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Article

Control Problem Related to a 2D Parabolic-Elliptic Chemo-Repulsion System with Nonlinear Production

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Abstract: In this paper we study a bilinear optimal control problem related to a 2D parabolic-elliptic chemo-repulsion system consider a nonlinear chemical signal production term. We prove the existence of global optimal solutions with bilinear control, and applying a result on the existence of Lagrange multipliers in Banach spaces, we derive an optimality system for a local optimal solution.

Keywords: chemo-repulsion system; optimal control problem; optimality conditions

1. Introduction

One very interesting feature of living organisms is their interaction with the environment in which they reside. Frequently, the form of interaction involves the movement of living organisms generated by an external stimulus, the response to such stimulus is called *taxis*. The process which leads to taxis it is divided into three steps [1]: First, the cell detects the extracellular signal by specific receptors on its surface then, the cell processes the signal and, finally, the cell alters its motile behavior. There exist different types of taxis, which depend on the nature of the stimulus (see, for instance, [2]), one of them is *chemotaxis*.

The chemotaxis phenomenon is understood as the movement of living organisms induced by the presence of certain chemical substances. In 1970, Keller and Segel [3] proposed a mathematical model that describes the chemotactic aggregation of cellular slime molds which move preferentially towards relatively high concentrations of a chemical secreted by the amoebae themselves. Such phenomenon is called *chemo-attraction*. In contrast, the phenomenon is called *chemo-repulsion*, if a region of high chemical concentration generate a repulsive effect on the organisms. The most classical model in the framework of chemotactic movements is the Keller-Segel system [3,4], which is given by the following system of partial differential equations:

$$\begin{cases} \partial_t u &= \alpha_u \Delta u - \nabla(\chi u \nabla v) & \text{in } Q, \\ \partial_t v &= \alpha_v \Delta v - \beta v + h(u) & \text{in } Q, \\ u(0, x) &= u_0(x), v(0, x) = v_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= \frac{\partial v}{\partial \mathbf{n}} & \text{on } (0, T) \times \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded domain with smooth boundary $\partial\Omega$, $(0, T)$ is a time interval with $0 < T < \infty$, $Q := (0, T) \times \Omega$ is the time-space region, and \mathbf{n} denotes the outward unit normal vector to $\partial\Omega$. The unknowns are the cell density $u := u(t, x) \geq 0$ and a chemical concentration $v := v(t, x) \geq 0$. The cell flux and chemical are given respectively by $\chi u \nabla v - \alpha_u \nabla u$ and $\alpha_v \nabla v$, where $\alpha_u, \alpha_v > 0$ and $\chi \neq 0$ are real constants. Therefore, the cells perform a biased random walk in the direction of the chemical gradient, and the chemical diffuses (it is produced by the cells, and it degrades) [5]. The term $\chi u \nabla v$ models the transport of cells; if $\chi > 0$ towards the higher concentrations of chemical (chemo-attraction) and if $\chi < 0$ towards the lower concentrations of chemical (chemo-repulsion). The term $-\beta v + h(u)$ models the consumption-production rate of the chemical, where β is a real parameter

which measures the self-degradation of the chemical, and the function $h(u)$ is the cells production term, which is nonnegative when $u \geq 0$.

In this paper we study a bilinear optimal control problem related to the parabolic-elliptic system associated to problem (1), considering nonlinear chemical signal production and a proliferation/degradation coefficient acting on a control subdomain ω . Specifically, we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega \in C^{2,1}$ and a time interval $(0, T)$, with $0 < T < \infty$. Then, we study an optimal control problem related to the following coupled system of partial differential equations in the time-space region $Q := (0, T) \times \Omega$:

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (u \nabla v), \\ -\Delta v + v &= u^p + f v 1_\omega, \end{cases} \quad (2)$$

where $\omega \subset \Omega$ is the control subdomain, f denotes a bilinear control that acts on the subdomain ω and 1_ω is the characteristic function of ω . The control lies in a nonempty, closed and convex set \mathcal{F} . We observe that, in the subdomain ω , where $f \geq 0$ we inject chemical substance and, where $f \leq 0$ we extract chemical substance; thus, we can see the control f as proliferation/degradation coefficient acting on the subregion ω . The term, u^p , for $p > 1$, is the nonlinear chemical signal production term.

The system (2) is completed with the initial condition

$$u(0, x) = u_0(x) \geq 0 \quad \text{in } \Omega, \quad (3)$$

and the Neumann boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on } (0, T) \times \partial\Omega. \quad (4)$$

Studies on the existence of solutions related to system (2)-(4) can be consulted in [6–15]. In particular, the case when $f \equiv 0$ and $p = 1$ has been analyzed by Mock in [6,7], in which the author proved the existence and uniqueness of global-in-time classical solutions and that the respective solutions are uniformly bounded and converge an exponential rate to steady-state. The parabolic-parabolic system related to problem (2)-(4) has been studied in [8–14]. In [8,9] considering linear production and $f \equiv 0$ has been studied, where Ciésłak et al. [8] proved the existence and uniqueness of smooth classical solutions in two-dimensional domains, as well as, the existence of weak solutions in spaces of dimension 3 and 4. In [9], the author delimits his analysis to a n -dimensional convex domain ($n \geq 3$) and changes the chemotactic term $\nabla \cdot (u \nabla v)$ by $\nabla \cdot (h(u) \nabla v)$, where $h(u)$ is an adequate smooth function. With this modification, the author proved the existence of a unique global-in-time classical solution and that the pair solution (u, v) converges to $\frac{1}{|\Omega|} (\int_\omega u_0, \int_\omega u_0)$, as t goes to ∞ . Moreover, in [10,11], the authors proved, for a quadratic production term ($p = 2$), the existence of weak solutions in 3D domains and global-in-time strong solutions assuming a regularity criteria in 1D and 2D domains; furthermore, they analyzed some numerical schemes to approximate the weak solutions. In [12], for $p = 1$ and $f \neq 0$, the authors proved the existence and uniqueness of strong solutions in 2D domains, and deduced that the solution (u, v) does not blow-up at finite time. The same authors in [13] extend the results obtained in [12] to 3D domains, and proved the existence of weak solutions and established a regularity criterion to get global-in-time strong solutions. In [14], consider, in a two-dimensional domain, the nonlinear case for $p \in (1, 2]$ and $f \neq 0$ and proved the existence and uniqueness of strong solutions. The existence and uniqueness of strong solutions for problem (2)-(4), considering $p \in (1, 2)$ and $f \in L^q(\omega)$, has been proved by Ancoma-Huarachi et al. [15]. For the stationary case and linear production term, we can refer to recent study developed by Lorca et al. [16].

In the context of the optimal control problems related to chemotaxis system, we can refer to [12–14,16–22] in all these works it proved the existence of at least one global optimal solution and derived an optimality system, in particular obtain first-order necessary optimality conditions. In [19] it studied an optimal control problem with state equations drive by a chemoattractive-Navier-Stokes

evolution system in 3D domains where stated first-order necessary optimality conditions, using that the state is differentiable with respect to the control. Rodríguez-Bellido et al. [20] analyzed a distributed optimal control problem related to a stationary chemotaxis model coupled with the Navier-Stokes equations. Also, they derived an optimality system through a penalty method, because the relation control-to-state is multivalued. In [22] consider a 2D chemotaxis model with logistic source and proved the existence of weak solutions for the dynamical equation, the solvability of the optimal control problem and derive an optimality system using a generic result on the existence of Lagrange multipliers. The works [12–14] are dedicated to study control problems related to chemo-repulsion models and considered the parabolic-parabolic system associated with problem (2)-(4). In [12], the authors explored a bilinear optimal control problem in 2D domains, and proved the existence of global optimal solutions and derived an optimality system. The same authors in [13] studied the 3D version of [12]. Guillén-González et al. [14] extended the results of [13] for superlinear production term; that is, for $p \in (1, 2]$. Recently, in [16] a bilinear optimal control problem related to a stationary version system of (2)-(4) has been studied for n -dimensional domains, with $n = 1, 2, 3$. In this work the authors proved the existence of global optimal solutions and derived first-order necessary optimality conditions for local optimal solutions. As far as we known, optimal control problems related with system (2)-(4) has not been considered in the literature.

The paper is organized as follows: In Section 2 we fix some notations, introduce the function spaces which will be used through the work, give the concept of strong solutions of problem (2)-(4), present a result concerning to the existence and uniqueness of global-in-time strong solutions of (2)-(4), and establish two regularity (parabolic and elliptic) results for heat-Neumann problem that will used to achieve our results. Finally, in Section 3 we analyze the bilinear optimal control problem and obtain the several important results, which includes the existence of global optimal solutions, the derivation of the an optimality system for a local optimal solution, via a result on existence of Lagrange multipliers in Banach spaces, and obtain some extra regularity properties of the Lagrange multipliers.

2. Preliminaries

In this section we establish some notations, definitions and preliminaries results that will be used throughout this work. We will use the classical Lebesgue spaces $L^s := L^s(\Omega)$, for $1 \leq s \leq \infty$, with norm denoted by $\|\cdot\|_{L^s}$. In particular, for $s = 2$, the L^2 -norm and the respective L^2 -inner product will be denoted by $\|\cdot\|$ and (\cdot, \cdot) . Moreover, we use the Sobolev spaces $W^{m,s} := W^{m,s}(\Omega) = \{u \in L^2 : \|\partial^\alpha u\|_{L^s}, \forall |\alpha| \leq m\}$, with norm denoted by $\|\cdot\|_{W^{m,s}}$. When $s = 2$, we denote by $H^m := W^{m,2}$ and the respective norm by $\|\cdot\|_{H^m}$. Also, we will use the space $W_n^{m,s} := \{u \in W^{m,s} : \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$, for $m > 1 + \frac{1}{s}$, with nor denoted by $\|\cdot\|_{W_n^{m,s}}$. Moreover, if X is a Banach space, we will denote by $L^s(X)$ the space of functions $u : [0, T] \rightarrow X$ that are integrable in the Bochner sense, and its norm will be denoted by $\|\cdot\|_{L^s(X)}$. For simplicity, we will denote $L^s(Q) := L^s(L^s)$ and its norm by $\|\cdot\|_{L^s(Q)}$. Also, $C(X)$ denotes the space of continuous functions $u : [0, T] \rightarrow X$, where X is a Banach space, and its respective norm by $\|\cdot\|_{C(X)}$. The topological dual space of a Banach space X will be denoted by X' , and the respective duality for a pair X and X' by $\langle \cdot, \cdot \rangle_{X'}$ or simply by $\langle \cdot, \cdot \rangle$ unless this leads ambiguity. Finally, as usual, the different letters C, K, C_1, K_1, \dots , denote positive constants independent of (u, v) , but its value may change from line to line.

We are interested in studying a bilinear optimal control problem related with the strong solutions of problem (2)-(4). The following definition establishes the concept of strong solutions of system (2)-(4), more details can be consulted in [15].

Definition 1. (Strong solutions) Let $f \in L^q(\omega)$, for $2 < q < \infty$, $u_0 \in H^1$ with $u_0 \geq 0$ a.e. in Ω . A pair (u, v) is called strong solution of system (2)-(4) in $(0, T)$, if $u \geq 0, v \geq 0$ a.e. in Q ,

$$u \in S_u := \{u \in L^\infty(H^1) \cap L^2(H_n^2) : \partial_t u \in L^2(Q)\}, \quad (5)$$

$$v \in S_v := L^q(W_n^{2,q}), \quad (6)$$

the pair (u, v) satisfies pointwisely a.e. $(t, x) \in Q$ the system

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (u \nabla v), \\ -\Delta v + v &= u^p + f v \mathbf{1}_\omega, \end{cases}$$

and the initial and boundary conditions (3) and (4) are satisfied, respectively.

Some properties that can be extracted directly from system (2)-(4) and that are key to obtaining the existence of strong solutions are the following:

- System (2)-(4) is conservative in u . Integrating (2)₁ in the spatial variable we have

$$\frac{d}{dt} \left(\int_{\Omega} u \right) = 0; \text{ hence, } \int_{\Omega} u(t) = \int_{\omega} u_0 := m_0 \quad \forall t > 0. \quad (7)$$

- Integrating (2)₂ in Ω we have

$$\int_{\Omega} v = \int_{\Omega} u^p + \int_{\omega} f v. \quad (8)$$

Now, we present a result related to the existence and uniqueness of strong solutions to problem (2)-(4). This result is valid only when $p \in (1, 2)$. For this reason, we restrict our analysis to $1 < p < 2$ (see [15], for more details).

Theorem 1. (Strong solutions [15, Theorem 2.7]) Assume that $p \in (1, 2)$. Let $u_0 \in H^1$ with $u_0 \geq 0$ in Ω and $f \in L^q(\omega)$ for $2 < q < \infty$. Suppose that there exists a constant $\beta > 0$ such that $\|f\|_{L^q(\omega)}$ is small enough satisfying

$$\|f\|_{L^q(\omega)} < \beta \leq \widehat{K}, \quad (9)$$

where $\widehat{K} := \widehat{K}(|\Omega|, q, p) > 0$ is a constant. Then there exists a unique strong solution (u, v) of system (1)-(2) in sense of Definition 1. Moreover, there exists a positive constant $K := K(m_0, T, \|f\|_{L^q(\omega)}, \widehat{K})$ such that

$$\|u\|_{S_u} + \|v\|_{S_v} \leq K. \quad (10)$$

Remark 1. The constant \widehat{K} , given in Theorem 1, is mainly related to the Sobolev embeddings $H^1 \hookrightarrow L^2$, for $1 \leq s < \infty$, and $W^{2,q} \hookrightarrow L^\infty$ and the continuous injection $L^q \hookrightarrow L^2$.

Throughout this paper, we frequently use the following equivalent norms in the spaces H^1 and H^2 (see, for instance, [23]):

$$\|u\|_{H^1}^2 \simeq \left(\|\nabla u\|^2 + \left(\int_{\Omega} u \right)^2 \right) \quad \forall u \in H^1, \quad (11)$$

$$\|u\|_{H^2}^2 \simeq \left(\|\Delta u\|^2 + \left(\int_{\Omega} u \right)^2 \right) \quad \forall u \in H_{\mathbf{n}}^2, \quad (12)$$

and the classical 2D interpolation inequality

$$\|u\|_{L^4} \leq C \|u\|^{1/2} \|u\|_{H^1}^{1/2} \quad \forall u \in H^1. \quad (13)$$

Moreover, we will apply the following results concerning to parabolic and elliptic regularity for the heat-Neumann problem:

Theorem 2. (Parabolic-regularity [25, Theorem 10.22]) Let $\Omega \in C^2$ be a bounded domain in \mathbb{R}^n , $n = 2, 3$, $u_0 \in \widehat{W}^{2-2/s,s}$ and $g \in L^s(Q)$, for $s \in (1, \infty)$ with $s \neq 3$. Then, there exists a unique strong solution u of problem

$$\begin{cases} \partial_t u - \Delta u &= g \text{ in } Q, \\ u(0, x) &= u_0(x) \text{ in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \text{ on } (0, T) \times \partial\Omega, \end{cases}$$

such that

$$u \in L^\infty(\widehat{W}^{2-2/s,s}) \cap L^s(W^{2,s}), \partial_t u \in L^s(Q).$$

Moreover, there exists a constant $C := C(|\Omega|, T) > 0$ such that

$$\|u\|_{L^\infty(\widehat{W}^{2-2/s,s})} + \|u\|_{L^s(W^{2,s})} + \|\partial_t u\|_{L^s(Q)} \leq C \left(\|u_0\|_{\widehat{W}^{2-2/s,s}} + \|g\|_{L^s(Q)} \right).$$

Here, the space $\widehat{W}^{2-2/s,s} := W^{2-2/s,s}$ for $s < 3$ and $\widehat{W}^{2-2/s,s} := W_n^{2-2/s,s}$ for $s > 3$.

Theorem 3. (Elliptic-regularity [26, Theorem 2.4.2.7]) Let $\Omega \in C^{1,1}$ be a bounded domain in \mathbb{R}^n , $n = 2, 3$, and $h \in L^s$, with $1 < s < \infty$. Then the elliptic system

$$\begin{cases} -\Delta u + u &= h \text{ in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \text{ on } \partial\Omega, \end{cases}$$

admits a unique solution u in the class $W^{2,s}$. Moreover, there exists a positive constant $C := C(|\Omega|)$ such that

$$\|u\|_{W^{2,s}} \leq C \|h\|_{L^s}.$$

3. The Bilinear Optimal Control Problem

This section is dedicated to the study of a bilinear optimal control problem related with the strong solutions of system (2)-(4). Firstly, we establish the statement of the bilinear control problem under analysis. Indeed, we assume that the controls set is \mathcal{F} which is a nonempty, closed and convex subset of $\mathcal{B}(\widehat{K})$, where $\mathcal{B}(\widehat{K}) \subset L^q(\omega)$, for $2 < q < \infty$, is the open ball

$$\mathcal{B}(\widehat{K}) := \{f \in L^q(\omega) : \|f\|_{L^q(\omega)} < \beta \leq \widehat{K}\}, \quad (14)$$

where β and \widehat{K} are the constants given in (9) (see Theorem 1 above) and $\omega \subset \Omega$ is the control domain. We consider the initial data $u_0 \in H^1$ with $u_0 \geq 0$ and the function $f \in \mathcal{F}$ that describes a bilinear control acting on the chemical equation (2)₂.

Furthermore, we consider the Banach spaces

$$\mathbb{X} := S_u \times S_v \times L^q(\omega) \text{ and } \mathbb{Y} := L^2(Q) \times L^q(Q) \times H^1(\Omega),$$

the functional $J : \mathbb{X} \rightarrow \mathbb{R}$ defined by

$$J(u, v, f) := \frac{\alpha_u}{2} \int_0^T \|u - u_d\|^2 + \frac{\alpha_v}{2} \int_0^T \|v - v_d\|^2 + \frac{\alpha_f}{q} \|f\|_{L^q(\omega)}^q \quad (15)$$

and the operator $R := (R_1, R_2, R_3) : \mathbb{X} \rightarrow \mathbb{Y}$, where $R_i : \mathbb{X} \rightarrow \mathbb{Y}$, for $i = 1, 2, 3$, are defined at each point $s := (u, v, f) \in \mathbb{X}$ by

$$\begin{cases} R_1(s) &= \partial_t u - \Delta u - \nabla \cdot (u \nabla v), \\ R_2(s) &= -\Delta v + v - u^p - f v \mathbf{1}_\omega, \\ R_3(s) &= u(0) - u_0. \end{cases} \quad (16)$$

In the functional J , defined in (15), the pair $(u_d, v_d) \in L^2(Q) \times L^2(Q)$ represents the desired states and the nonnegative real numbers α_u, α_v and α_f measure the cost of the states (u, v) and the control f , respectively. These real numbers are nonzero simultaneously. The functional J describes the deviation of the cell density u from a cell density u_d and the deviation of the chemical concentration v from a desired chemical v_d , plus the cost of the control measured in the L^q -norm.

Then, taking $S := S_u \times S_v \times \mathcal{F}$ we formulate the following bilinear optimal control problem:

$$\min_{s \in S} J(s) \text{ subject to } R(s) = \mathbf{0}. \quad (17)$$

notice that $S \subset \mathbb{X}$ is a closed and convex set and that the functional J is weakly lower-semicontinuous on S . The set of the admissible solutions of control problem (17) is given by

$$\mathcal{S}_{ad} := \{s = (u, v, f) \in S : R(s) = \mathbf{0}\},$$

which, by virtue of Theorem 1, is a nonempty set.

We are interested in proving the existence of global optimal solutions to problem (17) and derive the so-called first-order necessary optimality conditions for any local optimal solution of control problem (17). In the following definitions we present the concepts of global optimal solutions and local optimal solutions of problem (17), respectively.

Definition 2. (Global optimal solutions) An element $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ is called a global optimal solution of control problem (17) if

$$J(\tilde{s}) = \min_{s \in \mathcal{S}_{ad}} J(s). \quad (18)$$

Definition 3. (Local optimal solutions) We say that a triplet $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ is a local optimal solution of problem (17), if there exists $\varepsilon > 0$ such that for any $s = (u, v, f) \in \mathcal{S}_{ad}$ satisfying

$$\|\tilde{u} - u\|_{S_u} + \|\tilde{v} - v\|_{S_v} + \|\tilde{f} - f\|_{L^q(\omega)} \leq \varepsilon,$$

one has that $J(\tilde{s}) \leq J(s)$.

3.1. Existence of Optimal Solutions

In this subsection we will prove the existence of at least one global optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ for control problem (17). Specifically, we will prove the following result:

Theorem 4. (Existence of global optimal solutions) Consider the assumptions of Theorem 1. Then, the optimal control problem (17) has at least one global optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$.

Proof. Since $f \in \mathcal{B}(\widehat{K})$ (hence, in particular, $\|f\|_{L^q(\omega)} < \beta \leq \widehat{K}$), from Theorem 1 we deduce that the admissible set \mathcal{S}_{ad} is nonempty. Moreover, considering that the functional J is bounded from below, we deduce that there exists a minimizing sequence $\{s_m\}_{m \geq 1} := \{(u_m, v_m, f_m)\}_{m \geq 1} \subset \mathcal{S}_{ad}$ such that

$$\lim_{m \rightarrow \infty} J(s_m) = \inf_{s \in \mathcal{S}_{ad}} J(s).$$

Now, from the definition of J and that the control set \mathcal{F} is bounded in $L^q(\omega)$, we have that the sequence

$$\{f_m\}_{m \geq 1} \text{ is bounded in } L^q(\omega). \quad (19)$$

On the other hand, by definition of the admissible set \mathcal{S}_{ad} , for each $m \in \mathbb{N}$, the triplet (u_m, v_m, f_m) satisfies system (2)-(4). Thus, from estimate (9) we conclude that there exists a positive constant C , independent of m , such that

$$\|u_m\|_{S_u} + \|v_m\|_{S_v} \leq C. \quad (20)$$

Therefore, from (19)-(20) and the fact that the control set \mathcal{F} is a closed and convex subset of $L^q(\omega)$; then, by Mazur lemma (see [24]), is weakly closed in $L^q(\omega)$, we deduce that there exists a limit element $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in S_u \times S_v \times \mathcal{F}$ and a subsequence of $\{s_m\}_{m \geq 1}$, which, for simplicity, is still denoted by $\{s_m\}_{m \geq 1}$, such that the following convergences hold, as $m \rightarrow \infty$:

$$\begin{cases} u_m \rightarrow \tilde{u} \text{ weakly in } L^2(H_n^2) \text{ and weakly}^* \text{ in } L^\infty(H^1), \\ v_m \rightarrow \tilde{v} \text{ weakly in } L^q(W_n^{2,q}), \\ \partial_t u_m \rightarrow \partial_t \tilde{u} \text{ weakly in } L^2(Q), \\ f_m \rightarrow \tilde{f} \text{ weakly in } L^q(\omega) \text{ with } \tilde{f} \in \mathcal{F}. \end{cases} \quad (21)$$

In particular, following the arguments given in [14], we have that u_m converges strongly to \tilde{u} in $L^4(Q)$, which implies that

$$u_m^p \rightarrow \tilde{u}^p \text{ weakly in } L^2(Q). \quad (22)$$

Furthermore, from (21)₁, (21)₃, the Aubin-Lions lemma (see [27, Theorem 5.1]) and [28, Corollary 4] we have

$$u_m \rightarrow \tilde{u} \text{ strongly in } C(L^2) \cap L^2(H^1). \quad (23)$$

Therefore, considering the convergences (21)-(23) and following a standard argument (see, for instance, [14]), we can pass to the limit in system (2)-(4) writing by (u_m, v_m, f_m) , as m goes to ∞ ; and thus, we deduce that $(\tilde{u}, \tilde{v}, \tilde{f})$ is a solution of (2)-(4). Consequently, the triplet $(\tilde{u}, \tilde{v}, \tilde{f})$ belongs to \mathcal{S}_{ad} and

$$\lim_{m \rightarrow \infty} J(u_m, v_m, f_m) = \inf_{(u,v,f) \in \mathcal{S}_{ad}} J(u, v, f) \leq J(\tilde{u}, \tilde{v}, \tilde{f}). \quad (24)$$

Also, taking into account that the cost functional J is weakly lower semi-continuous on \mathcal{S}_{ad} , we have

$$J(\tilde{u}, \tilde{v}, \tilde{f}) \leq \liminf_{m \rightarrow \infty} J(u_m, v_m, f_m);$$

which together with (24) implies (18). Therefore, the triplet $(\tilde{u}, \tilde{v}, \tilde{f})$ is a global optimal solution of problem (17). \square

3.2. Optimality System

In this subsection we will obtain first-order necessary optimality conditions and derive an optimality system for a local optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ of control problem (17), using a generic result on the existence of Lagrange multipliers in Banach spaces. This result, concerning on the existence of Lagrange multipliers, has been established by Zowe and Kurcyusz in 1979 (see [29]).

The following results related to the differentiability of the functional J and the operator R can be easily deduced.

Lemma 1. *The functional $J : \mathbb{X} \rightarrow \mathbb{R}$ is Fréchet-differentiable and the Fréchet derivative of J at the point $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathbb{X}$ in the direction $r = (U, V, F) \in \mathbb{X}$ is given by*

$$J'(\tilde{s})[r] = \alpha_u \int_0^T \int_\Omega (\tilde{u} - u_d)U + \alpha_v \int_0^T \int_\Omega (\tilde{v} - v_d)V + \alpha_f \int_\omega \text{sgn}(\tilde{f})|\tilde{f}|^{q-1}F. \quad (25)$$

Lemma 2. The operator $R : \mathbb{X} \rightarrow \mathbb{Y}$, defined in (16), is continuously Fréchet-differentiable and its Fréchet derivative at the point $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathbb{X}$, in the direction $r = (U, V, F) \in \mathbb{X}$, is the linear and continuous operator $R'(\tilde{s})[r] := (R'_1(\tilde{s})[r], R'_2(\tilde{s})[r], R'_3(\tilde{s})[r])$ defined by

$$\begin{cases} R'_1(\tilde{s})[r] &= \partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{v}) - \nabla \cdot (\tilde{u} \nabla V), \\ R'_2(\tilde{s})[r] &= -\Delta V + V - p\tilde{u}^{p-1}U - \tilde{f}V 1_\omega - F\tilde{v}, \\ R'_3(\tilde{s})[r] &= U(0). \end{cases} \quad (26)$$

By adapting the abstract sense given in [29], we have the following definition:

Definition 4. An admissible element $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ is a regular point for the optimal control problem (17) if for each triplet $(f_u, f_v, U_0) \in \mathbb{Y}$ there exists $r = (U, V, F) \in S_u \times S_v \times C(\tilde{f})$ such that

$$R'(\tilde{s})[r] = (f_u, f_v, U_0). \quad (27)$$

Here, $C(\tilde{f}) := \{\delta(f - \tilde{f}) : \delta \geq 0, f \in \mathcal{F}\}$ is the conical hull of \tilde{f} in \mathcal{F} .

Our aim is to prove the existence of Lagrange multipliers, which is guaranteed if a local optimal solution of control problem (17) is a regular point. The following result goes in that direction.

Proposition 1. Suppose that the assumptions of Theorem 4 hold. If $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in S_{ad}$, then \tilde{s} is a regular point for the optimal control problem (17).

Proof. Let $\tilde{s} \in S_{ad}$ be a fixed element and $(f_u, f_v, U_0) \in \mathbb{Y}$. Notice that 0 belongs to the conical hull $C(\tilde{f})$; hence it suffices to prove the existence of a pair $(U, V) \in S_u \times S_v$ such that

$$\begin{cases} \partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{v}) - \nabla \cdot (\tilde{u} \nabla V) &= f_u \text{ in } Q, \\ -\Delta V + V - p\tilde{u}^{p-1}U - \tilde{f}V 1_\omega &= f_v \text{ in } Q, \\ U(0) &= U_0 \text{ in } \Omega, \\ \frac{\partial U}{\partial \mathbf{n}} &= \frac{\partial V}{\partial \mathbf{n}} = 0 \text{ on } (0, T) \times \partial\Omega. \end{cases} \quad (28)$$

Now, we define the linear operator $S : (\bar{U}, \bar{V}) \in W_u \times W_v \mapsto (U, V) \in S_u \times S_v \hookrightarrow W_u \times W_v$, where (U, V) is the solution of the problem

$$\begin{cases} \partial_t U - \Delta U &= \nabla \cdot (\tilde{u} \nabla V) + \nabla \cdot (\bar{U} \nabla V) + f_u \text{ in } Q, \\ -\Delta V + V &= p\tilde{u}^{p-1}\bar{U} + \tilde{f}\bar{V} 1_\omega + f_v \text{ in } Q, \end{cases} \quad (29)$$

endowed with the respective initial and boundary conditions (28)₃ and (28)₄. The weakly spaces W_u and W_v are defined as follows:

$$W_u := C(L^2) \cap L^{\frac{2q}{q-2}}(H^1) \text{ and } W_v := L^q(L^\infty), \text{ with } 2 < q < \infty.$$

Following [15] we can prove easily that operator S is well-defined from $W_u \times W_v$ to $S_u \times S_v$ and completely continuous from $W_u \times W_v$ onto itself (see [15, Lemma 3.2]). Also, from [15, Lemma 3.1] we have that the space $S_u \times S_v$ is compactly embedded in $W_u \times W_v$.

On the other hand, we consider the set

$$S_\alpha := \{(U, V) \in S_u \times S_v : (U, V) = \alpha S(U, V) \text{ for some } \alpha \in [0, 1]\}.$$

The set S_α is bounded in $S_u \times S_v$, independently of the parameter $\alpha \in [0, 1]$. Indeed, let $(U, V) \in S_\alpha$ and $\alpha \in (0, 1]$ (the case $\alpha = 0$ is clear). Then, since operator S is well-defined from $W_u \times W_v$ to $S_u \times S_v$, we deduce that $(U, V) \in S_u \times S_v$ and satisfies point-wisely a.e. in Q the following problem:

$$\begin{cases} \partial_t U - \Delta U &= \nabla \cdot (\tilde{u} \nabla V) + \alpha \nabla \cdot (U \nabla \tilde{v}) + \alpha f_u, \\ -\Delta V + V &= \alpha p \tilde{u}^{p-1} U + \alpha \tilde{f} V 1_\omega + \alpha f_v, \end{cases} \quad (30)$$

endowed with corresponding initial and boundary conditions. Then, testing (30)₁ by U and considering that $\nabla \cdot (\tilde{u} \nabla V) = \tilde{u} \Delta V + \nabla \tilde{u} \cdot \nabla V$, we have

$$\frac{1}{2} \frac{d}{dt} \|U\|^2 + \|\nabla U\|^2 \leq |(\tilde{u} \Delta V, U)| + |(\nabla \tilde{u} \cdot \nabla V, U)| + \alpha |(U \nabla \tilde{v}, \nabla U)| + \alpha |(f_u, U)|. \quad (31)$$

Now, we will bound the terms on the right-hand side of (31). Applying the Hölder and young inequality and taking into account the 2D interpolation inequality (13) and that $\alpha \leq 1$, we can obtain

$$\begin{aligned} |(\tilde{u} \Delta V, U)| &\leq \|\tilde{u}\|_{L^4} \|\Delta V\| \|U\|_{L^4} \leq \varepsilon \|\Delta V\|^2 + C \|\tilde{u}\|_{L^4}^2 \|U\| \|U\|_{H^1} \\ &\leq \varepsilon (\|\Delta V\|^2 + \|U\|_{H^1}^2) + C \|\tilde{u}\|_{L^4}^4 \|U\|^2, \end{aligned} \quad (32)$$

$$\begin{aligned} |(\nabla \tilde{u} \cdot \nabla V, U)| &\leq \|\nabla V\| \|\nabla \tilde{u}\|_{L^4} \|U\|_{L^4} \\ &\leq \varepsilon (\|\nabla V\|^2 + \|U\|_{H^1}^2) + C \|\nabla \tilde{u}\|_{L^4}^4 \|U\|^2, \end{aligned} \quad (33)$$

$$\begin{aligned} \alpha |(U \nabla \tilde{v}, \nabla U)| &\leq \|U\| \|\nabla \tilde{v}\|_{L^\infty} \|\nabla U\| \\ &\leq \varepsilon \|\nabla U\|^2 + C \|\nabla \tilde{v}\|_{L^\infty}^2 \|U\|^2, \end{aligned} \quad (34)$$

$$\alpha |(f_u, U)| \leq \|f_u\| \|U\| \leq \|f_u\|^2 + \|U\|^2, \quad (35)$$

where $\varepsilon > 0$ is arbitrary. Then, replacing (32)-(35) in (31) and adding to both sides $\|U\|^2$ in order to complete the H^1 -norm, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U\|^2 + \|U\|_{H^1}^2 &\leq \varepsilon (\|\nabla U\|^2 + \|U\|_{H^1}^2 + \|\nabla V\|^2 + \|\Delta V\|^2) \\ &\quad + C (\|\tilde{u}\|_{L^4}^4 + \|\nabla \tilde{u}\|_{L^4}^4 + \|\nabla \tilde{v}\|_{L^\infty}^2 + 1) \|U\|^2 + \|f_u\|^2. \end{aligned} \quad (36)$$

Also, testing equation (30)₂ by $(V - \Delta V)$ we have

$$\begin{aligned} &\|V\|_{H^1}^2 + \|\Delta V\|^2 + \|\nabla V\|^2 \\ &\leq \alpha p |(\tilde{u}^{p-1} U, V)| + \alpha |(\tilde{f} V 1_\omega, V)| + \alpha |(f_v, V)| \\ &\quad + \alpha p |(\tilde{u}^{p-1} U, \Delta V)| + \alpha |(\tilde{f} V 1_\omega, \Delta V)| + \alpha |(f_v, \Delta V)|. \end{aligned} \quad (37)$$

Thus, working in a similar way as we did to obtain the estimate (36), we arrive at

$$\begin{aligned} \frac{1}{2} (\|V\|_{H^1}^2 + \|V\|_{H^2}^2) &\leq \varepsilon (\|U\|_{H^1}^2 + \|V\|^2 + \|\Delta V\|^2) + \frac{1}{4} (\|V\|^2 + \|\Delta V\|^2) \\ &\quad + C \|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} \|U\|^2 + 2K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2 \|V\|_{H^1}^2 + C \|f_v\|_{L^q}^2. \end{aligned} \quad (38)$$

Here, the positive constant $K_1 := K_1(|\Omega|)$ is given by the Sobolev embedding $H^1 \hookrightarrow L^s$, for $s \in (2, \infty)$. This injection is necessary to estimate the terms $\alpha |(\tilde{f} V 1_\omega, V)|$ and $\alpha |(\tilde{f} V 1_\omega, \Delta V)|$. Indeed, from the Hölder inequality we have the estimate $\alpha |(\tilde{f} V 1_\omega, V)| \leq \|\tilde{f}\|_{L^q} \|V\|_{L^s} \|V\|$, with $\frac{1}{q} + \frac{1}{s} = \frac{1}{2}$. Thus, using that $H^1 \hookrightarrow L^s$, we have that there exists a constant $K_1 > 0$ such that $\|V\|_{L^s} \leq K_1 \|V\|_{H^1}$; consequently, we deduce that $\alpha |(\tilde{f} V 1_\omega, V)| \leq K_1 \|\tilde{f}\|_{L^q} \|V\|_{H^1} \|V\|$. Similarly, we can obtain that $\alpha |(\tilde{f} V 1_\omega, \Delta V)| \leq K_1 \|\tilde{f}\|_{L^q} \|V\|_{H^1} \|\Delta V\|$.

Now, adding inequalities (36) and (38), and choosing $\varepsilon > 0$ suitably we can obtain the following estimate

$$\begin{aligned} \frac{d}{dt} \|U\|^2 + C \|V\|_{H^1}^2 + \frac{1}{2} \|V\|_{H^2}^2 &\leq C \left(\|\tilde{u}\|_{L^4}^4 + \|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + \|\nabla \tilde{u}\|_{L^4}^4 + \|\nabla \tilde{v}\|_{L^\infty}^2 + 1 \right) \|U\|^2 \\ &\quad + 4K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2 \|V\|_{H^1}^2 + C \left(\|f_u\|^2 + \|f_v\|_{L^q}^2 \right) \\ &\leq C \left(\|\tilde{u}\|_{L^4}^4 + \|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + \|\nabla \tilde{u}\|_{L^4}^4 + \|\nabla \tilde{v}\|_{L^\infty}^2 + 1 \right) \|U\|^2 \\ &\quad + 4K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2 \|V\|_{H^2}^2 + C \left(\|f_u\|^2 + \|f_v\|_{L^q}^2 \right), \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dt} \|U\|^2 + C \|V\|_{H^1}^2 + \left(\frac{1 - 8K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2}{2} \right) \|V\|_{H^2}^2 \\ \leq C \left(\|\tilde{u}\|_{L^4}^4 + \|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + \|\nabla \tilde{u}\|_{L^4}^4 + \|\nabla \tilde{v}\|_{L^\infty}^2 + 1 \right) \|U\|^2 \\ + C \left(\|f_u\|^2 + \|f_v\|_{L^q}^2 \right). \end{aligned} \quad (39)$$

From assumption (9) given in Theorem 1 we deduce that $\|\tilde{f}\|_{L^q(\omega)} < \frac{1}{2\sqrt{2}K_1}$; thus, we conclude that $1 - 8K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2 > 0$. Hence, from (39) and Gronwall lemma we have that U is bounded in $L^\infty(L^2) \cap L^2(H^1)$. Moreover, integrating (39) in $(0, T)$ we obtain that $V \in L^2(H^2)$.

It remains to prove that the pair (U, V) is bounded in $S_u \times S_v$. Indeed, notice that due to $\tilde{u} \in L^\infty(H^1) \cap L^2(H_n^2)$, from Sobolev embeddings we deduce that $\tilde{u} \in L^s(Q)$ for any $s \in [1, \infty)$. Also, since $p \in (1, 2)$ we have $\tilde{u}^{p-1} \in L^s(Q)$, and since $U \in L^2(L^2) \cap L^2(H^1) \hookrightarrow L^4(Q)$, we obtain that $\alpha p \tilde{u}^{p-1} U$ belongs to $L^q(Q)$ for any $q \in (2, \infty)$. Moreover, using that $V \in L^2(H_n^2)$ (in particular, from Sobolev embeddings, $V \in L^2(L^\infty)$), $(\tilde{f}, f_v) \in L^q(\omega) \times L^q(\Omega)$, we deduce that

$$\alpha p \tilde{u}^{p-1}(t, \cdot) U(t, \cdot) + \alpha \tilde{f} V(t, \cdot) 1_\omega + \alpha f_v \in L^q \text{ for any } t \in (0, T).$$

Therefore, applying Theorem 3 (for $s = q > 2$) we conclude that $V(t, \cdot) \in W_n^{2,q}$ for any time $t \in (0, T)$, satisfies the elliptic problem

$$\begin{cases} -\Delta V + V = \alpha p \tilde{u}^{p-1} U + \alpha \tilde{f} V 1_\omega + \alpha f_v & \text{in } \Omega, \\ \frac{\partial V}{\partial \mathbf{n}} = 0 & \text{on } \partial \Omega, \end{cases}$$

and the following estimate holds

$$\begin{aligned} \|V\|_{W^{2,q}} &\leq \alpha C \left(p \|\tilde{u}^{p-1} U\|_{L^q} + \|\tilde{f}\|_{L^q(\omega)} \|V\|_{L^\infty} + \|f_v\|_{L^q} \right) \\ &\leq C \left(p \|\tilde{u}^{p-1} U\|_{L^q} + \|f_v\|_{L^q} \right) + K_2 \|\tilde{f}\|_{L^q(\omega)} \|V\|_{W_n^{2,q}}, \end{aligned}$$

where $K_2 := K_2(|\Omega|) > 0$ is a constant given by the Sobolev injection $W_n^{2,q} \hookrightarrow L^\infty$. Thus, we have

$$\left(1 - K_2 \|\tilde{f}\|_{L^q(\omega)} \right) \|V\|_{W_n^{2,q}} \leq C \left(p \|\tilde{u}^{p-1} U\|_{L^q} + \|f_v\|_{L^q} \right). \quad (40)$$

Moreover, from Theorem 1 we deduce that $K_2 \|\tilde{f}\|_{L^q(\omega)} < 1$; hence, from (40) we conclude that

$$\|V\|_{W_n^{2,q}}^q \leq \frac{C}{\left(1 - K_2 \|\tilde{f}\|_{L^q(\omega)} \right)^q} \left(p \|\tilde{u}^{p-1} U\|_{L^q} + \|f_v\|_{L^q} \right)^q. \quad (41)$$

Since $\tilde{u}^{p-1}U \in L^q(Q)$, after integrating (41) in time, we deduce $V \in L^q(W_{\mathbf{n}}^{2,q}) = S_v$.

On the other hand, testing (28)₁ by $-\Delta U$ we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla U\| + \|\Delta U\|^2 &\leq |(\tilde{u}\Delta V, \Delta U)| + |(\nabla \tilde{u} \cdot \nabla V, \Delta U)| + \alpha |(U\Delta \tilde{v}, \Delta U)| \\ &\quad + \alpha |(\nabla U \cdot \nabla \tilde{v}, \Delta U)| + \alpha |(f_u, \Delta U)|, \end{aligned}$$

and working in a similar way as we did to obtain the estimate (36), we can obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla U\| + \|\Delta U\|^2 &\leq \varepsilon \|\Delta U\|^2 + C \left(\|\nabla \tilde{v}\|_{L^\infty}^2 + \|\Delta \tilde{v}\|_{L^q}^2 \right) \|U\|_{H^1}^2 \\ &\quad + C \|\tilde{u}\|_{L^s}^2 \|\Delta V\|_{L^q}^2 + C \|\nabla \tilde{u}\|^2 \|\nabla V\|_{L^\infty}^2 + C \|f_u\|^2, \end{aligned} \quad (42)$$

where $\varepsilon > 0$ is arbitrary and $s \in (2, \infty)$ is chosen in such a way that $\frac{1}{s} + \frac{1}{q} = \frac{1}{2}$. Also, integrating the U -equation (28)₁ in the spatial variable we obtain

$$\frac{d}{dt} \int_{\Omega} U = \alpha \int_{\Omega} f_u,$$

which implies the following inequalities

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} U \right)^2 \leq C \left(\int_{\Omega} f_u \right)^2 + C \left(\int_{\Omega} U \right)^2 \leq C |\Omega| \left(\|f_u\|^2 + \|U\|^2 \right), \quad (43)$$

$$\left| \int_{\Omega} U(t) \right|^2 = \left| \int_{\Omega} U_0 + \alpha \int_0^t \int_{\Omega} f_u \right|^2 \leq C. \quad (44)$$

Adding estimates (42)-(44) and taking into account the equivalent norms given in (11)-(12) we deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U\|_{H^1}^2 + \|U\|_{H^2}^2 &\leq \varepsilon \|\Delta U\|^2 + C \left(\|\nabla \tilde{v}\|_{L^\infty}^2 + \|\Delta \tilde{v}\|_{L^q}^2 + |\Omega| \right) \|U\|_{H^1}^2 \\ &\quad + C \|\tilde{u}\|_{L^s}^2 \|\Delta V\|_{L^q}^2 + C \|\nabla \tilde{u}\|^2 \|\nabla V\|_{L^\infty}^2 + C \|f_u\|^2 + C; \end{aligned}$$

which, for $\varepsilon > 0$ suitably, implies

$$\begin{aligned} \frac{d}{dt} \|U\|_{H^1}^2 + C \|U\|_{H^2}^2 &\leq C \left(\|\nabla \tilde{v}\|_{L^\infty}^2 + \|\Delta \tilde{v}\|_{L^q}^2 + |\Omega| \right) \|U\|_{H^1}^2 \\ &\quad + \|\tilde{u}\|_{L^s}^2 \|\Delta V\|_{L^q}^2 + C \|\nabla \tilde{u}\|^2 \|\nabla V\|_{L^\infty}^2 + C \|f_u\|^2 + C. \end{aligned} \quad (45)$$

Then, from (45), Gronwall lemma and taking into account that $\tilde{u} \in L^\infty(H^1) \cap L^2(H_{\mathbf{n}}^2)$ (hence $\tilde{u} \in L^\infty(L^s)$, for any $s \in (2, \infty)$, and $\nabla \tilde{u} \in L^\infty(L^2)$), we conclude that $U \in L^\infty(H^1) \cap L^2(H_{\mathbf{n}}^2)$; thus, using that $\Delta U \in L^2(Q)$, $\tilde{u}, U \in L^\infty(H^1) \cap L^2(H_{\mathbf{n}}^2)$ and $\tilde{v}, V \in L^q(W_{\mathbf{n}}^{2,q})$, we deduce that $\partial_t U = \Delta U + \nabla \cdot (\tilde{u}\nabla V) + \alpha \nabla \cdot (U\nabla \tilde{v}) + \alpha f_u \in L^2(Q)$. Then,

$$\begin{aligned} \|\partial_t U\|_{L^2(Q)} &\leq \|\Delta U + \nabla \cdot (\tilde{u}\nabla V) + \alpha \nabla \cdot (U\nabla \tilde{v}) + \alpha f_u\|_{L^2(Q)} \\ &\leq \|\Delta U\|_{L^2(Q)} + \|\nabla \cdot (\tilde{u}\nabla V)\|_{L^2(Q)} + \|\nabla \cdot (U\nabla \tilde{v})\|_{L^2(Q)} + \|f_u\|_{L^2(Q)} \\ &\leq C. \end{aligned}$$

Therefore, $U \in S_u$.

Consequently, we deduce that the operator S and the set S_α satisfy the conditions of the Leray-Schauder fixed-point theorem. Thus, there exists a pair $(U, V) \in S_u \times S_v$ such that $S(U, V) = (U, V)$, which is a solution of system (28). Moreover, using a classical comparison argument we can deduce the uniqueness of solution of problem (28). \square

We are now in a position to prove the existence of Lagrange multipliers for optimal control problem (17).

Theorem 5. *Suppose that assumptions of Theorem 4 hold. Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution of the control problem (17). Then, there exists a triplet of the Lagrange multipliers $(\varphi, \psi, \chi) \in L^2(Q) \times (L^q(Q))' \times (H^1(\Omega))'$ such that for all $(U, V, F) \in S_u \times S_v \times \mathcal{C}(\tilde{f})$ one has*

$$\begin{aligned} & \alpha_u \int_0^T \int_{\Omega} (\tilde{u} - u_d)U + \alpha_v \int_0^T \int_{\Omega} (\tilde{v} - v_d)V + \alpha_f \int_{\omega} \operatorname{sgn}(\tilde{f})|\tilde{f}|^{q-1}F \\ & \quad - \int_0^T \int_{\Omega} (\partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{v}) - \nabla \cdot (\tilde{u} \nabla V)) \varphi \\ & - \int_0^T \int_{\Omega} (-\Delta V + V - p\tilde{u}^{p-1}U - \tilde{f}V) \psi - \int_{\Omega} U(0)\chi + \int_0^T \int_{\omega} F\tilde{v}\varphi \geq 0. \end{aligned} \quad (46)$$

Proof. From Proposition 1 we have that $\tilde{s} \in \mathcal{S}_{ad}$ is a regular point. Then, from [29, Theorem 3.1] we deduce that there exist Lagrange multipliers $(\varphi, \psi, \chi) \in L^2(Q) \times (L^q(Q))' \times (H^1(\Omega))'$ such that the following variational inequality holds

$$J'(\tilde{s})[r] - \langle R'_1(\tilde{s})[r], \varphi \rangle - \langle R'_2(\tilde{s})[r], \psi \rangle_{(L^q)'} - \langle R'_3(\tilde{s})[r], \chi \rangle_{(H^1)'} \geq 0 \quad (47)$$

for all $r = (U, V, F) \in S_u \times S_v \times \mathcal{C}(\tilde{f})$. Therefore, inequality (46) follows from (25), (26) and (47). \square

From Theorem 5 we can derive an optimality system for the control problem (17), for this purpose we will consider the following linear subspace of S_u :

$$\widehat{S}_u := \{u \in S_u : u(0) = 0\}. \quad (48)$$

The choice of the space \widehat{S}_u permit us to focus our analysis on the Lagrange multipliers φ and ψ .

Corollary 1. *Under assumptions of Theorem 4. Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution of control problem (17). Then, the Lagrange multipliers $(\varphi, \psi) \in L^2(Q) \times (L^q(Q))'$, provided by Theorem 5, satisfy the following variational formulation*

$$\begin{aligned} & \int_0^T \int_{\Omega} (\partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{v})) \varphi - p \int_0^T \int_{\Omega} \tilde{u}^{p-1}U\psi \\ & = \alpha_u \int_0^T \int_{\Omega} (\tilde{u} - u_d)U \quad \forall U \in \widehat{S}_u, \end{aligned} \quad (49)$$

$$\begin{aligned} & \int_0^T \int_{\Omega} (-\Delta V + V) \psi - \int_0^T \int_{\omega} \tilde{f}V\psi - \int_0^T \int_{\Omega} \nabla \cdot (\tilde{u} \nabla V) \varphi \\ & = \alpha_v \int_0^T \int_{\Omega} (\tilde{v} - v_d)V \quad \forall V \in S_v \end{aligned} \quad (50)$$

and the optimality condition

$$\int_{\omega} \left(\alpha_f \operatorname{sgn}(\tilde{f})|\tilde{f}|^{q-1} + \int_0^T \tilde{v}\varphi \right) (f - \tilde{f}) \geq 0 \quad \forall f \in \mathcal{F}. \quad (51)$$

Proof. Notice that $\widehat{S}_u \times S_v$ is a vector space. Hence, (49) can be obtained by taking $(V, F) = (0, 0)$ into (46). Similarly, taking $(U, F) = (0, 0)$ in (46) we deduce (50). Finally, taking $(U, V) = (0, 0)$ in (46) we obtain

$$\alpha_f \int_{\omega} \operatorname{sgn}(\tilde{f})|\tilde{f}|^{q-1}F + \int_0^T \int_{\omega} F\tilde{v}\varphi \geq 0 \quad \forall F \in \mathcal{C}(\tilde{f}),$$

which implies

$$\int_{\omega} \left(\alpha_f \operatorname{sgn}(\tilde{f}) |\tilde{f}|^{q-1} + \int_0^T \tilde{v} \varphi \right) F \geq 0 \quad \forall F \in \mathcal{C}(\tilde{f}). \quad (52)$$

Therefore, choosing $F = (f - \tilde{f}) \in \mathcal{C}(\tilde{f})$ in (52) we deduce inequality (51). \square

Finally, we will derive an optimality system for a local optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ of control problem (17). Firstly, we must improve the regularity of the Lagrange multipliers obtained in Theorem 5. The following result goes in that direction.

Theorem 6. *Suppose that assumptions of Theorem 4 hold. Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution of control problem (17). If $\|\tilde{u}\|_{L^{4(p-1)}}$ and $\|\tilde{f}\|_{L^q(\omega)}$ are small enough such that*

$$\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + \|\tilde{f}\|_{L^q(\omega)}^2 < \frac{1}{\min\{C, K_1^2, \widehat{K}_2^2\}}, \quad (53)$$

where C , K_1 and \widehat{K}_2 are positive constants which depend on $|\Omega|$. Then the Lagrange multipliers $(\varphi, \psi) \in L^2(Q) \times (L^q(Q))'$, provided by Theorem 5, have the following strong regularity:

$$\begin{cases} \varphi \in S_{\varphi} & := \{ \varphi \in L^{\infty}(H^1) \cap L^2(H_{\mathbf{n}}^2) : \partial_t \varphi \in L^2(Q) \}, \\ \psi \in S_{\psi} & := L^r(W^{2,r}), \text{ for any } r \in (1, 2). \end{cases} \quad (54)$$

Proof. Notice that the pair of functions $(\varphi, \psi) \in L^2(Q) \times (L^q(Q))'$, obtained in Theorem 5, corresponds with the concept of very weak solution of the following adjoint system

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + \nabla \varphi \cdot \nabla \tilde{v} - p \tilde{u}^{p-1} \psi & = \alpha_u (\tilde{u} - u_d) \text{ in } Q, \\ -\Delta \psi - \nabla \cdot (\tilde{u} \nabla \varphi) + \psi - \tilde{f} \psi 1_{\omega} & = \alpha_v (\tilde{v} - v_d) \text{ in } Q, \\ \varphi(T) & = 0 \text{ in } \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} & = \frac{\partial \psi}{\partial \mathbf{n}} = 0 \text{ on } (0, T) \times \partial \Omega. \end{cases} \quad (55)$$

Thus, first we will analyze the regularity of the solutions of problem (55) and then we will improve the regularity of the pair of the Lagrange multipliers (φ, ψ) . Indeed, let $\tau := T - t$, with $t \in (0, T)$, and $\widehat{\varphi}(\tau) = \varphi(t)$. Then, system (55) is equivalent to the following forward problem

$$\begin{cases} \partial_{\tau} \widehat{\varphi} - \Delta \widehat{\varphi} + \nabla \widehat{\varphi} \cdot \nabla \tilde{v} - p \tilde{u}^{p-1} \psi & = \alpha_u (\tilde{u} - u_d) \text{ in } Q, \\ -\Delta \psi - \nabla \cdot (\tilde{u} \nabla \widehat{\varphi}) + \psi - \tilde{f} \psi 1_{\omega} & = \alpha_v (\tilde{v} - v_d) \text{ in } Q, \\ \widehat{\varphi}(0) & = 0 \text{ in } \Omega, \\ \frac{\partial \widehat{\varphi}}{\partial \mathbf{n}} & = \frac{\partial \psi}{\partial \mathbf{n}} = 0 \text{ on } (0, T) \times \partial \Omega. \end{cases} \quad (56)$$

Since system (56) is a linear problem, we argue in a formal sense, proving that any regular enough solution is bounded in the space $S_{\varphi} \times S_{\psi}$ (a rigorous proof can be performed using the Leray-Schauder fixed point theorem, similar to what was done for the proof of Proposition 1).

Testing (56)₁ by $\widehat{\varphi} - \Delta \widehat{\varphi}$ and applying the Hölder, and Young inequalities and taking into account the 2D interpolation inequality (13) for the L^4 -norm, we can obtain the following estimate

$$\begin{aligned} & \frac{1}{2} \frac{d}{d\tau} \|\widehat{\varphi}\|_{H^1}^2 + \|\nabla \widehat{\varphi}\|^2 + \|\Delta \widehat{\varphi}\|^2 \\ & \leq \varepsilon \left(\|\Delta \widehat{\varphi}\|^2 + \|\nabla \widehat{\varphi}\|^2 + \|\psi\|_{H^1}^2 \right) + C \left(\|\tilde{u}^{p-1}\|_{L^4}^2 + \|\nabla \tilde{v}\|_{L^{\infty}}^2 + 1 \right) \|\widehat{\varphi}\|_{H^1}^2 \\ & \quad + C \|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} \|\psi\|_{H^1}^2 + C \|\tilde{u} - u_d\|^2, \end{aligned} \quad (57)$$

where $\varepsilon > 0$ is arbitrary. Similarly, testing the ψ -equation (56) by ψ we can arrive at

$$\begin{aligned} \|\psi\|_{H^1}^2 &\leq \varepsilon \left(\|\psi\|^2 + \|\nabla\psi\|^2 + \|\widehat{\varphi}\|_{H^2}^2 \right) + C\|\tilde{u}\|_{L^4}^4 \|\nabla\widehat{\varphi}\|^2 \\ &\quad + K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2 \|\psi\|_{H^1}^2 + C\|\tilde{v} - v_d\|^2, \end{aligned} \quad (58)$$

where $\varepsilon > 0$ is arbitrary and the constant $K_1 := K_1(|\Omega|) > 0$ is given by the Sobolev embedding $H^1 \hookrightarrow L^s$ for $s \in (2, \infty)$.

Now, summing estimates (57) and (58) and then adding $\|\widehat{\varphi}\|^2$ on both sides of the resulting inequality, with the aim of completing the H^2 -norm, it is possible to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{d\tau} \|\widehat{\varphi}\|_{H^1}^2 + \|\widehat{\varphi}\|_{H^2}^2 + \|\psi\|_{H^1}^2 \\ &\leq \varepsilon \left(\|\Delta\widehat{\varphi}\|^2 + \|\nabla\widehat{\varphi}\|^2 + \|\widehat{\varphi}\|_{H^2}^2 + \|\psi\|^2 + \|\nabla\psi\|^2 + \|\psi\|_{H^1}^2 \right) \\ &\quad + C \left(\|\tilde{u}^{p-1}\|_{L^4}^2 + \|\tilde{u}\|_{L^4}^4 + \|\nabla\tilde{v}\|_{L^\infty}^2 + 1 \right) \|\widehat{\varphi}\|_{H^1}^2 \\ &\quad + \left(C\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2 \right) \|\psi\|_{H^1}^2 + C \left(\|\tilde{u} - u_d\|^2 + \|\tilde{v} - v_d\|^2 \right). \end{aligned}$$

Thus, choosing $\varepsilon > 0$ suitably in the last inequality, we can obtain

$$\begin{aligned} &\frac{d}{d\tau} \|\widehat{\varphi}\|_{H^1}^2 + C\|\widehat{\varphi}\|_{H^2}^2 + \left(1 - \left(C\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2 \right) \right) \|\psi\|_{H^1}^2 \\ &\leq C \left(\|\tilde{u}^{p-1}\|_{L^4}^2 + \|\tilde{u}\|_{L^4}^4 + \|\nabla\tilde{v}\|_{L^\infty}^2 + 1 \right) \|\widehat{\varphi}\|_{H^1}^2 + C \left(\|\tilde{u} - u_d\|^2 + \|\tilde{v} - v_d\|^2 \right). \end{aligned} \quad (59)$$

Notice that assumption (53) implies that $1 - \left(C\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + K_1^2 \|\tilde{f}\|_{L^q(\omega)}^2 \right) > 0$. Hence, from (59), Gronwall lemma and taking into account that the terms $\|\tilde{u}^{p-1}\|_{L^4}^2$, $\|\tilde{u}\|_{L^4}^4$, $\|\nabla\tilde{v}\|_{L^\infty}^2$, $\|\tilde{u} - u_d\|^2$, $\|\tilde{v} - v_d\|^2$ are integrable in time, we deduce that $\widehat{\varphi} \in L^\infty(H^1) \cap L^2(H_n^2)$. Similarly, integrating in time (59), for $\tau \in (0, T)$, we have that $\psi \in L^2(H^1)$.

Now, using that $(\tilde{u}, \tilde{v}) \in \widehat{S}_u \times S_v$ (in particular $\tilde{u}^{p-1} \in L^\infty(L^s)$, for any $s \in (1, \infty)$, and $\nabla\tilde{v} \in L^q(L^\infty)$) and that $\nabla\widehat{\varphi} \in L^\infty(L^2)$, we deduce that

$$p\tilde{u}^{p-1}\psi + \alpha_u(\tilde{u} - u_d) - \nabla\widehat{\varphi} \cdot \nabla\tilde{v} \in L^2(Q).$$

Hence, applying Theorem 2 to (56)₁ (for $s = 2$), we deduce that $\widehat{\varphi} \in S_\varphi$, which implies that $\varphi \in S_\varphi$. Moreover, using that $\tilde{u} \in L^s(Q)$, for any $s \in (1, \infty)$, and $\Delta\widehat{\varphi} \in L^2(Q)$, we deduce that $\nabla \cdot (\tilde{u}\nabla\widehat{\varphi}) + \tilde{f}\psi 1_\omega + \alpha_v(\tilde{v} - v_d) \in L^r(Q)$, for any $r \in (1, 2)$. Thus,

$$\nabla \cdot (\tilde{u}(\tau, \cdot)\nabla\widehat{\varphi}(\tau, \cdot)) + \tilde{f}\psi(\tau, \cdot) 1_\omega + \alpha_v(\tilde{v}(\tau, \cdot) - v_d(\tau, \cdot)) \in L^r(\Omega),$$

for any $r \in (1, 2)$ and any time $\tau \in (0, T)$. Then, applying elliptic regularity to (56), for $s = 2$, (see Theorem 3) we conclude that $\psi \in W^{2,r}$ and satisfies the estimate

$$\begin{aligned} \|\psi\|_{W^{2,r}} &\leq C \left(\|\nabla \cdot (\tilde{u}\nabla\widehat{\varphi})\|_{L^r} + \|\tilde{v} - v_d\|_{L^r} + \|\tilde{f}\|_{L^q(\omega)} \|\psi\|_{L^\infty} \right) \\ &\leq C \left(\|\nabla \cdot (\tilde{u}\nabla\widehat{\varphi})\|_{L^r} + \|\tilde{v} - v_d\| \right) + \widehat{K}_2 \|\tilde{f}\|_{L^q(\omega)} \|\psi\|_{W^{2,r}}, \end{aligned}$$

where $\widehat{K}_2 := \widehat{K}_2(|\Omega|)$ is a constant given by the embedding $W^{2,r} \hookrightarrow L^\infty$. Thus, we obtain

$$\left(1 - \widehat{K}_2 \|\tilde{f}\|_{L^q(\omega)} \right) \leq C \left(\|\nabla \cdot (\tilde{u}\nabla\widehat{\varphi})\|_{L^r} + \|\tilde{v} - v_d\| \right). \quad (60)$$

From assumption (53) we deduce that $\widehat{K}_2 \|\tilde{f}\|_{L^q(\omega)} < 1$; then, from (60) we have

$$\|\psi\|_{W^{2,r}}^r \leq \frac{C}{(1 - \widehat{K}_2 \|\tilde{f}\|_{L^q(\omega)})^r} (\|\nabla \cdot (\tilde{u} \nabla \widehat{\varphi})\|_{L^r} + \|\tilde{\sigma} - v_d\|)^r. \quad (61)$$

Therefore, integrating (61) in time we conclude that $\psi \in L^r(W^{2,r}) = S_\psi$ for any $r \in (1, 2)$. Moreover, following and classical comparison argument we can deduce the uniqueness of the pair (φ, ψ) solving the adjoint system (55).

It remains to prove that the pair of Lagrange multipliers, provided by Theorem 5, have the strong regularity (54). Indeed, let $(\tilde{\varphi}, \tilde{\psi}) \in S_\varphi \times S_\psi$ the unique solution of adjoint problem (55) and $(U, V) \in S_u \times S_v$ the unique solution of the linear problem (28) with data $f_u := (\varphi - \tilde{\varphi}) \in L^2(Q)$ and $f_v := \text{sgn}(\psi - \tilde{\psi})|\psi - \tilde{\psi}|^{\frac{1}{q-1}} \in L^q(Q)$. We recall that $\psi \in (L^q(Q))'$; that is, $\psi \in L^{\frac{q}{q-1}}$, thus $f_v = \text{sgn}(\psi - \tilde{\psi})|\psi - \tilde{\psi}|^{\frac{1}{q-1}} \in L^q(Q)$ make sense. Then, it suffices to identify (φ, ψ) with $(\tilde{\varphi}, \tilde{\psi})$ in order to prove that (φ, ψ) satisfies the regularity (54). Now, writing (55) for $(\tilde{\varphi}, \tilde{\psi})$ instead of (φ, ψ) , then testing the first equation by U and the second by V , after integrating by parts, we can obtain

$$\int_0^T \int_\Omega (\partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{\sigma})) \tilde{\varphi} - p \int_0^T \int_\Omega \tilde{u}^{p-1} U \tilde{\psi} = \alpha_u \int_0^T \int_\Omega (\tilde{u} - u_d) U, \quad (62)$$

$$\int_0^T \int_\Omega (-\Delta V + V) \tilde{\psi} - \int_0^T \int_\omega \tilde{f} V \tilde{\psi} - \int_0^T \int_\Omega \nabla \cdot (\tilde{u} \nabla V) = \alpha_v \int_0^T \int_\Omega (\tilde{\sigma} - v_d) V. \quad (63)$$

Taking the difference between (49) and (62) and between (50) and (63), and summing the respective equalities, we deduce

$$\begin{aligned} & \int_0^T \int_\Omega (\partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{\sigma}) - \nabla \cdot (\tilde{u} \nabla V)) (\varphi - \tilde{\varphi}) \\ & + \int_0^T \int_\Omega (-\Delta V + V - p \tilde{u}^{p-1} U) (\psi - \tilde{\psi}) - \int_0^T \int_\omega \tilde{f} V (\psi - \tilde{\psi}) = 0. \end{aligned} \quad (64)$$

Thus, considering that the pair (U, V) is the unique solution of system (28) for $(\varphi - \tilde{\varphi}) \in L^2(Q)$ and $\text{sgn}(\psi - \tilde{\psi})|\psi - \tilde{\psi}|^{\frac{1}{q-1}} \in L^q(Q)$, from (64) we conclude that

$$\|\varphi - \tilde{\varphi}\|_{L^2(Q)}^2 + \|\psi - \tilde{\psi}\|_{L^{\frac{q}{q-1}}}^{\frac{q}{q-1}} = 0.$$

Therefore, $(\varphi, \psi) = (\tilde{\varphi}, \tilde{\psi}) \in L^2(\Omega) \times (L^q(Q))'$. Consequently, the Lagrange multiplier (φ, ψ) , provided by Theorem 5, have the strong regularity (54). \square

Theorem 6 allow us derive an optimality system for the control problem (17).

Corollary 2. Under conditions of Theorem 6. Let $\tilde{s} = (\tilde{u}, \tilde{\sigma}, \tilde{f}) \in S_{ad}$ be a local optimal solution of optimal control problem (17). Then the pair of Lagrange multipliers $(\varphi, \psi) \in S_\varphi \times S_\psi$ satisfies the following optimality system

$$\left\{ \begin{array}{l} -\partial_t \varphi - \Delta \varphi + \nabla \varphi \cdot \nabla \tilde{\sigma} - p \tilde{u}^{p-1} \psi = \alpha_u (\tilde{u} - u_d) \text{ a.e. } (t, x) \in Q, \\ -\Delta \psi - \nabla \cdot (\tilde{u} \nabla \varphi) + \psi - \tilde{f} \psi 1_\omega = \alpha_v (\tilde{\sigma} - v_d) \text{ a.e. } (t, x) \in Q, \\ \varphi(T) = 0 \text{ in } \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} = \frac{\partial \psi}{\partial \mathbf{n}} = 0 \text{ on } (0, T) \times \partial \Omega, \\ \int_\omega \left(\alpha_f \text{sgn}(\tilde{f}) |\tilde{f}|^{q-1} + \int_0^T \tilde{\sigma} \varphi \right) (f - \tilde{f}) \geq 0 \quad \forall f \in \mathcal{F}. \end{array} \right. \quad (65)$$

4. Conclusions

In this article, we have studied an optimal control problem for a 2D parabolic-elliptic chemo-repulsion model with nonlinear chemical signal production term. We controlled the system, applying a bilinear control on a subdomain $\omega \subset \Omega$, which act on the chemical equation (2)₂ as degradation/proliferation coefficient. We proved the existence of at least one global optimal solution and derive first-order necessary optimality conditions for a local optimal solution, applying a generic result on the existence of Lagrange multipliers. Also, for the Lagrange multipliers obtained, we improve its regularity, which allows us to conclude that they satisfy point-wisely an adjoint system related to the primal problem (2)-(4).

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