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The Structure of Semiconic Idempotent Commutative Residuated Lattices †

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Abstract: In this paper, we study semiconic idempotent commutative residuated lattices. After giving some properties of such residuated lattices, we obtain a structure theorem for semiconic idempotent commutative residuated lattices. As an application, we make use of the structure theorem to prove that the variety of strongly semiconic idempotent commutative residuated lattices has the amalgamation property.

Keywords: residuated lattices; idempotent semigroup; chain; construction; amalgamation

MSC: 06F05; 20M10

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1. Introduction

A *commutative residuated lattice* is defined as an algebra $(L, \wedge, \vee, \cdot, \rightarrow, e)$ of type $(2, 2, 2, 2, 0)$ satisfying the following conditions:

$(RL1)$ $(L, \wedge, \vee)$ is a lattice;
$(RL2)$ $(L, \cdot, e)$ is a commutative monoid with identity e; and
$(RL3)$ $(\forall x, y, z \in L)$ $x \cdot y \leq z \Longleftrightarrow y \leq x \rightarrow z$, where $\leq$ is the lattice ordering.

Some time, commutative residuated lattices are also called *commutative residuated lattice-ordered monoids* and abbreviated by CRLs. It is well known that $(RL3)$ holds if and only if $\leq$ is compatible with $\cdot$ and for all $a, b \in L$, $\{p \in L : a \cdot p \leq b\}$ contains a greatest element (denoted by $a \rightarrow b$).

A CRL $\mathbf{L}$ is called *idempotent* if for all $a \in L$, $a \cdot a = a$; is called *integral* if for all $a \in L$, $a \leq e$; is called *totally ordered* if for all $a, b \in L$, $a \leq b$ or $a \geq b$; is called *semilinear* when it is a subdirect product of totally ordered CRLs; is called *conic* if for all $a \in L$, $a \leq e$ or $a \geq e$ (see [4,9,12]). A semilinear idempotent CRL is said to be an odd Sugihara monoid if for all $a \in L$, $(a \rightarrow e) \rightarrow e = a$. An integral idempotent CRL is said to be a *Brouwerian algebra* if for all $a, b \in L$, $ab = a \wedge b$. As in [12], a CRL $\mathbf{L}$ is called *semiconic* when it is a subdirect product of conic CRLs.

Idempotent CRLs form an important tool both in algebra and logic (see [10]). Among them, semiconic ones make a valuable contribution, because they include several important algebraic counterparts of substructural logics (see [19]). Recently, algebra properties for semiconic CRLs have been given by many authors (see [4,5,7–10,12–21]). In [20], the authors obtain a structure theorem for semilinear idempotent CRLs. In this paper, we will investigated algebraic structure properties of semiconic idempotent CRLs. Idempotent CRLs are indeed ordered semigroups (for ordered semigroups, see [1]). The natural partial order play an important role in investigation of semigroups (see [16]). We will make use of the natural partial order to obtain some important properties and then establish a structure theorem of semiconic idempotent CRLs, which generalizes the main result of [20].

We proceed as follows: in Section 2, we present some definitions and facts used in the sequel. In Section 3, we obtain some properties of semiconic idempotent CRLs. In Section 4, we give a structure theorem for semiconic idempotent CRLs, which generalizes the main result of [20]. In Section 5, we

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prove that the variety of strongly semiconic idempotent CRLs has the amalgamation property, which generalizes the main result of [11].

2. Preliminaries

In this section, we will list some facts about CRLs.

A monoid $(M, \cdot, e)$ is said to be a *po-monoid* when it is also a poset $(M, \leq)$, and in which $\leq$ is compatible with $\cdot$, in the sense that $(\forall a, b, c \in M)\ a \leq b \Longrightarrow c \cdot a \leq cb, a \cdot c \leq bc$. A po-monoid $(M, \cdot, e, \leq)$ is said to be a *lattice-ordered monoid* when $(M, \leq)$ is a lattice. A lattice-ordered monoid $(M, \cdot, e, \leq)$ is said to be *idempotent* if for all $a \in M$, $a \cdot a = a$; is said to be *commutative* when the monoid reduct $(M, \cdot, e)$ is a commutative monoid; is said to be *conic*, if for all $a \in M$, $a \leq e$ or $a \geq e$. For convenience, we simply write $a \cdot b$ as ab for $a, b \in M$. The reader is referred to reference [1] for detailed information on lattice-ordered monoids.

We need the following results.

Lemma 1. [21] *Let $(M, \cdot, e, \leq)$ be an idempotent lattice-ordered monoid with identity e, and $a, b \in M$.*

(1) $a \wedge b \leq ab \leq a \vee b$.
(2) *If* $a, b \geq e$, *then* $ab = a \vee b$.
(3) *If* $a, b \leq e$, *then* $ab = a \wedge b$.
(4) *If* $a \leq e \leq ab$, *then* $ab = b$.
(5) *If* $ab \leq e \leq a$, *then* $ab = b$.

Let $(L, \wedge, \vee, \cdot, \rightarrow, e)$ be a CRL and $\leq$ shall always denote the lattice order of **L** in this paper.

Lemma 2. [10,14] *Let $(L, \wedge, \vee, \cdot, \rightarrow, e)$ be a* CRL *and $a, b, c \in L$.*

(1) $a(b \vee c) = ab \vee ac$.
(2) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$.
(3) $(b \vee c) \rightarrow a = (b \rightarrow a) \wedge (c \rightarrow a)$.
(4) $b(b \rightarrow a) \leq a$.
(5) $e \leq a \rightarrow a$.
(6) $((a \rightarrow b) \rightarrow b) \rightarrow b = a \rightarrow b$.
(7) $a \rightarrow (b \rightarrow c) = (ab) \rightarrow c$.

From now on, we denote $a \rightarrow e$ and $(a \rightarrow e) \rightarrow e$ by a^* and a^{**}, respectively. Next, we shall present some known facts on conic idempotent CRLs used in later proofs. More details on semiconic residuated lattices can be found in [4,12,13].

Lemma 3. [12] *Let **L** be a conic idempotent* CRL, *and $a, b \in L$.*

(1) *If a and b are incomparable, then $a^* = b^*$.*
(2) *The elements a and b^* are comparable.*
(3) $a \not\leq b$ *if and only if* $a \rightarrow b < e$.
(4) *If* $a \leq e (a > e)$, *then* $a^* = a \rightarrow a \geq e (a^* < e)$.
(5) $\{a^* : a \in L\}$ *is a chain in* $(L, \wedge, \vee)$.

3. Some Properties

To begin with, we obtain some properties of conic idempotent commutative lattice-ordered monoids.

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Now let $(L, \cdot, e, \leq)$ be a conic idempotent commutative lattice-ordered monoid. Since the monoid reduct of $\mathbf{L}$ is an idempotent commutative monoid, we define the *natural partial order* on $\mathbf{L}$ as follows: for $a, b \in L$,

$$a \leq_n b \text{ if and only if } ab = a.$$

It is clear that $(L, \leq_n)$ is a semilattice. For $a, b \in L$, $a \parallel b$ [resp. $a \parallel_n b$] means that a and b are incomparable under $\leq$ [resp. $\leq_n$]; $a \prec b$ [resp. $a \prec_n b$] means that $a < b$ [resp. $a <_n b$] and for any $c \in L$, $a \leq c \leq b$ [resp. $a \leq_n c \leq_n b$] implies either $a = c$ or $b = c$. Let $a \wedge_n b = \max\{c \in L : c \leq_n a, b\}$ and $a \vee_n b = \min\{c \in L : a, b \leq_n c\}$ if it exists in $(L, \leq_n)$.

Proposition 1. *Let $(L, \cdot, e, \leq)$ be a conic idempotent commutative lattice-ordered monoid. The following statements are true for $a, b \in L$:*

(1) *If $a, b \leq e$, then $a \leq_n b$ if and only if $a \leq b$.*
(2) *If $a, b \geq e$, then $a \leq_n b$ if and only if $a \geq b$.*
(3) *$a \parallel b$ if and only if $a \parallel_n b$.*
(4) *If $a \parallel b$ and $a < e$, then $a \wedge_n b = a \wedge b$.*
(5) *If $a \parallel b$ and $a > e$, then $a \wedge_n b = a \vee b$.*

Proof. (1) Let $a, b \in L$ be such that $a, b \leq e$. Then by Lemma 1(3), $a \leq_n b \iff ab = a \iff a \wedge b = a \iff a \leq b$.

(2): is similar to (1).

(3) If $a \parallel b$, then since $\mathbf{L}$ is conic, $a, b \leq e$ or $a, b \geq e$. If $a, b \leq e$, then by (1), $a \nparallel_n b$ is impossible. Thus $a \parallel_n b$. Similarly, if $a, b \geq e$, then $a \parallel_n b$. Conversely, let $a, b \in L$ such that $a \parallel_n b$. Suppose that $a \leq e \leq b$ or $b \leq e \leq a$. Then since $\mathbf{L}$ is conic, by Lemma 1(4,5), $ab = a$ or $ab = b$, which implies that $a \leq_n b$ or $b \leq_n a$, a contradiction. Hence $a, b \leq e$ or $e \leq a, b$. Thus, by (1) and (2), $a \parallel b$.

(4) Let $a, b \in L$ such that $a \parallel b$ and $a < e$. Then by Lemma 1(3), $a \wedge b = ab$. Let $c \in L$ such that $c \leq_n a$ and $c \leq_n b$. Then $ca = c$ and $cb = c$, so $cab = cb = c$. Thus $c \leq_n ab$. Since $ab \leq_n a$ and $ab \leq_n b$, $a \wedge b = ab = a \wedge_n b$.

(5) Let $a, b \in L$ such that $a \parallel b$ and $a > e$. Then by Lemma 1(2), $ab = a \vee b > a, b > e$. So by (2), $a \vee b \leq_n a$ and $a \vee b \leq_n b$. Let $c \in L$ such that $c \leq_n a$ and $c \leq_n b$. Then $ca = c$ and $cb = c$, so $cab = cb = c$. Thus $c \leq_n ab = a \vee b$. Therefore $a \wedge_n b = a \vee b$. □

Secondly, we obtain some properties of conic idempotent CRLs.

Proposition 2. *Let $\mathbf{L}$ be a conic idempotent* CRL. *The following statements are true for $a, b \in L$:*

(1) *If $a < e$, then $a <_n a^*$ and $a^* \nparallel_n b$.*
(2) *If $a < e$ and $a <_n b <_n a^*$, then $b < e$.*
(3) *If $a > e$, then $a^* <_n a$ and $a^* \nparallel_n b$.*
(4) *If $a > e$ and $a^* <_n b <_n a$, then $b > e$.*
(5) *If $a \parallel b$ and $a < e$, then $a \vee_n b = a \vee b$ and $(a \wedge b)^* = a^*$.*

Proof. (1) Let $a, b \in L$ such that $a < e$. Then $ae = a < e$ and so $a < e \leq a \to e = a^*$. Since $a(a \to e) \leq e$ by Lemma 2(4), $a(a \to e) = a$ by Lemma 1(5), which implies that $a <_n a \to e = a^*$. Since by Lemma 3(2), $a^* \nparallel b$, by Proposition 1(3), $a^* \nparallel_n b$.

(2) Let $a \in L$ such that $a < e$ and $a <_n b <_n a^* = a \to e$. Then $ab = a < e$ and so $b \leq a \to e = a^*$. Suppose that $b \geq e$. Then since by Lemma 3(4), $a^* \geq e$, by Proposition 1(2), $a^* \leq_n b$, contrary to $b <_n a^*$. Thus $b < e$.

(3): is similar to (1).

(4): is similar to (2).

(5) Let $a, b \in L$ such that $a, b < e$ and $a \parallel b$. Since $a, b \leq a \vee b \leq e$, by (1), $a \leq_n a \vee b$ and $b \leq_n a \vee b$. Let $d \in L$ such that $a \leq_n d$ and $b \leq_n d$. Then $ad = a$ and $bd = b$, so $d(a \vee b) = da \vee db = a \vee b$ by

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Lemma 2(1). Thus $a \vee b \leq_n d$. Therefore $a \vee_n b = a \vee b$. Because $a \wedge b \leq a < e$, $a^* = a \rightarrow e \leq (a \wedge b) \rightarrow e = (a \wedge b)^*$ by Lemma 2(3). Suppose that $a^* < (a \wedge b)^*$. Then $a \wedge b < a < a^* < (a \wedge b)^*$, so by (2), $a^* < e$. But since $a < e$, $a^* \geq e$ by Lemma 3(4), a contradiction. Thus $a^* = (a \wedge b)^*$. □

Proposition 3. [4] *Let* **L** *be a conic idempotent* CRL, *and let $a, b \in L$ such that $a \leq e$. If $b < a$ or $a \parallel b$, then $a \rightarrow b = b$ or $a \rightarrow b \parallel a$.*

Let $(L, \leq)$ be a join-semilattice, and let $L_\perp = L \cup \{\perp\}$ such that $\perp \leq a$ for all $a \in L$. **L** is said to be an *upper pre-lattice* when **L** isn't a lattice and $(L_\perp, \leq)$ is a lattice. Let **L** be a lattice and $C \subseteq L$. **C** is said to be an *upper pre-sublattice of* **L** if **C** is an upper pre-lattice and there exists $a \in L$ such that $(C \cup \{a\}, \leq)$ is a sublattice of **L**. Similarly, we can define *lower pre-lattice* and *lower pre-sublattice.*

Let **L** be a conic idempotent CRL. We define the following sets: $L^+ = \{a \in L : a > e\}$, $L^- = \{b \in L : b \leq e\}$, $L^* = \{j \in L : (\exists a \in L)\ j = a^*\}$, $L^{*-} = \{j \in L^* : j \leq e\}$, $L^{*+} = \{j \in L^* : j > e\} = \{j \in L^* : (\exists i \in L^{*-})\ j = i^*\}$. For every $j \in L^*$, let $L_j = \{c \in L : c^{**} = j\}$. By Lemma 3(4), $L_j \subseteq L^+$ for all $j \in L^{*+}$. Since $a^* <_n a^{**} \leq_n a <_n e$ for all $a > e$ by Proposition 2(1,3) and Lemma 3(4), $a^{**} > e$ by Proposition 2(4). It follows that $L_i \subseteq L^-$ for all $i \in L^{*-}$. Because $L^{*-} \subseteq L^*$ and $(L^*, \leq)$ is a chain by Lemma 3(5), $(L^{*-}, \leq)$ is a chain. It is clear that $L^* = L^{*-} \cup L^{*+}$.

We have the following result, which generalizes [7, Theorem 3.2].

Theorem 1. *Let* **L** *be a conic idempotent* CRL.

(1) *If $a \in L$, then $a \in L^*$ if and only if $a^{**} = a$.*

(2) *If $i, l \in L^{*-}$, then $i = l$ if and only if $i^* = l^*$. In addition, $i \prec_n i^*$ for all $i \in L^{*-} \setminus \{e\}$.*

(3) *If $j, s \in L^*$ such that $j \neq s$, then $L_j \cap L_s = \emptyset$.*

(4) *If $i \in L^{*-}$, then $\mathbf{L}_i$ is a sublattice of $(L, \wedge, \vee)$ and $(L_i, \wedge, \vee, \cdot, \rightarrow^{L_i}, i)$ is a Brouwerian algebra, where $\rightarrow^{L_i}$ is given by $x \rightarrow^{L_i} y = (x \rightarrow y) \wedge i$ for all $x, y \in L_i$.*

(5) *If $j = i^* \in L^{*+}$, then $\mathbf{L}_j$ has a greatest element j and is either a sublattice of* **L** *or an upper pre-sublattice of* **L**.

(6) *If $i, l \in L^{*-}$ such that $i \neq l$, $a \in L_i, b \in L_l$ and $c \in L_{i^*}, d \in L_{l^*}$ then $i < l \Longleftrightarrow a < b \Longleftrightarrow c > d$.*

(7) *If $i \in L^{*-}$ and $j = i^*$ such that $\mathbf{L}_j$ is an upper pre-sublattice of* **L**, *then there exists $l \in L^{*-}$ such that $i \prec l$ in L^{*-} and $(L_j \cup \{l^*\}, \leq)$ is a sublattice of* **L** *with a least element l^*.*

(8) *If* **L** *satisfies that $(x \wedge y)^* = x^* \vee y^*$, then $\mathbf{L}_i$ is a sublattice of* **L** *for all $i \in L^{*+}$.*

(9) *If $i \in L^{*-}, l \in L^{*+}$ and $a \in L_i, b \in L_l$, then $i <_n l \Longleftrightarrow a <_n b$.*

(10) **L** *is finitely subdirectly irreducible if and only if $\mathbf{L_e}$ is a finitely subdirectly irreducible Brouwerian algebra.*

(11) $\mathbf{L}^*$ *is a totally ordered odd Sugihara monoid and subalgebra of* **L**, *that we call its skeleton.*

Proof. (1) We only need to verify the necessity because the sufficiency is clear. Suppose that $a \in L^*$. Then there exists $c \in L$ such that $c^* = a$. Thus by Lemma 2(6), $a = c^* = c^{***} = a^{**}$.

(2) We only need to prove the sufficiency because the necessity is obvious. Suppose that $i^* = l^*$. Since $i, l \in L^{*-}$ by assumption, $i = i^{**} = l^{**} = l$ by (1). Let $i \in L^{*-} \setminus \{e\}$. Then by Proposition 2(1), $i <_n i^*$. Let $a \in L$ such that $i \leq_n a \leq_n i^*$. Suppose that $i <_n a <_n i^*$. If $a \geq e$, then by Proposition 2(2), $a < e$, a contradiction. If $a < e$, then since $i = i^{**} <_n a <_n i^*$, $a > e$ by Proposition 2(4), a contradiction. Consequently, $i \prec_n i^*$.

(3) It is obvious.

(4) Let $i \in L^{*-}$ and let $x, y \in L_i$. Then $x^{**} = y^{**} = i$ which together with $(x \vee y)^* = x^* \wedge y^* \in \{x^*, y^*\}$ by Lemmas 2(3) and 3(5), derives that $(x \vee y)^{**} = i$, whence $x \vee y \in L_i$. If $x \leq y$ or $y < x$, then $(xy)^* = (x \wedge y)^* = x^*$ or $(xy)^* = (x \wedge y)^* = y^*$, and so $(xy)^{**} = (x \wedge y)^{**} = i$, which implies that $xy = x \wedge y \in L_i$. If $x \parallel y$, then by Proposition 2(5), $(xy)^* = (x \wedge y)^* = x^*$ and so $(xy)^{**} = (x \wedge y)^{**} = i$, which implies that $xy = x \wedge y \in L_i$. Thus L_i is a sublattice of $(L, \wedge, \vee)$. By (1), $i \in L_i$. Let $c \in L_i$. Then $c \leq (c \rightarrow e) \rightarrow e = c^* \rightarrow e = c^{**} = i$. Thus i is the greatest element of L_i and so $(L_i, \cdot, i, \leq)$ is

an integral idempotent commutative lattice-ordered monoid with an identity i. We can claim that $\max\{z \in L_i : xz \leq y\} = (x \to y) \wedge i$ for all $x, y \in L_i$. To prove this, we consider the following cases:

- If $x \leq y$, then $i \leq e \leq x \to y$ by Lemma 3(3) and so $\max\{z \in L_i : xz \leq y\} = i = (x \to y) \wedge i$.
- If $x > y$ or $x \parallel y$, then by Proposition 3, $x \to y = y$ or $x \to y \parallel x$. So $(x \to y)^{**} = y^{**} = i$ or $(x \to y)^{**} = x^{**} = i$ by Lemma 3(1). Thus $x \to y \in L_i$, whence $\max\{z \in L_i : xz \leq y\} = x \to y = (x \to y) \wedge i$

We define $x \to^{L_i} y = (x \to y) \wedge i$ for all $x, y \in L_i$. Thus $(L_i, \wedge, \vee, \cdot, \to^{L_i}, i)$ is a Brouwerian algebra.

(5) Let $j = i^* \in L^{*+}$. By similar arguments as in the proof of (4), j is the greatest element of L_j and $b \vee c \in L_j$ for all $b, c \in L_j$, so $\mathbf{L}_j$ is a join-semilattice with a greatest element j. Suppose that $\mathbf{L}_j$ isn't a sublattice of $\mathbf{L}$. Then there exist $b, b' \in L_j$ such that $b \parallel b'$ and $d = b \wedge b' \notin L_j$. Hence $e \leq d < b$ and $d^{**} < b^{**} = j$ by Lemma 2(3). Let $c \in L_j$. Suppose that $c \parallel d$. Then $d^{**} = c^{**} = j$ by Lemma 3(1), which is contrary to $d^{**} < j$. Assume that $c < d$. Then $d^{**} > c^{**} = b^{**} = j$ by Lemma 2(3), which is contrary to $d^{**} < j$. Thus for all $c \in L_j$, $d < c$. Similarly, if $g, g' \in L_j$ such that $g \wedge g' \notin L_j$, then for all $c \in L_j$, $g \wedge g' < c$. It follows that $d = g \wedge g'$. Therefore $(L_j \cup \{d\}, \leq)$ is a sublattice of $\mathbf{L}$. Consequently, $\mathbf{L}_j$ is an upper pre-sublattice.

(6) Let $i, l \in L^{*-}$ such that $i \neq l$ and let $a \in L_i$, $b \in L_l$. If $i < l$, then by (4), $a \leq i < l$ and $b \leq l$. Suppose that $b \leq i$. Then $b^{**} \leq i^{**} = i < j$, which is contrary to $b^{**} = j$. Thus $i < b$ by Lemma 3(2), whence $a < b$. Conversely, if $a < b$, then $a \leq i$ and $a < b \leq l$. Suppose that $l < i$. Then $a^{**} \leq l^{**} = l < i$, which is contrary to $a^{**} = i$. Thus $i < l$. Similarly, $i < l \iff c > d$.

(7) Let $i \in L^{*-}$ and $j = i^*$ such that $\mathbf{L}_j$ is an upper pre-sublattice of $\mathbf{L}$. Then there exist $b, b' \in L_j$ such that $b \parallel b'$ and $d = b \wedge b' \notin L_j$. Let $l = d^*$. Then $d \in L_{l^*}$. Since $d < b$, $i < l$ by (6). Let $k \in L^{*-}$ such that $i \leq k \leq l$. Suppose that $i < k < l$. Then by (6) $d < k^* < b, b'$, contrary to $d = b \wedge b'$. Thus $i \prec l$ in L^*. We have $dl = dd^* = d(d \to e) \leq e \Longrightarrow d \leq d^* \to e = d^{**} = l^*$. We claim that $l^* = d$. Otherwise, if $d < l^*$, then since $i < l$, $j = i^* > l^*$ by (6) and so $l^* < b, b'$. It follows that $l^* \leq b \wedge b' = d$. It's a contradiction. Thus $l^* = d^{**} = d = a \wedge b$. Consequently, $(L_j \cup \{l^*\}, \leq)$ is a sublattice of $\mathbf{L}$ with a least element l^*.

(8) Let $i \in L^{*+}$ and $a, b \in L_i$. Then $(a \wedge b)^{**} = (a^* \vee b^*)^* = a^{**} \wedge b^{**} = i$ and so $a \wedge b \in L_i$. It follows that $\mathbf{L}_i$ is a sublattice of $\mathbf{L}$.

(9) Since $a \in L_i$ and $b \in L_l$, $a \leq i \leq e < b \leq l$ by $(4-5)$ and $i^* \neq e$. Then $a \leq_n i$ and $l \leq_n b$ by Proposition 1(1-2). Suppose that $i <_n l$. Then $a <_n b$. Conversely, assume that $a <_n b$. We claim that $a^* \neq e$. Otherwise if $a^* = e$, then $a <_n b \Longrightarrow ab = a \leq e \Longrightarrow b \leq a \to e = a^* = e$, which contrary to $b > e$. Consequently, $a^* > e$ by Lemma 3(4). By Proposition 2(1,3), $a \leq_n i = a^{**} <_n a^*$ and $b^* <_n l = b^{**} \leq_n b$. Suppose that $l <_n i$. If $a <_n l$, then $a <_n l <_n i = a^{**} <_n a^*$, and so by Proposition 2(2), $l < e$, which contrary to $l > e$. If $l <_n a$, then $b^* <_n l = b^{**} <_n a <_n b$, and so by Proposition 2(4), $a > e$, which contrary to $a \leq e$. Consequently, $i <_n l$.

(10) Suppose that $\mathbf{L}$ is finitely subdirectly irreducible, then e is join-irreducible in L. Since $\mathbf{L}_e$ is a sublattice of $\mathbf{L}$, e is join-irreducible in L_e, which implies that $\mathbf{L}_e$ is finitely subdirectly irreducible. Conversely, $\mathbf{L}_e$ is finitely subdirectly irreducible. Then e is join-irreducible in L_e. By (6), we have that $b < a$ for all $a \in L_e$ and $b \in L_i$ such that $i \in L^{*-} \setminus \{e\}$, which implies that e is join-irreducible in L. Thus $\mathbf{L}$ is finitely subdirectly irreducible.

(11) By Lemma 3(5), $(L^*, \leq)$ is a totally ordered set, which implies that $\mathbf{L}^*$ is a sublattice of $\mathbf{L}$. Let $a^*, b^* \in L^*$. If $a^*, b^* \leq e$, then $a^* b^* = a^* \wedge b^* \in L^*$. If $a^*, b^* \geq e$, then $a^* b^* = a^* \vee b^* \in L^*$. If $a^* \leq e, b^* > e$ or $a^* > e, b^* \leq e$, then by Lemma 1(4,5), $a^* b^* \in \{a^*, b^*\} \subseteq L^*$. Thus L^* is closed with respect to multiplication. By Lemma 2(7), we have $a^* \to b^* = a^* \to (b \to e) = (a^* b) \to e \in L^*$. Consequently, $\mathbf{L}^*$ is a subalgebra of $\mathbf{L}$. By Lemma 2(6), $(a^* \to e) \to e = ((a \to e) \to e) \to e = a \to e = a^*$. It follows that $\mathbf{L}^*$ is a totally ordered odd Sugihara monoid. □

Theorem 2. *Let $\mathbf{L}, \mathbf{K}$ be conic idempotent* CRL*s, and $f : L \to K$ be a homomorphism between conic idempotent* CRL*s.*

(1) *$f(L^*) \subseteq K^*$ and $f(L_i) \subseteq K_{f(i)}$ for all $i \in L^*$.*

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(2) *If $i \in L^*$ such that $\mathbf{L}_i$ is an upper pre-sublattice of $\mathbf{L}$ and $f(i) \neq e$, then $K_{f(i)}$ is an upper pre-sublattice of $\mathbf{K}$ and there exists $j \in L^*$ such that $j \prec i$ in L^* and $f(j) \prec f(i)$ in K^*.*

Proof. (1) Let $a \in L^*$. Then there exists $b \in L$ such that $a = b^*$. Since f is a homomorphism, $f(a) = f(b^*) = f(b)^* \in K^*$, which implies that $f(L^*) \subseteq K^*$. Let $i \in L^*$ and $a \in L_i$. Then $a^{**} = i$ and so $f(a)^{**} = f(a^{**}) = f(i)$, which implies that $f(a) \in K_{f(i)}$. It follows that $f(L_i) \subseteq K_{f(i)}$.

(2) Since L_i is an upper pre-sublattice of $\mathbf{L}$, $i > e$ by Theorem 1(4) and there exist $a, b \in L_i$ such that $a \wedge b \notin L_i$. Let $j = a \wedge b$. By the proof of Theorem 1(7), $e \leq a \wedge b = j \prec i$ in L^* and so $j < i$ in L. Hence $i^* < j^*$ by Theorem 1(6). It follows that $i <_n j$ and $i^* <_n j^*$ by Theorem 1(2). We claim that $i <_n j^*$. Otherwise if $j^* <_n i$, then $i^* <_n j^* <_n i$ and so by Proposition 2(4), $j^* > e$, which contrary to $j^* = (a \wedge b)^* \leq e$. Thus $i <_n j^*$. We have $f(a), f(b) \in K_{f(i)}$ and $f(a) \wedge f(b) = f(a \wedge b) = f(j) \in K_{f(j)}$ by (1). Suppose that $f(j) = f(i)$. Then $f(j) = f(i) = f(ij^*) = f(i)f(j^*) = f(j)f(j)^* = f(j)^*$ and so by Proposition 2(1,3), $f(i) = f(j) = e$ which contrary to $f(i) \neq e$. Consequently, $f(j) \neq f(i)$. It follows that $K_{f(i)}$ is an upper pre-sublattice of $\mathbf{K}$ and by the proof of Theorem 1(7), $f(j) = f(a \wedge b) = f(a) \wedge f(b) \prec f(i)$ in K^*. $\square$

4. The Construction Theorem

In this section we shall show how to construct a conic idempotent CRL and then prove that any conic idempotent CRL is isomorphic to some conic idempotent CRL constructed in this way.

To start with, we introduce some new concepts.

Definition 1. *Let $(I, \leq)$ be a chain with a greatest element e. Let $I^+ = \{i^+ : i \in I \setminus \{e\}\}$ such that $I \cap I^+ = \emptyset$ and $i^+ \neq l^+$ for every pair $i, l \in I \setminus \{e\}$ such that $i \neq l$. Let $J = I \cup I^+$. Let $\mathcal{A} = \{(A_j, \leq_{A_j}) : j \in J\}$ be a family of pairwise disjoint nonempty poset indexed by J. $(I, I^+, J; \mathcal{A})$ is called a chain expansion-system (abbreviated by CE-system) if the following conditions hold:*

(CE1) *If $i \in I$, then $(A_i, \leq_{A_i})$ is a Brouwerian algebra with a greatest element i.*
(CE2) *If $i^+ \in I^+$, then $(A_{i^+}, \leq_{A_{i^+}})$ is either a lattice with a greatest element i^+ or an upper pre-lattice with a greatest element i^+.*
(CE3) *If $i^+ \in I^+$ such that $(A_{i^+}, \leq_{A_{i^+}})$ is an upper pre-lattice, then there exists $j \in I$ such that $i \prec j$ in I.*

Given a CE-system $(I, I^+, J; \mathcal{A})$, put $L = \bigcup_{j \in J} A_j$. Define a binary relation $\leq$ on the set L as follows. Let $a \in A_j, b \in A_k$. $a \leq b$ in L if one of the following conditions is satisfied:

(P1) $j = k \in J$ and $a \leq_{A_j} b$.
(P2) $j, k \in I$ and $j < k$.
(P3) $j = i_1^+ \in I^+, k = i_2^+ \in I^+$ and $i_2 < i_1$ in I.
(P4) $j \in I$ and $k \in I^+$.

Lemma 4. *$(L, \leq)$ is a lattice.*

Proof. Firstly, we will prove that $(L, \leq)$ is a poset. Obviously, $\leq$ is reflexive. Next we prove that $\leq$ is antisymmetric. To see this, let $a \in A_j, b \in A_k$ such that $a \leq b$ and $b \leq a$ .We consider four cases:

- If $j = k \in J$, then by $(P1)$, $a \leq_{A_j} b$ and $b \leq_{A_j} a$. Since $(A_j, \leq_{A_j})$ is a poset, $a = b$.
- Suppose $j \neq k$ and $j, k \in I$. Then since $a \leq b$ and $b \leq a$, $j < k$ and $k < j$, a contradiction. Thus $j \neq k$ and $j, k \in I$ is impossible.
- By similar arguments as in the previous case, $j \neq k$ and $j, k \in I^+$ is impossible.
- Similarly, either $j \in I, k \in I^+$ or $k \in I, j \in I^+$ is impossible.

Next, we prove that $\leq$ is transitive. Let $a \in A_j, b \in A_k, c \in A_s$ be such that $a \leq b$ and $b \leq c$. We consider four cases:

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- $j = k = s \in J$. Then by $(P1)$, $a \leq_{A_j} b$ and $b \leq_{A_j} c$. Since $(A_j, \leq_{A_j})$ is a poset, $a \leq_{A_j} c$. Thus by $(P1)$, $a \leq c$.
- $j = k \neq s$. If $k, s \in I$ and $k < s$, then $j < s$ and so by $(P2)$, $a \leq c$. If $k = i_1^+, s = i_2^+ \in I^+$ such that $i_2 < i_1$ in I, then $j = i_1^+$ and so by $(P3)$, $a \leq c$. If $k \in I$ and $s \in I^+$, then $j \in I$ and so by $(P4)$, $a \leq c$.
- $j \neq k = s$. Then by similar arguments as in the prior case , $a \leq c$.
- $j \neq k$ and $k \neq s$. If $j, k, s \in I$, then $j < k$ and $k < s$, and so $j < s$, which implies that $a \leq c$ by $(P2)$. If $j \in I$ and $s \in I^+$, then by $(P4)$, $a \leq c$. If $j, k, s \in I^+$ such that $j = i_1^+, k = i_2^+, s = i_3^+$, then $i_3 < i_2$ and $i_2 < i_1$ in I by $(P3)$. Since $(I, \leq)$ is a chain, $i_3 < i_2$ and so by $(P3)$, $a \leq c$.

We conclude $a \leq c$, and whence $\leq$ is transitive.

Finally, we will prove that for all $a, b \in L$, $a \vee b$ and $a \wedge b$ exist in L. Let $a \in A_j, b \in A_k$. We consider three cases:

- If $a \leq b$, then $a \vee b = b$ and $a \wedge b = a$ in L.
- If $b \leq a$, then $a \vee b = a$ and $a \wedge b = b$ in L.
- If $a \parallel b$, then by the definition of $\leq$, $j = k$. If $j, k \in I$, then since $(A_j, \leq_{A_j})$ is a Brouwerian algebra, $a \vee^{A_j} b$ exist in A_j. Let $c \in A_s$ such that $a, b \leq c$. If $s = j$, then by $(P1)$, $a \leq_{A_j} c$ and $b \leq_{A_j} c$, and so $a \vee^{A_j} b \leq_{A_j} c$. Thus by $(P1)$, $a \vee^{A_j} b \leq c$. If $s \neq j$, then since $a \leq c$, either $s \in I$ and $j < s$ or $s \in I^+$, which together with $a \vee^{A_j} b \in A_j$, derives that $a \vee^{A_j} b \leq c$. It follows that $a \vee b = a \vee^{A_j} b$ in L. Similarly, $a \wedge b = a \wedge^{A_j} b$ in L. If $j = k = i^+ \in I^+$, then $(A_{i^+}, \leq_{A_{i^+}})$ is either a lattice or a pre-lattice by (CE2). If $(A_{i^+}, \leq_{A_{i^+}})$ is a lattice or an upper pre-lattice and $a \wedge^{A_{i^+}} b$ exists, then by similar arguments as in the prior case, $a \vee b = a \vee^{A_{i^+}} b$ and $a \wedge b = a \wedge^{A_{i^+}} b$ in L. If $(A_{i^+}, \leq_{A_{i^+}})$ is an upper pre-lattice and $a \wedge^{A_{i^+}} b$ doesn't exist, then by similar arguments in the prior case, $a \vee b = a \vee^{A_{i^+}} b$ in L. By (CE3), there exists $t \in I$ such that $i \prec t$ in I. We claim that $t^+ = a \wedge b$ in L. Because t^+ is the greatest element of $(A_{t^+}, \leq_{A_{t^+}})$ by $(CE2)$, $t^+ \leq a, b$ by $(P3)$. Let $c \in A_s$ such that $c \leq a, b$. Since $(A_{i^+}, \leq_{A_{i^+}})$ is an upper pre-lattice and $a \wedge^{A_{i^+}} b$ doesn't exist, $c \notin A_{i^+}$, and so by $(P3-4)$, either $s \in I$ or there exists $l \in I$ such that $s = l^+$ and $i < l$ in I, which implies that either $s \in I$ or $s = l^+$ such that $t \leq l$. It follows that $c \leq t^+$ by $(P3-4)$. Thus $a \wedge b = t^+$ in L.

□

We define a multiplication $\circ$ on L in the following ways: for $a \in A_j, b \in A_k$,

$$a \circ b = \begin{cases} a \wedge b & \text{if } j, k \in I, \\ a \vee b & \text{if } j, k \in I^+, \\ a & \text{if } j = i^+ \in I^+, k \in I, i < k \text{ or } j \in I, k = l^+ \in I^+, j \leq l, \\ b & \text{if } j = i^+ \in I^+, k \in I, i \geq k \text{ or } j \in I, k = l^+ \in I^+, j > l. \end{cases}$$

Lemma 5. *$(L, \wedge, \vee, \circ, e)$ is a conic lattice-ordered idempotent commutative monoid with identity e.*

Proof. It is clear that $a \circ a = a$ and $a \circ b = b \circ a$ for $a, b \in L$.

Let $a \in A_j$. If $j \in I$, then since e is the greatest of I, $j \leq e$ which together with e is the greatest element of A_e by $(CE1)$, derives that $a \leq e$ and $a \circ e = a$. If $j = i^+ \in I^+$, then $i < e$, so $a > e$ and $a \circ e = a$. Now, we will show that $\circ$ satisfies the associative law. Let $a \in A_j, b \in A_k, c \in A_s$. We consider the following cases:

- If $j, k, s \in I$, then $(a \circ b) \circ c = (a \wedge b) \circ c = a \wedge b \wedge c$ and $a \circ (b \circ c) = a \circ (b \wedge c) = a \wedge b \wedge c$, whence $a \circ (b \circ c) = (a \circ b) \circ c$.
- If $j, k \in I$ and $s = i^+ \in I^+$, then

$$(a \circ b) \circ c = (a \wedge b) \circ c = \begin{cases} a \wedge b & \text{if } j \wedge k \leq i, \\ c & \text{if } i < j \wedge k; \end{cases}$$

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and

$$a \circ (b \circ c) = \begin{cases} a \circ b = a \wedge b & \text{if } k \leq i, \\ a \circ c = a = a \wedge b & \text{if } j \leq i < k, \\ c & \text{if } i < j \wedge k; \end{cases} = \begin{cases} a \wedge b & \text{if } j \wedge k \leq i, \\ c & \text{if } i < j \wedge k. \end{cases}$$

It follows that $a \circ (b \circ c) = (a \circ b) \circ c$.

- If $j \in I, k = i^+, s = l^+ \in I^+$, then

$$(a \circ b) \circ c = \begin{cases} a \circ c = a & \text{if } j \leq i \wedge l, \\ b \circ c = b \vee c & \text{if } i < j, \\ a \circ c = c & \text{if } l < j \leq i; \end{cases} = \begin{cases} a & \text{if } j \leq i \wedge l, \\ b \vee c & \text{if } j > i \wedge l; \end{cases}$$

and

$$a \circ (b \circ c) = a \circ (b \vee c) = \begin{cases} a & \text{if } j \leq i \wedge l, \\ b \vee c & \text{if } j > i \wedge l. \end{cases}$$

However, $a \circ (b \circ c) = (a \circ b) \circ c$.

- If $j, k, s \in I^+$ then $(a \circ b) \circ c = (a \vee b) \circ c = a \vee b \vee c$ and $a \circ (b \circ c) = a \circ (b \vee c) = a \vee b \vee c$, whence $a \circ (b \circ c) = (a \circ b) \circ c$.

Finally, we will show that $\leq$ is compatible with $\circ$. Let $a, b \in L$ be such that $a \leq b$. We need only to prove that $a \circ c \leq b \circ c$ for every $c \in L$. Suppose that $a \in A_j, b \in A_k, c \in A_s$. We need to consider the following cases:

(1) If $j, k, s \in I$, then by the definition of $\circ$, $a \circ c = a \wedge c$ and $b \circ c = b \wedge c$. Since $a \leq b$, $a \circ c \leq b \circ c$.

(2) If $j, k \in I$ and $s = i^+ \in I^+$, then $a \leq b < c$ and $j \leq k$. The following subcases need be considered:

- If $i < j$, then $i < k$ and so by the definition of $\circ$, $a \circ c = c$ and $b \circ c = c$, whence $a \circ c \leq b \circ c$.
- If $j \leq i$, then by the definition of $\circ$, $a \circ c = a$ and $b \circ c \in \{b, c\}$. It follows that $a \circ c \leq b \circ c$.

(3) If $j, s \in I$ and $k \in I^+$, then by the definition of $\circ$, $a \circ c = a \wedge c \leq c < b$ and $b \circ c \in \{b, c\}$, whence $a \circ c \leq b \circ c$.

(4) If $j \in I$ and $k, s \in I^+$, then $a < b \leq b \vee c$, so by the definition of $\circ$, $a \circ c \in \{a, c\}$ and $b \circ c = b \vee c$, whence $a \circ c \leq b \circ c$.

(5) If $j = i^+, k = l^+ \in I^+$ and $s \in I$, then since $a \leq b$, $l \leq i$ in I by $(P3)$. The following subcases need be considered:

- If $s \leq l$, then by the definition of $\circ$, $a \circ c = c$ and $b \circ c = c$, whence $a \circ c \leq b \circ c$.
- If $s > l$, then by the definition of $\circ$, $a \circ c \in \{a, c\}$ and $b \circ c = b$. It follows that $a \circ c \leq b \circ c$.

(6) If $j, k, s \in I^+$, then by the definition of $\circ$, $a \circ c = a \vee c$ and $b \circ c = b \vee c$, whence $a \circ c \leq b \circ c$. $\square$

We may define a binary operation $\rightarrow$ on L in the following way: for $a, b \in L$ such that $a \in A_j, b \in A_k$,

$$a \rightarrow b = \begin{cases} j^+ & \text{if } j, k \in I \text{ and } a \leq b, \text{ or } j \in I, k = i^+ \in I^+ \text{ and } j \leq i, \\ b & \text{if } j, k \in I \text{ and } j > k, \text{ or } j \in I, k = i^+ \in I^+ \text{ and } j > i, \\ a \rightarrow^{A_j} b & \text{if } j = k \in I \text{ such that } a \parallel b \text{ or } a > b, \\ b & \text{if } j, k \in I^+ \text{ and } a \leq b, \text{ or } j = i^+ \in I^+, k \in I \text{ and } i \geq k, \\ i & \text{if } j = i^+, k \in I^+ \text{ and } a \nleq b, \text{ or } j = i^+ \in I^+, k \in I \text{ and } i < k. \end{cases}$$

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We denote by $\mathbf{J} \bigotimes \mathcal{A}$ the above $(L, \wedge, \vee, \circ, \rightarrow, e)$.

Theorem 3. *$L = J \bigotimes \mathcal{A}$ is a conic idempotent* CRL.

Proof. We need only to prove that for all $a, b \in L, a \rightarrow b = \max\{c : a \circ c \leq b\}$. Suppose that $a \in A_j, b \in A_k$. We need to consider the following cases:

Case 1. $j, k \in I$ and $a \leq b$, or $j \in I, k = i^+ \in I^+$ and $j \leq i$. We need only to check the following subcases:

(1) If $j, k \in I$ and $a \leq b$, then by the definition of $\circ$, $a \circ (a \rightarrow b) = a \circ j^+ = a \leq b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. If $s \in I$, then by $(P4)$, $c \leq j^+ = a \rightarrow b$. If $s = l^+ \in I^+$, then $b < c$ by $(P4)$, and by the definition of $\circ$, $a \circ c \in \{a, c\}$, which together with $a \circ c \leq b$, derives that $a \circ c = a$. Thus $j \leq l$, whence $c \leq j^+ = a \rightarrow b$.

(2) If $j \in I, k = i^+ \in I^+$ and $j \leq i$, then by the definition of $\circ$ and $(P4)$, $a \circ (a \rightarrow b) = a \circ j^+ = a \leq b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. If $s \in I$, then by $(P4)$, $c \leq j^+ = a \rightarrow b$. If $s = l^+ \in I^+$, then by the definition of $\circ$, $a \circ c \in \{a, c\}$. Assume that $a \circ c = c$. Then by the definition of $\circ$, $l < j \leq i$, so $b < c = a \circ c$ by $(P3)$, which is contrary to $a \circ c \leq b$. Thus $a \circ c = a$, which implies that $j \leq l$ and so $c \leq j^+$ by $(P1)$ and $(P3)$.

Case 2. $j, k \in I$ and $j > k$, or $j \in I, k = i^+ \in I^+$ and $j > i$. We need only to check the following subcases:

(1) If $j, k \in I$ and $j > k$, then $a > b$ by $(P2)$, and so by the definition of $\circ$, $a \circ (a \rightarrow b) = a \circ b = a \wedge b = b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. Suppose that $s \in I^+$. Then $c > b$ by $(P4)$, and by the definition of $\circ$, $a \circ c \in \{a, c\}$, which implies that $a \circ c > b$, a contradiction. Suppose that $s \in I$ such that $s \geq j$. Then $a \circ c = a \wedge c \in A_j$, which implies that $a \circ c > b$, a contradiction. Thus $s \in I$ and $s < j$, whence $c = a \wedge c = a \circ c \leq b = a \rightarrow b$.

(2) If $j \in I, k = i^+ \in I^+$ and $j > i$, then by the definition of $\circ$, $a \circ (a \rightarrow b) = a \circ b = b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. If $s \in I$, then $c \leq b = a \rightarrow b$ by $(P4)$. If $s = l^+ \in I^+$ such that $l \geq j$, then $l > i$ and so $c \leq b = a \rightarrow b$ by $(P3)$. If $s = l^+ \in I^+$ such that $l < j$, then by the definition of $\circ$, $c = a \circ c \leq b = a \rightarrow b$.

Case 3. $j = k \in I$ such that $a \parallel b$ or $a > b$. Then by the definition of $\circ$ and $(CE1)$, $a \circ (a \rightarrow b) = a \circ (a \rightarrow^{A_j} b) = a \wedge (a \rightarrow^{A_j} b) = a \wedge^{A_j} (a \rightarrow^{A_j} b) \leq_{A_j} b$, which implies that $a \circ (a \rightarrow b) \leq b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. Suppose that $s \in I^+$ or $s \in I$ such that $s > j$. Then $c > b$ by $(P2, 4)$, and by the definition of $\circ$, $a \circ c \in \{a, c\}$, which implies that $a \circ c \nleq b$, a contradiction. If $s \in I$ such that $s < j$, then $c \leq a \rightarrow^{A_j} b = a \rightarrow b$ by $(P2)$. If $s \in I$ such that $s = j$, then by the definition of $\circ$, $a \wedge^{A_j} c = a \wedge c = a \circ c \leq b$, which implies that $a \wedge^{A_j} c \leq_{A_j} b$, so $c \leq_{A_j} a \rightarrow^{A_j} b$ by $(CE1)$. Thus $c \leq a \rightarrow^{A_j} b = a \rightarrow b$.

Case 4. $j, k \in I^+$ and $a \leq b$, or $j = i^+ \in I^+, k \in I$ and $i \geq k$. We need only to check the following subcases:

(1) If $j, k \in I^+$ and $a \leq b$, then by the definition of $\circ$, $a \circ (a \rightarrow b) = a \circ b = a \vee b = b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. If $s \in I$, then by $(P4), c \leq b$. If $s \in I^+$, then by the definition of $\circ$, $c \leq a \vee c = a \circ c \leq b = a \rightarrow b$.

(2) If $j = i^+ \in I^+, k \in I$ and $i \geq k$, then by the definition of $\circ$, $a \circ (a \rightarrow b) = a \circ b = b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. Suppose that $s \in I^+$. Then by the definition of $\circ$, $a \circ c = a \vee c > b$, a contradiction. If $s \in I$, then by the definition of $\circ$, $a \circ c \in \{a, c\}$, which together with $a > b$, derives that $c = a \circ c \leq b = a \rightarrow b$.

Case 5. $j = i^+, k \in I^+$ and $a \nleq b$, or $j = i^+ \in I^+, k \in I$ and $i < k$. We need only to check the following subcases:

(1) If $j = i^+, k \in I^+$ and $a \nleq b$, then by the definition of $\circ$ and $(P4)$, $a \circ (a \rightarrow b) = a \circ i = i \leq b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. Suppose that $s \in I^+$. Then $a \circ c = a \vee c \nleq b$, a contradiction. If $s \in I$, then by the definition of $\circ$, $a \circ c \in \{a, c\}$, which together with $a \nleq b$, derives that $c = a \circ c \leq b$.

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(2) If $j = i^+ \in I^+, k \in I$ and $i < k$, then by the definition of $\circ$ and $(P2)$, $a \circ (a \to b) = a \circ i = i \leq b$. Let $c \in A_s \subseteq L$ such that $a \circ c \leq b$. Suppose that $s \in I^+$. Then by the definition of $\circ$, $a \circ c = a \vee c > b$, a contradiction. If $s \in I$, then by the definition of $\circ$, $a \circ c \in \{a, c\}$, which together with $a > b$, derives that $a \circ c = c$. Thus $s \leq i$, whence $c \leq i$. □

Next we shall prove that any conic idempotent CRL is isomorphic to some $\mathbf{J} \bigotimes \mathcal{A}$. Suppose that $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, e)$ is a conic idempotent CRL. Let $L^* = \{j \in L : (\exists a \in L) j = a^*\}$, $I = \{i \in L^* : i \leq e\} = L^{*-}$ and $I^* = \{i^* : i \in I \setminus \{e\}\} = L^{*+}$. Let $\mathcal{Y} = \{(L_j, \leq) : j \in L^*\}$. By Proposition 2, for all $i \in I \setminus \{e\}$, $i^* > e$, so $I^* \cap I = \emptyset$. If $i, l \in I$ such that $i \neq l$, then there exist $a, b \in L$ such that $a^* = i$ and $b^* = l$, so $i^{**} = a^{***} = a^* = i \neq l = b^* = b^{***} = l^{**}$. Thus $i^* \neq l^*$.

Lemma 6. *$(I, I^*, L^*; \mathcal{Y})$ is a CE-system.*

Proof. By Theorem 1(1-5,7), $(I, I^*, L^*; \mathcal{Y})$ is a CE-system. □

Theorem 4. *$\mathbf{L}$ is equal to $\mathbf{L}^* \bigotimes \mathcal{Y}$.*

Proof. For convenience, we denote by $\leq_1$ the imposed ordering on $L^* \bigotimes \mathcal{Y}$. We need only to prove that for all $a, b \in L$, $\leq = \leq_1$ and $a \cdot b = a \circ b$.

We now prove $\leq = \leq_1$. Let $a, b \in L$. Assume that $a \leq b$. We need to consider three cases:

(1) If $a \leq e, b \leq e$, then $a^{**}, b^{**} \in I$ by Lemma 3(4) and by Theorem 1(6), $a^{**} \leq b^{**}$, which together with $a \in L_{a^{**}}$ and $b \in L_{b^{**}}$, derives that $a \leq_1 b$ by $(P1-2)$.

(2) If $a \geq e, b \geq e$, then $a^*, b^* \in I$, which together with $a \in L_{a^{**}}$ and $b \in L_{b^{**}}$, derives that $a^* \geq b^*$ by Theorem 1(6). Thus by $(P3)$, $a \leq_1 b$.

(3) If $a \leq e$ and $b > e$, then by Lemma 3(4), $a^{**} \leq e$ and $b^{**} > e$, so $a^{**} \in I$ and $b^{**} \in I^*$, whence by $(P4)$, $a \leq_1 b$.

Thus $\leq \subseteq \leq_1$.

Suppose that $a \leq_1 b$. We need to consider four cases:

(1) If $a^{**} = b^{**} \in I$, then $a \leq b$ by $(P1)$.

(2) If $a^{**}, b^{**} \in I$ such that $a^{**} < b^{**}$, then by Theorem 1(6), $a \leq b$.

(3) If $a^{**}, b^{**} \in I^*$ such that $a^* > b^*$, then by Theorem 1(6), $a \leq b$.

(4) If $a^{**} \in I$ and $b^{**} \in I^*$, then by Lemma 3(4), $a \leq e$ and $b > e$, so $a \leq b$.

Thus $\leq_1 \subseteq \leq$, whence $\leq_1 = \leq$

It remains to verify $a \cdot b = a \circ b$ for all $a, b \in L$. For this, we need to consider three cases:

(1) If $a \leq e, b \leq e$, then by Lemma 1(3), $a \cdot b = a \wedge b$. On the other hand, by the definition of $\circ$ and $\leq = \leq_1$, $a \circ b = a \wedge b$, whence $a \cdot b = a \circ b$.

(2) If $a > e, b > e$, then by similar arguments as in (1), $a \cdot b = a \circ b$.

(3) $a > e$ and $b \leq e$.

- If $b \leq a^*$, then $a^*, b^{**} \in I$ by Lemma 3(4) and $b^{**} \leq a^{***} = a^*$ by Lemma 2(3), which together with $a \in L_{(a^*)^*}$ and $b \in L_{b^{**}}$ derives $a \circ b = b$ by the definition of $\circ$. On the other hand, $a \cdot b = a \cdot a^* \cdot b = a^* \cdot b = b$ by Proposition 2(3). Hence $a \cdot b = b = a \circ b$.
- If $b > a^*$, then $a^*, b^{**} \in I$ by Lemma 3(4) and $b^{**} \geq b > a^{***} = a^*$ by Theorem 1(4), which together with $a \in L_{(a^*)^*}$ and $b \in L_{b^{**}}$ derives $a \circ b = a$ by the definition of $\circ$. Suppose that $a \cdot b = b$. Then $a^* <_n b <_n a$, so by Proposition 2(4), $b > e$, a contradiction. Thus $a \cdot b = a$ by Lemma 1(4,5). Hence $a \cdot b = a = a \circ b$.

□

By Theorem 4, we have the following result, which generalizes [20, Theorem 20].

Theorem 5. *Let $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, e)$ be a* CRL. *The following conditions are equivalent:*

(I) *$\mathbf{L}$ is a subdirectly irreducible idempotent semiconic* CRL.

(II) *There exists a CE-system $(I, I^+, J; \mathcal{A})$ such that*

(1) *A_e is a nontrivial subdirectly irreducible Brouwerian algebra or $A_e = \{e\}$ and there exists $i \in I$ such that $i \prec e$ in I;*

(2) $\boldsymbol{L} \cong \boldsymbol{J} \bigotimes \mathcal{A}$.

Proof. Let $\mathbf{L}$ be a subdirectly irreducible semiconic idempotent CRL. Then since semiconic idempotent CRL is the variety generated by conic idempotent CRLs, $\mathbf{L}$ is conic. By Theorem 4, $\mathbf{L} \cong \mathbf{L}^* \bigotimes \mathcal{Y}$, where $(I, I^*, L^*; \mathcal{Y})$ is a CE-system. Because $\mathbf{L}$ is a subdirectly irreducible CRL, the set $\{a \in L : a < e\}$ has a greatest element. Let $i = \max\{a \in L : a < e\}$. If $i \in L_e$, then $i = \max\{a \in L_e : a < e\}$, so by Theorem 1(4), $\mathbf{L}_e$ is a nontrivial subdirectly irreducible Brouwerian algebra. If $i \notin L_e$, then since $L_e \subseteq L^-$, $L_e = \{e\}$ and $i \prec e$, so $i^{**} < e$, which implies that $i^{**} \le i$. On the other hand, by Proposition 2(1), $ii^* = i \le e$, so $i \le i^* \to e = i^{**}$. Thus $i = i^{**}$, whence $i \in I$ by Theorem 1(1).

Conversely, let $(I, I^+, J; \mathcal{A})$ be a CE-system such that (1) and (2). Then by Theorem 3, $\mathbf{L}$ is a conic idempotent CRL. If A_e is a nontrivial subdirectly irreducible Brouwerian algebra, then $\max\{a \in A_e : a < e\}$ exists and so $\max\{a \in A_e : a < e\} = \max\{a \in L : a < e\}$, which implies that $\mathbf{L}$ is a subdirectly irreducible semiconic idempotent CRL. If $A_e = \{e\}$ and there exists $i \in I$ such that $i \prec e$, then by $(P1, 2)$, $\max\{a \in L : a < e\} = i$, which implies that $\mathbf{L}$ is a subdirectly irreducible semiconic idempotent CRL. □

5. The Amalgamation Property

In this section we will use the structure theorem of conic idempotent CRLs to give some new result about the amalgamation property of the variety of semiconic idempotent CRLs, which generalize the main results of [11].

Let $\mathbf{K}$ be a class of algebras. A *span* is a pair of embeddings $\langle i_1 : \mathbf{A} \hookrightarrow \mathbf{B}, i_2 : \mathbf{A} \hookrightarrow \mathbf{C} \rangle$ between algebras $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{K}$. The class $\mathbf{K}$ is said to have the *amalgamation property* if for every span of $\mathbf{K}$, there exist an *amalgam* $\mathbf{D} \in \mathbf{K}$ and embeddings $j_1 : \mathbf{B} \hookrightarrow \mathbf{D}$ and $j_2 : \mathbf{C} \hookrightarrow \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$.

Example 1. *Let $A = \{a_2, a_1, e, a_{-1}, a_{-2}\}$. We define an order relation $\le_A$ on A by $a_{-2} <_A a_{-1} <_A e <_A a_1 <_A a_2$, see Figure 1.1. We can define a multiplication operation on A by for all $i, j \in \{1, 2, -1, -2\}$,*

$$a_i a_j = a_j a_i = \begin{cases} a_i & \text{if } |j| < |i|, \\ a_i & \text{if } i = j, \\ a_i & \text{if } i = -j < 0; \end{cases}$$

and $ae = ea = a$ for all $a \in A$. Let $B = \{x_{-2}, x_{-1}, e, x_1, y_2, z_2, x_2\}$. We define an order relation $\le_B$ on B by $x_{-2} <_B x_{-1} <_B e <_B x_1 <_B y_2, z_2 <_B x_2$, see Figure 1.2. We can define a multiplication operation on B by for all $i, j \in \{1, 2, -1, -2\}$ and $b \in \{y, z\}$,

$$x_i x_j = x_j x_i = \begin{cases} x_i & \text{if } |j| < |i|, \\ x_i & \text{if } i = j, \\ x_i & \text{if } i = -j < 0; \end{cases}$$

$$x_i b_2 = b_2 x_i = \begin{cases} b_2 & \text{if } |i| < 2, \\ x_i & \text{if } |i| = 2; \end{cases}$$

$y_2 z_2 = z_2 y_2 = x_2$ and $ce = ec = c$ for all $c \in B$. Let $C = \{m_{-3}, m_{-2}, m_{-1}, e, m_1, m_2, n_3, k_3, m_3\}$. We define an order relation $\le_C$ on C by $m_{-3} <_C m_{-2} <_C m_{-1} <_C e <_C m_1 <_C m_2 <_C n_3, k_3 <_C m_3$, see Figure 1.3. We can define a multiplication operation on C by for all $i, j \in \{1, 2, 3, -1, -2, -3\}$ and $b \in \{n, k\}$,

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$$m_i m_j = m_j m_i = \begin{cases} m_i & \text{if } |j| < |i|, \\ m_i & \text{if } i = j, \\ m_i & \text{if } i = -j < 0; \end{cases}$$

$$m_i b_3 = b_3 m_i = \begin{cases} b_3 & \text{if } |i| < 3, \\ m_i & \text{if } |i| = 3; \end{cases}$$

$n_3 k_3 = k_3 n_3 = m_3$ and $ce = ec = c$ for all $c \in C$. We define a division operation on P by $a \to b = \max\{p \in P \mid ap \leq b\}$ for all $a, b \in P$, where $P \in \{A, B, C\}$. It's easy to see that $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are subdirectly irreducible semiconic idempotent CRLs*. We define two maps as follows: $\varphi_1 : \mathbf{A} \longrightarrow \mathbf{B}$; $e \mapsto e$ and $a_i \mapsto x_i$ for $i \in \{-2, -1, 1, 2\}$; $\varphi_2 : \mathbf{A} \longrightarrow \mathbf{C}$; $e \mapsto e$, $a_i \mapsto m_i$ for $i \in \{-1, 1\}$; $a_2 \mapsto m_3$ and $a_{-2} \mapsto m_{-3}$. It's clear that φ_1, φ_2 are embeddings of $\mathbf{A}$ into $\mathbf{B}, \mathbf{C}$, respectively. We claim that there doesn't an amalgam in $\mathbf{K}$ where $\mathbf{K}$ is the class of all conic idempotent* CRLs*. Suppose that there exist an amalgam $\mathbf{D} \in \mathbf{K}$ and embeddings $\psi_1 : \mathbf{B} \hookrightarrow \mathbf{D}$ and $\psi_2 : \mathbf{C} \hookrightarrow \mathbf{D}$ such that $\psi_1 \varphi_1 = \psi_2 \varphi_2$. Then $\psi_1(x_1) = \psi_1 \varphi_1(a_1) = \psi_2 \varphi_2(a_1) = \psi_2(m_1)$ and $\psi_1(x_2) = \psi_1 \varphi_1(a_2) = \psi_2 \varphi_2(a_2) = \psi_2(m_3)$. Hence by Theorem 2, $\psi_2(m_1) = \psi_1(x_1) \prec \psi_1(x_2) = \psi_2(m_3)$ in D^*. But $\psi_2(m_1) < \psi_2(m_2) < \psi_2(m_3)$ in D^*. It's a contradiction. We conclude that the span $\langle \varphi_1 : \mathbf{A} \longrightarrow \mathbf{B}, \varphi_2 : \mathbf{A} \longrightarrow \mathbf{C} \rangle$ hasn't an amalgam in $\mathbf{K}$.*

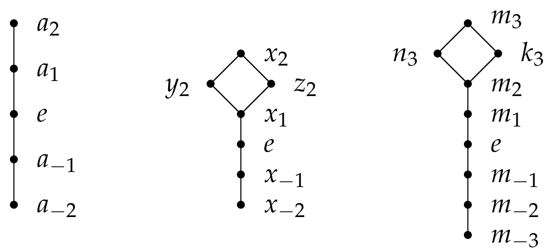

Fig. 1.1. $(A, \leq_A)$ Fig. 1.2. $(B, \leq_B)$ Fig. 1.3. $(C, \leq_C)$

Figure 1

By Example 1, we have the following result.

Proposition 4. *The class of all conic idempotent* CRLs *and the class of subdirectly irreducible semiconic idempotent* CRLs *haven't the amalgamation property.*

Definition 2. *The variety of strongly semiconic idempotent* CRLs *consists of the semiconic idempotent* CRLs *that satisfy $(x \wedge y)^* = x^* \vee y^*$.*

Proposition 5. *A conic idempotent* CRL $\mathbf{L}$ *is a strongly semiconic idempotent* CRL *if and only if L_i is a lattice for all $i \in L^*$.*

Let $\mathbf{L}$ be a CRL. A lattice filter F of $\mathbf{L}$ is called *normal* if it contains e and it's closed under multiplication. A normal filter F of $\mathbf{L}$ is said to be *prime* if it is prime in the usual lattice-theoretic sense; that is, whenever $x \vee y \in F$, then $x \in F$ or $y \in F$. Let F and Θ be a normal filter and a congruence of $\mathbf{L}$ respectively. It's well known that $\Theta_F = \{(x, y) \in L^2 \mid (x \to y) \wedge (y \to x) \in F\}$ is a congruence of $\mathbf{L}$ and the upper set $F_\Theta = \uparrow [e]_\Theta$ of the equivalence class $[e]_\Theta$ is a normal filter. Moreover:

Lemma 7. [2] *The lattice $\mathcal{NF}(\mathbf{L})$ of normal filters of a* CRL $\mathbf{L}$ *is isomorphic to its congruence lattice $Con(\mathbf{L})$. The isomorphism is given by the mutually inverse maps $F \mapsto \Theta_F$ and $\Theta \mapsto \uparrow [e]_\Theta$.*

Lemma 8. [2] *Let $\mathbf{L}$ be a* CRL *and let F be a normal filter of $\mathbf{L}$. Then $[e]_{\Theta_F} = \{x \mid x \wedge (x \to e) \wedge e \in F\} = \{x \mid \exists a \in F^-, a \leq x \leq a \to e\}$.*

In what follows, if F is a normal filter of $\mathbf{L}$, $\mathbf{L}/F$ shall always denote the quotient algebra $\mathbf{L}/\Theta_F$. Given a element $x \in L$, we write $[x]_F$ or $[x]$ if no confusion for the equivalence class of x in L/F.

Lemma 9. *Let $\mathbf{L}$ be a semiconic* CRL, *and let F be a normal filter of $\mathbf{L}$. Then the following statements are equivalent*

(1) *F is prime.*
(2) *For all $a, b \in L^-$, whenever $a \vee b \in F$, then $a \in F$ or $b \in F$.*
(3) *$\mathbf{L}/F$ is a finitely subdirectly irreducible conic* CRL.

Proof. (1) $\Rightarrow$ (2) By specialization.

(2) $\Rightarrow$ (3) Suppose that (2) holds, and let $a \in L$. Since $\mathbf{L}$ is semiconic, $(a \wedge e) \vee (a \to e \wedge e) = e \in F$. It follows that either $a \wedge e \in F$ or $a \to e \wedge e \in F$. If $a \wedge e \in F$, then by Lemma 8, $[a \wedge e] = [e] \Longrightarrow [a] \wedge [e] = [e] \Longrightarrow [a] \geq [e]$. If $a \to e \wedge e \in F$, then $[a \to e \wedge e] = [e] \Longrightarrow [a] \to [e] \wedge [e] = [e] \Longrightarrow [a] \to [e] \geq [e] \Longrightarrow [a] \leq [e]$. Thus $\mathbf{L}/F$ is a conic CRL. Let $a, b \in L$ such that $[a] \vee [b] = [e]$. Then since $\mathbf{L}$ is conic, $([a] \vee [b]) \wedge [e] = [e] \Longrightarrow ([a] \wedge [e]) \vee ([b] \wedge [e]) = [e] \Longrightarrow [(a \wedge e) \vee (b \wedge e)] = [e]$, which implies that $(a \wedge e) \vee (b \wedge e) \in F$. Hence $a \wedge e \in F$ or $b \wedge e \in F$, which derives that $[a \wedge e] = [e]$ or $[b \wedge e] = [e]$. Since $[a \vee b] = [e], [a] \leq [e]$ and $[b] \leq [e]$. It follows that $[a] = [a] \wedge [e] = [a \wedge e] = [e]$ or $[b] = [b] \wedge [e] = [b \wedge e] = [e]$. Consequently, $\mathbf{L}/F$ is a finitely subdirectly irreducible conic CRL.

(3) $\Rightarrow$ (1) Assume that (3) holds, and let $a, b \in L$ such that $a \vee b \in F$. Then $(a \vee b) \wedge e = (a \wedge e) \vee (b \wedge e) \in F$. It follows that $[(a \wedge e) \vee (b \wedge e)] = [e] \Longrightarrow [a \wedge e] \vee [b \wedge e] = [e] \Longrightarrow [a \wedge e] = [e]$ or $[b \wedge e] = [e] \Longrightarrow a \wedge e \in F$ or $b \wedge e \in F \Longrightarrow a \in F$ or $b \in F$. Thus F is prime. $\square$

Lemma 10. [2] *Let $\mathbf{L}$ be a residuated lattice and $\{a_i \mid 1 \leq i \leq n\}, \{b_j \mid 1 \leq j \leq m\} \subseteq L^-$ finite subsets of the negative cone of $\mathbf{L}$ with the property that $a_i \vee b_j = e$, for any i and j. Then $(\prod_{i=1}^n a_i) \vee (\prod_{j=1}^m b_j) = e$.*

Lemma 11. [15] *Let $\mathbf{U}$ be a subclass of a variety $\mathbf{V}$ satisfying the following conditions:*

(i) *Every subdirectly irreducible member of $\mathbf{V}$ is in $\mathbf{U}$.*
(ii) *$\mathbf{U}$ is closed under isomorphisms and subalgebras.*
(iii) *For any algebra $\mathbf{B} \in \mathbf{V}$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if $\Theta \in Con(\mathbf{A})$ and $\mathbf{A}/\Theta \in \mathbf{U}$, then there exists $\Phi \in Con(\mathbf{B})$ such that $\Phi \bigcap A^2 = \Theta$ and $\mathbf{B}/\Phi \in U$.*
(iV) *Every span in $\mathbf{U}$ has an amalgam in $\mathbf{V}$.*

Then $\mathbf{V}$ has the amalgamation property.

We have the following result, which generalizes [15, Theorem 49] in the commutative case.

Theorem 6. *Let $\mathbf{V}$ be a variety of semiconic* CRLs, *and suppose that the class of finitely subdirectly irreducible conic* CRLs *in $\mathbf{V}$ has the amalgamation property. Then $\mathbf{V}$ has the amalgamation property.*

Proof. It is well known that every subdirectly irreducible semiconic CRL is a finitely subdirectly irreducible conic CRL. It is clear that the class of finitely subdirectly irreducible conic CRL is closed under isomorphisms and subalgebras. By Lemma 11, we need only to prove that for any $\mathbf{B} \in \mathbf{V}$, any subalgebra $\mathbf{A}$ of $\mathbf{B}$, and $P \in \mathcal{NF}(\mathbf{A})$ such that $\mathbf{A}/P$ is a finitely subdirectly irreducible conic CRL, there is $Q \in \mathcal{NF}(\mathbf{B})$ such that $Q \cap A = P$ and $\mathbf{B}/Q$ is a finitely subdirectly irreducible conic CRL. Since $\mathbf{V}$ has the congruence extension property, there is a normal filter F of $\mathbf{B}$, such that $P = F \cap A$. Let

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$\mathcal{X}$ denote the poset, under set-inclusion, of all set-inclusion, of all normal filters of $\mathbf{B}$ whose intersection with A is P. Since $F \in \mathcal{X}$, $\mathcal{X} \neq \emptyset$. By Zorn's lemma, element Q. Next, we shall show that Q is a prime normal filter of $\mathbf{B}$. Suppose otherwise, and let $x, y \in B^-$ be such that $x \vee y \in Q$ but $x \notin Q$ and $y \notin Q$. Let Q_x and Q_y be the normal filters of $\mathbf{B}$ generated by $Q \cup \{x\}$ and $Q \cup \{y\}$, respectively. Then, by the maximality of Q, P is a proper subset of the normal filters $Q_x \cap A$ and $Q_y \cap A$ of $\mathbf{A}$ and so there exist elements $c, d \in A \setminus P$, $q, r \in Q^-$ and $n, m \in \mathbb{Z}^+$ such that $qx^n \leq c \leq e$, and $ry^m \leq d \leq e$. Hence by Lemma 8, $[q]_Q = [r]_Q = [e]_Q$ and $x \vee y \in Q \cap B^- \implies [x \vee y]_Q = [e]_Q$. Thus by Lemma 10, $[e]_Q = [x^n]_Q \vee [y^m]_Q = [q]_Q[x^n]_Q \vee [r]_Q[y^m]_Q = [qx^n]_Q \vee [ry^m]_Q = [qx^n \vee ry^m]_Q \leq [c \vee d]_Q \leq [e]_Q$. It follows that $[c \vee d]_Q = [e]_Q$. Since $P = Q \cap A$, the map $\varphi : \mathbf{A}/P \to \mathbf{B}/Q$ is an embedding, which together with $c \vee d \in A$ derives that $[c]_P \vee [d]_P = [c \vee d]_P = [e]_P$. Because $\mathbf{A}/P$ is a finitely subdirectly irreducible conic CRL, $[c]_P = [e]_P$ or $[d]_P = [e]_P$. Then by Lemma 8, $c \in P$, or $d \in P$. But $c, d \notin P$, which is a contradiction. Thus Q is a prime normal filter of $\mathbf{B}$, and by Lemma 9, $\mathbf{B}/Q$ is a finitely subdirectly irreducible conic CRL. The proof of the theorem is complete. □

Lemma 12. [11] *The class of totally ordered Sugihara monoids has the amalgamation property.*

The following result is essentially due to Maksimova (see [6, Chapter 6]).

Lemma 13. *(Maksimova) The variety all Brouwerian algebras has the amalgamation property and the class of finitely subdirectly irreducible Brouwerian algebras has the amalgamation property.*

Theorem 7. *The class of finitely subdirectly irreducible strongly conic idempotent* CRLs *has the amalgamation property.*

Proof. Let $\langle i_1 : \mathbf{A} \hookrightarrow \mathbf{B}, i_2 : \mathbf{A} \hookrightarrow C\rangle$ be a span of finitely subdirectly irreducible strongly conic idempotent CRLs, assuming without loss of generality that i_1 and i_2 are inclusion maps and that $B \cap C = A$. Then using Theorem 1(11), we also have inclusions between their skeletons $\mathbf{A}^* \hookrightarrow \mathbf{B}^*$ and $\mathbf{A}^* \hookrightarrow C^*$. Since by Theorem 1(11), these skeletons are totally ordered odd Sugihara monoids, Lemma 12 yields an amalgam J for this span that is also a totally ordered odd Sugihara monoid. Moreover, we may assume that $J = B^* \cup C^*$. Let $J^- = \{j \in J \mid j \leq e\}$ and $J^+ = \{j \in J \mid j > e\}$.

Consider $i \in A^*$. Recalling that $A_i = \{x \in A \mid x^{**} = i\}$, clearly $A_i \subseteq B_i = \{x \in B \mid x^{**} = i\}$ and $A_i \subseteq C_i = \{x \in C \mid x^{**} = i\}$. If $i = e$, then by Theorem 1(10), $\mathbf{A}_e$, $\mathbf{B}_e$ and $\mathbf{C}_e$ are finitely subdirectly irreducible Brouwerian algebras and by Theorem 2, $\mathbf{A}_e$ is a subalgebra of $\mathbf{B}_e$ and $\mathbf{C}_e$. Hence by Lemma 13, there exists a finitely subdirectly irreducible Brouwerian algebra $\mathbf{D}_e$ as an amalgam with $D_e = B_e \cup C_e$. If $i < e$, then by Lemma 13, there exists a Brouwerian algebra $\mathbf{D}_i$ as an amalgam with $\mathbf{D}_i = \mathbf{B}_i \cup C_i$. If $i > e$, then by Proposition 5, each of $\mathbf{B}_i$ and $\mathbf{C}_i$ is a lattice. It is well known that class of lattices has the amalgamation property. It follows that there exists a lattice $\mathbf{D}_i$ as an amalgam with $D_i = B_i \cup C_i$. Since i is the greatest element of A_i, B_i and C_i, it is also the greatest element of D_i. Now, for all $j \in B^* \setminus A^*$ and $k \in C^* \setminus A^*$, let $D_j = B_j$ and $D_k = C_k$. Let $\mathcal{X} = \{(D_j, \leq_{D_j}) \mid j \in J\}$. By construction, $(J^-, J^+, J; \mathcal{X})$ is a CE-system. Thus $\mathbf{D} = \mathbf{J} \bigotimes \mathcal{X}$ is a conic idempotent CRL. Since $\mathbf{D}_e$ is a finitely subdirectly irreducible Brouwerian algebra, $\mathbf{D} = \mathbf{J} \bigotimes \mathcal{X}$ is a finitely subdirectly irreducible conic idempotent CRL. By Proposition 5, $\mathbf{D} = \mathbf{J} \bigotimes \mathcal{X}$ is strongly finitely subdirectly irreducible conic idempotent CRL. To show that $\mathbf{D}$ is an amalgam of the original span, it suffices to check that $\mathbf{B}$ and $\mathbf{C}$ are subalgebras of $\mathbf{D}$. Consider $x, y \in B$ with $x \in B_i, y \in B_j$. Then $i, j \in J$. If $i < j$ in $B^* \subseteq J$, then $x \leq_B y$ and $x \leq_D y$, so $x \vee^D y = y = x \vee^B y$ and $x \wedge^D y = x = x \wedge^B y$. If $i = j \in B^{*+} \subseteq J^+$, then since $\mathbf{D}_i$ is a lattice and $\mathbf{B}_i$ is a sublattice of $\mathbf{D}_i$, $x \vee^D y = x \vee^{D_i} y = x \vee^{B_i} y = x \vee^B y$ and $x \wedge^D y = x \wedge^{D_i} y = x \wedge^{B_i} y = x \wedge^B y \in B$. If $i = j \in B^{*-} \subseteq J^-$, then since $\mathbf{D}_i$ is a Brouwerian algebra and $\mathbf{B}_i$ is a subalgebra of $\mathbf{D}_i$, $x \vee^D y = x \vee^{D_i} y = x \vee^{B_i} y = x \vee^B y$ and $x \wedge^D y = x \wedge^{D_i} y = x \wedge^{B_i} y = x \wedge^B y$. Thus $\mathbf{B}$ is a sublattice of $\mathbf{D}$. By the definition of $\mathbf{D}$, we have

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$$x \circ^B y = \begin{cases} x \wedge^B y & \text{if } i,j \in B^{*-} \subseteq J^-, \\ x \vee^B y & \text{if } i,j \in B^{*+} \subseteq J^+, \\ x & \text{if } i \in B^{*+} \subseteq J^+, j \in B^{*-} \subseteq J^-, i^* <_{B^*} j \text{ or } i \in B^{*-} \subseteq J^-, j \in B^{*+} \subseteq J^+, i \leq_{B^*} j^*, \\ y & \text{if } i \in B^{*+} \subseteq J^+, j \in B^{*-} \subseteq J^-, i^* \geq_{B^*} j \text{ or } i \in B^{*-} \subseteq J^-, j \in B^{*+} \subseteq J^+, i >_{B^*} j^*. \end{cases}$$

and

$$x \circ^D y = \begin{cases} x \wedge^D y & \text{if } i,j \in J^-, \\ x \vee^D y & \text{if } i,j \in J^+, \\ x & \text{if } i \in J^+, j \in J^-, i^* <_J j \text{ or } i \in J^-, j \in J^+, i \leq_J j^*, \\ y & \text{if } i \in J^+, j \in J^-, i^* \geq_J j \text{ or } i \in J^-, j \in J^+, i >_J j^*. \end{cases}$$

Thus $x \circ^D y = x \circ^B y$.
By the similar arguments, we have $x \to^D y = x \to^B y$.
The proof that **C** is a subalgebra of **D** is symmetrical. □

Since every variety of commutative residuated lattices has the congruence extension property, by Theorem 6, we have the following result ,which generalizes [11, Theorem 5.6].

Theorem 8. *The variety of strongly semiconic idempotent* CRLs *has the amalgamation property.*

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