \mathbb{R} \setminus T$, where $T = \{t \in \mathbb{R} \mid t = \|a\|(\epsilon + \|\varphi\|)\}$.*

Proof. 1. If $t \leq 0$, then $N_{\varphi,\epsilon}(a,t) = 0$ for all $a \in A$. So $N_{\varphi,\epsilon}(.,t)$ is a constant function on A that is continuous.

2. If $t > 0$ and $a \in S$, then $\|a\| = \frac{t}{\epsilon+\|\varphi\|}$ and $N_{\varphi,\epsilon}(a,t) = 0$. Set $a_n = \frac{t-\frac{t}{2n}}{\|a\|(\epsilon+\|\varphi\|)}a$ for all $n \in \mathbb{N}$. So $a_n \longrightarrow a$ as $n \longrightarrow \infty$, and
$\|a_n\| = \frac{t-\frac{t}{2n}}{\|a\|(\epsilon+\|\varphi\|)}\|a\| < \frac{t}{(\epsilon+\|\varphi\|)}$ for all $n \in \mathbb{N}$. This shows that
$t > \|a_n\|(\epsilon+\|\varphi\|)$ for all $n \in \mathbb{N}$.
Hence

$$\begin{aligned}\lim_{n\to\infty} N_{\varphi,\epsilon}(a_n,t) &= \lim_{n\to\infty}\frac{t-|\varphi(a_n)|}{t+|\varphi(a_n)|}\\ &= \frac{t-|\varphi(a)|}{t+|\varphi(a)|}\\ &\neq N_{\varphi,\epsilon}(a,t) = 0,\end{aligned}$$

since, $t = \|a\|(\epsilon+\|\varphi\|) > \|a\|\|\varphi\| \geq |\varphi(a)|$. Therefore $N_{\varphi,\epsilon}(.,t)$ is discontinuous at every $a \in S$. Let $a \notin S$ and let $a_n \longrightarrow a$ as $n \longrightarrow \infty$. So $t > \|a\|(\epsilon+\|\varphi\|)$ or $t < \|a\|(\epsilon+\|\varphi\|)$. If $t > \|a\|(\epsilon+\|\varphi\|)$, then there exists an $N \in \mathbb{N}$ such that $t > \|a_n\|(\epsilon+\|\varphi\|)$ for all $n \geq N$. Hence

$$\begin{aligned}\lim_{n\to\infty} N_{\varphi,\epsilon}(a_n,t) &= \lim_{n\to\infty}\frac{t-|\varphi(a_n)|}{t+|\varphi(a_n)|}\\ &= \frac{t-|\varphi(a)|}{t+|\varphi(a)|}\\ &= N_{\varphi,\epsilon}(a,t).\end{aligned}$$

If $t < \|a\|(\epsilon+\|\varphi\|)$, then there exists an $N \in \mathbb{N}$ such that
$t < \|a_n\|(\epsilon+\|\varphi\|)$ for all $n \geq N$. So

$$\lim_{n\to\infty} N_{\varphi,\epsilon}(a_n,t) = \lim_{n\to\infty} 0 = 0 = N_{\varphi,\epsilon}(a,t).$$

Consequently in the case where $t > 0$, $N_{\varphi,\epsilon}(.,t)$ is continuous at every point $a \in A \setminus S$.

3. Let $t \in T$. So $t = \|a\|(\epsilon+\|\varphi\|)$ and $N_{\varphi,\epsilon}(a,t) = 0$.
Set $t_n = (\|a\| + \frac{1}{n})(\epsilon+\|\varphi\|)$ for all $n \in \mathbb{N}$. Clearly $t_n \longrightarrow t$ as $n \longrightarrow \infty$, and $t_n > \|a\|(\epsilon+\|\varphi\|)$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}\lim_{n\to\infty} N_{\varphi,\epsilon}(a,t_n) &= \lim_{n\to\infty}\frac{t_n-|\varphi(a)|}{t_n+|\varphi(a)|}\\ &= \begin{cases}1 & a = 0\\ \frac{t-|\varphi(a)|}{t+|\varphi(a)|} & a \neq 0.\end{cases}\end{aligned}$$

It follows that $\lim_{n\to\infty} N_{\varphi,\epsilon}(a,t_n) \neq N_{\varphi,\epsilon}(a,t) = 0$. Note that if $t = \|a\|(\epsilon+\|\varphi\|)$ and $a \neq 0$, then $t - |\varphi(a)| \neq 0$, since

$$t = \|a\|(\epsilon+\|\varphi\|) > \|a\|\|\varphi\| \geq |\varphi(a)|.$$

We shall show that $N_{\varphi,\epsilon}(a,.)$ is continuous at every $t \in \mathbb{R} \setminus T$.
Let $t \in \mathbb{R} \setminus T$ and let $t_n \longrightarrow t$ as $n \longrightarrow \infty$. Then $t <\|a\|(\epsilon+\|\varphi\|)$ or $t >\|a\|(\epsilon+\|\varphi\|)$. If $t <\|a\|(\epsilon+\|\varphi\|)$, then there exists an $N \in \mathbb{N}$ such that $t_n <\|a\|(\epsilon+\|\varphi\|)$ for all $n \geq N$. Hence

$$\lim_{n\to\infty} N_{\varphi,\epsilon}(a,t_n) = \lim_{n\to\infty} 0 = 0 = N_{\varphi,\epsilon}(a,t).$$

If $t >\|a\|(\epsilon+\|\varphi\|)$, then there exists an $N \in \mathbb{N}$ such that $t_n >\|a\|(\epsilon+\|\varphi\|)$ for all $n \geq N$. So

$$\begin{aligned}\lim_{n\to\infty} N_{\varphi,\epsilon}(a,t_n) &= \lim_{n\to\infty} \frac{t_n - |\varphi(a)|}{t_n + |\varphi(a)|} \\ &= \frac{t - |\varphi(a)|}{t + |\varphi(a)|} \\ &= N_{\varphi,\epsilon}(a,t).\end{aligned}$$

This shows that $N_{\varphi,\epsilon}(a,.)$ is continuous at every $t \in \mathbb{R} \setminus T$.
□

Theorem 3.3. *Let A be a normed algebra and $\psi : A \longrightarrow \mathbb{F}$ be a continuous homomorphism. If $N^{(1)}_{\psi} : A \times \mathbb{R} \longrightarrow [0,1]$ and $N^{(2)}_{\psi} : A \times \mathbb{R} \longrightarrow [0,1]$ are defined by*

$$N^{(1)}_{\psi}(a,t) = \begin{cases} 0 & t \leq\|a\| + |\psi(a)| \\ \frac{t}{t+\|a\|+|\psi(a)|} & t >\|a\| + |\psi(a)|, \end{cases}$$

$$N^{(2)}_{\psi}(a,t) = \begin{cases} 0 & t \leq |\psi(a)| \\ \frac{t}{t+\|a\|+|\psi(a)|} & t > |\psi(a)|, \end{cases}$$

and in the case where $\ker\psi = \{0\}$, $N^{(3)}_{\psi} : A \times \mathbb{R} \longrightarrow [0,1]$ *is defined by*

$$N^{(3)}_{\psi}(a,t) = \begin{cases} 0 & t \leq |\psi(a)| \\ \frac{t}{t+|\psi(a)|} & t > |\psi(a)|, \end{cases}$$

then

1. *the maps $N^{(1)}_{\psi}(.,t)$, $N^{(2)}_{\psi}(.,t)$, and $N^{(3)}_{\psi}(.,t)$ are continuous on A for all $t \leq 0$.*
2. *if $t > 0$, then the map $N^{(1)}_{\psi}(.,t)$ is continuous at every $a \in A \setminus S_1$, where $S_1 = \{a \in A \mid t =\|a\| + |\psi(a)|\}$.*
3. *if $t > 0$, then the maps $N^{(2)}_{\psi}(.,t)$ and $N^{(3)}_{\psi}(.,t)$ are continuous at every $a \in A \setminus S_2$, where $S_2 = \{a \in A \mid t = |\psi(a)|\}$.*
4. *for $a \in A$, the map $N^{(1)}_{\psi}(a,.)$ is continuous at every $t \in \mathbb{R} \setminus T_1$, where $T_1 = \{t \in \mathbb{R} \mid t =\|a\| + |\psi(a)|\}$.*
5. *for $a \neq 0$, the map $N^{(2)}_{\psi}(a,.)$ is continuous at every $t \in \mathbb{R} \setminus T_2$, where $T_2 = \{t > 0 \mid t = |\psi(a)|\}$. Also the map $N^{(2)}_{\psi}(0,.)$ is continuous at every $t \in \mathbb{R}$ except $t = 0$.*
6. *for $a \in A$, the map $N^{(3)}_{\psi}(a,.)$ is continuous at every $t \in \mathbb{R} \setminus T_3$, where $T_3 = \{t \in \mathbb{R} \mid t = |\psi(a)|\}$.*

Proof. 1. It is obvious.

2. Let $t > 0$ and $a \in S_1$. So $t = \|a\| + |\psi(a)|$. Set $a_n = \frac{t - \frac{t}{2n}}{\|a\| + |\psi(a)|} a$ for all $n \in \mathbb{N}$. Obviously $a_n \longrightarrow a$ and so $\psi(a_n) \longrightarrow \psi(a)$ as $n \longrightarrow \infty$. Also

$$\begin{aligned} \|a_n\| &= \frac{t - \frac{t}{2n}}{\|a\| + |\psi(a)|} \|a\| \\ &< \frac{t}{\|a\| + |\psi(a)|} \|a\| \\ &= \|a\|, \quad n \in \mathbb{N}, \end{aligned}$$

and

$$\begin{aligned} |\psi(a_n)| &= \frac{t - \frac{t}{2n}}{\|a\| + |\psi(a)|} |\psi(a)| \\ &\leq \frac{t}{\|a\| + |\psi(a)|} |\psi(a)| \\ &= |\psi(a)|, \quad n \in \mathbb{N}. \end{aligned}$$

So $\|a_n\| + |\psi(a_n)| < \|a\| + |\psi(a)| = t$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} \lim_{n \to \infty} N_\psi^{(1)}(a_n, t) &= \lim_{n \to \infty} \frac{t}{t + \|a_n\| + |\psi(a_n)|} \\ &= \frac{t}{t + t} \\ &= \frac{1}{2} \\ &\neq N_\psi^{(1)}(a, t) \\ &= 0. \end{aligned}$$

This shows that $N_\psi^{(1)}(., t)$ is discontinuous at every $a \in S_1$. Now let $a \in A \setminus S_1$ and $\{z_n\}_{n=1}^\infty$ be a sequence such that $z_n \longrightarrow a$ as $n \longrightarrow \infty$. So $t > \|a\| + |\psi(a)|$ or $t < \|a\| + |\psi(a)|$. If $t > \|a\| + |\psi(a)|$, then there exists an $N \in \mathbb{N}$ such that $t > \|z_n\| + |\psi(z_n)|$ for all $n \geq N$. Hence

$$\begin{aligned} \lim_{n \to \infty} N_\psi^{(1)}(z_n, t) &= \lim_{n \to \infty} \frac{t}{t + \|z_n\| + |\psi(z_n)|} \\ &= \frac{t}{t + \|a\| + |\psi(a)|} \\ &= N_\psi^{(1)}(a, t). \end{aligned}$$

If $t < \|a\| + |\psi(a)|$, then there exists an $N \in \mathbb{N}$ such that $t < \|z_n\| + |\psi(z_n)|$ for all $n \geq N$. So

$$\lim_{n \to \infty} N_\psi^{(1)}(z_n, t) = \lim_{n \to \infty} 0 = 0 = N_\psi^{(1)}(a, t).$$

This shows that $N_\psi^{(1)}(., t)$ is continuous at every $a \in A \setminus S_1$.

3. Let $a \in S_2$. So $t = |\psi(a)|$, $a \neq 0$ and $N_\psi^{(2)}(a,t) = N_\psi^{(3)}(a,t) = 0$. Set $a_n = (1 - \frac{1}{2n})a$ for all $n \in \mathbb{N}$. Clearly $a_n \longrightarrow a$, $\psi(a_n) \longrightarrow \psi(a)$ as $n \longrightarrow \infty$. Also $|\psi(a_n)| = (1 - \frac{1}{2n})|\psi(a)| < |\psi(a)| = t$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} \lim_{n\to\infty} N_\psi^{(2)}(a_n,t) &= \lim_{n\to\infty} \frac{t}{t+\|a_n\| + |\psi(a_n)|} \\ &= \frac{t}{t+\|a\| + t} \\ &= \frac{t}{2t+\|a\|} \\ &\neq 0 \\ &= N_\psi^{(2)}(a,t). \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} N_\psi^{(3)}(a_n,t) &= \lim_{n\to\infty} \frac{t}{t + |\psi(a_n)|} \\ &= \frac{t}{t+t} \\ &= \frac{1}{2} \\ &\neq 0 \\ &= N_\psi^{(3)}(a,t). \end{aligned}$$

Hence $N_\psi^{(2)}(.,t)$ and $N_\psi^{(3)}(.,t)$ are discontinuous at every $a \in S_2$.
Now let $a \notin S_2$ and $z_n \longrightarrow a$ as $n \longrightarrow \infty$. Then $t < |\psi(a)|$ or $t > |\psi(a)|$. If $t < |\psi(a)|$, then there exists an $N \in \mathbb{N}$ such that $t < |\psi(z_n)|$ for all $n \geq N$. Hence

$$\lim_{n\to\infty} N_\psi^{(2)}(z_n,t) = 0 = N_\psi^{(2)}(a,t),$$

and

$$\lim_{n\to\infty} N_\psi^{(3)}(z_n,t) = 0 = N_\psi^{(3)}(a,t).$$

Also if $t > |\psi(a)|$, then there exists an $N \in \mathbb{N}$ such that
$t > |\psi(z_n)|$ for all $n \geq N$. So

$$\begin{aligned} \lim_{n\to\infty} N_\psi^{(2)}(z_n,t) &= \lim_{n\to\infty} \frac{t}{t+\|z_n\| + |\psi(z_n)|} \\ &= \frac{t}{t+\|a\| + |\psi(a)|} \\ &= N_\psi^{(2)}(a,t), \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} N_\psi^{(3)}(z_n,t) &= \lim_{n\to\infty} \frac{t}{t + |\psi(z_n)|} \\ &= \frac{t}{t + |\psi(a)|} \\ &= N_\psi^{(3)}(a,t). \end{aligned}$$

Hence $N_\psi^{(2)}(.,t)$ and $N_\psi^{(3)}(.,t)$ are continuous at every $a \in A \setminus S_2$.

4. Let $t \in T_1$. Then $t = \|a\| + |\psi(a)|$ and $N_\psi^{(1)}(a,t) = 0$. Set $t_n = \|a\| + |\psi(a)| + \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly $t_n > \|a\| + |\psi(a)|$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} t_n = t$. So

$$\begin{aligned} \lim_{n\to\infty} N_\psi^{(1)}(a,t_n) &= \lim_{n\to\infty} \frac{t_n}{t_n + \|a\| + |\psi(a)|} \\ &= \begin{cases} 1 & a = 0 \\ \frac{t}{t+t} = \frac{1}{2} & a \neq 0. \end{cases} \end{aligned}$$

   It follows that $\lim\limits_{n\to\infty} N_\psi^{(1)}(a,t_n) \neq N_\psi^{(1)}(a,t) = 0$. This shows that $N_\psi^{(1)}(a,.)$ is discontinuous at every $t \in T_1$. One can easily verify that if $t \in \mathbb{R} \setminus T_1$, then $N_\psi^{(1)}(a,.)$ is continuous at t.

5. Let $a \neq 0$. If $t \in T_2$, then $t = |\psi(a)| > 0$ and $N_\psi^{(2)}(a,t) = 0$. Set $t_n = (1 + \frac{1}{n})|\psi(a)|$ for all $n \in \mathbb{N}$. Clearly $t_n \longrightarrow t$ as $n \longrightarrow \infty$, and $t_n > |\psi(a)|$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} \lim_{n\to\infty} N_\psi^{(2)}(a,t_n) &= \lim_{n\to\infty} \frac{t_n}{t_n + \|a\| + |\psi(a)|} \\ &= \frac{t}{2t + \|a\|} \\ &\neq 0 \\ &= N_\psi^{(2)}(a,t). \end{aligned}$$

   This shows that $N_\psi^{(2)}(a,.)$ is discontinuous at every $t \in T_2$.
   Let $t \in \mathbb{R} \setminus T_2$. Then $t \leq 0$ or $t \neq |\psi(a)|$. In the case where $t < 0$ or $t \neq |\psi(a)|$, the continuity of $N_\psi^{(2)}(a,.)$ at t, can be obviously verified. If $t = 0$ and $t_n \longrightarrow 0$ as $n \longrightarrow \infty$, then for all subsequences $\{t_{n_k}\}_{k=1}^\infty \subseteq \{t_n\}_{n=1}^\infty$ satisfying $t_{n_k} \leq |\psi(a)|$ we have, $\lim\limits_{k\longrightarrow\infty} N_\psi^{(2)}(a,t_{n_k}) = 0 = N_\psi^{(2)}(a,0)$. Also for all subsequences $\{t_{n_k}\}_{k=1}^\infty \subseteq \{t_n\}_{n=1}^\infty$ satisfying $t_{n_k} > |\psi(a)|$ we have,

$$\lim_{k\to\infty} N_\psi^{(2)}(a,t_{n_k}) = \lim_{k\to\infty} \frac{t_{n_k}}{t_{n_k} + \|a\| + |\psi(a)|} = \frac{0}{0 + \|a\| + |\psi(a)|} = 0 = N_\psi^{(2)}(a,0).$$

   So $N_\psi^{(2)}(a,.)$ is continuous at $t = 0$.
   Clearly the map $N_\psi^{(2)}(0,.)$ is continuous at every $t \in \mathbb{R}$ except $t = 0$.

6. Inspired by part (5), the proof is obvious.

☐

Theorem 3.4. *Let A be a normed algebra and $\eta : A \longrightarrow \mathbb{F}$ be a continuous homomorphism. If $N_\eta^{(1)} : A \times \mathbb{R} \longrightarrow [0,1]$ is defined by*

$$N_\eta^{(1)}(a,t) = \begin{cases} 0 & t \leq \|a\| + |\eta(a)| \\ \frac{t - \|a\| - |\eta(a)|}{t} & t > \|a\| + |\eta(a)|, \end{cases}$$

and in the case where $\ker \eta = \{0\}$, $N_\eta^{(2)} : A \times \mathbb{R} \longrightarrow [0,1]$ is defined by

$$N_\eta^{(2)}(a,t) = \begin{cases} 0 & t \leq |\eta(a)| \\ \frac{t - |\eta(a)|}{t} & t > |\eta(a)|, \end{cases}$$

then

1. *the map $N_\eta^{(1)}(.,t)$ is continuous for all $t \in \mathbb{R}$.*

2. *the map* $N_{\eta}^{(1)}(a,.)$ *is continuous for all* $a \in A$ *except* $a = 0$.
3. *the map* $N_{\eta}^{(2)}(.,t)$ *is continuous for all* $t \in \mathbb{R}$.
4. *the map* $N_{\eta}^{(2)}(a,.)$ *is continuous for all* $a \in A$ *except* $a = 0$.

Proof. An argument similar to the proofs of the previous theorems can be applied. □

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