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Advancing Fractional Riesz Derivatives through Dunkl Operator

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Abstract: This work aims to introduce a novel concept: the Riesz-Dunkl fractional derivatives, within the context of Dunkl type operators. A particularly noteworthy revelation is that when a specific parameter κ equals zero, the Riesz-Dunkl fractional derivative smoothly reduces to both the well-known Riesz fractional derivative and the fractional second-order derivative. Furthermore, we introduce a new concept: the fractional Sobolev space. This space is defined and characterized using the versatile framework of the Dunkl transform.

Keywords: fractional calculus; difference-differential operator; special function

1. Intoduction

In the real line, the Dunkl operator $\mathscr{D}_\kappa$ is formed by combining the standard derivative with a term involving the parameter κ ($\kappa \in \mathbb{R}$) and the reflection operator [11]. Specifically, it takes the form:

$$\mathscr{D}_\kappa := \frac{d}{dx} + \frac{\kappa}{x}(1-s),$$

where the reflection operator s acts on a real variable function $f(x)$ as follows:

$$(sf)(x) := f(-x).$$

The term given by $\frac{\kappa}{x}(1-s)$ captures the interplay between the derivative and the reflection operation. This extension finds significant applications in areas such as mathematical physics, harmonic analysis and approximation theory [1–3,11–14,18]. One of the remarkable consequences of the Dunkl operator is the Dunkl transform [11,12,18]. Analogous to the Fourier transform, the Dunkl transform functions as an isometry that maps functions in the L^2_κ space onto themselves, preserving their inner product structure. Moreover, the introduction of the generalized translation operator τ^y by Trimeche [18] expands the notion of standard translation into the context of Dunkl operators. Particularly valuable when dealing with functions in the $L^2_\kappa(\mathbb{R})$ space, this operator facilitates meaningful operations in line with the deformations introduced by the Dunkl operator.

When considering the one-dimensional case, the fractional Laplacian [6,9] is commonly referred to as the Riesz fractional derivative. For our purposes, we will use the notation D_α to represent this operator [7]. Interestingly, the symbols associated with both the fractional Laplacian and the Riesz fractional derivative appear identical. This relationship is captured by the following equation [8]:

$$D_\alpha = \mathcal{F}^{-1}|x|^\alpha \mathcal{F} = -(-\frac{d^2}{dx^2})^{\alpha/2}. \tag{1.1}$$

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Building upon S. Bochner's investigations into the generalization of standard diffusion to encompass generalized diffusion equations for L'evy stable densities. Explicit expressions can be written in a regularized integral form valid for $\alpha \in (0,2)$ [10]:

$$D_\alpha f(x) = -(-\frac{d^2}{dx^2})^{\alpha/2} f(x) = \frac{\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \int_0^\infty \frac{2f(x) - f(x+y) + f(x-y)}{y^{\alpha+1}} dy. \tag{1.2}$$

Furthermore, the operator $-\frac{d^2}{dx^2}$ is widely recognized as an unbounded self-adjoint operator with its spectrum spanning the continuous interval $[0,\infty)$. It plays a pivotal role in the expansion of the cosine function. Intriguingly, a noteworthy result in [4] demonstrates that the fractional second-order derivative can also be interpreted as a pseudo-differential operator via the Fourier-cosine transform. This equivalence holds when considering even functions, resulting in the same representation as provided in equation (1.2). As a consequence, the Riesz fractional derivative is inherently a symmetric property, aligning seamlessly with the requirement of even functions.

The main objective of this work is to extend the representation (1.2) for the Riesz fractional derivative by introducing a one-parameter extension. Our methodology is inspired by the Dunkl transform and the generalized translation operator, drawing parallels with the approach presented by Butzer et al. in [5]. We initiate our exploration by establishing a pointwise formula for the Riesz-Dunkl fractional derivative. Notably, this formulation remains valid even for Schwartz functions. This foundational formula serves as the cornerstone of our subsequent analysis and results, underscoring its pivotal role in our investigation. Our work unveils a compelling alignment between the Riesz-Dunkl fractional derivative, the conventional Riesz fractional derivative, and the fractional second-order derivative. This equivalence holds true beyond the realm of even functions, extending the applicability of our findings within the Schwartz space. Moreover, our examination of even functions reveals a fascinating parallel between the Riesz-Dunkl fractional derivative and the Bessel fractional derivative. Furthermore, we introduce a characterization of the fractional Sobolev space associated with the Dunkl operator. This characterization not only sheds light on the behavior of the Dunkl operator within fractional Sobolev spaces but also deepens our understanding of its intrinsic properties.
The paper's structure is outlined as follows:
In Section 2, we lay the foundation by introducing key concepts. We delve into the realm of the Dunkl transform and the generalized translation operator. Section 3 presents the primary outcomes of our research. This section offers a concise summary of the significant findings we have achieved. Section 4 offers a comprehensive proof of the main results. Detailed derivation and explanation establish the validity of our findings, aiming to provide readers with a comprehensive understanding of the mathematical underpinnings.

2. Preliminaries

Before unveiling our main findings, it is crucial to lay the foundation by introducing essential notations and gathering relevant information about the Bessel operator. This section functions as a primer, shedding light on the importance of the Fourier-Bessel transform and the Delsarte translation. These concepts will play a pivotal role in our subsequent analysis.

Let $\kappa \geq 0$, and f, be a differentiable function $\mathbb{R}$. The Dunkl derivative $\mathscr{D}_\kappa f(x)$ is defined by [12,14]

$$\mathscr{D}_\kappa f(x) = \begin{cases} f'(x) + \kappa \frac{f(x)-f(-x)}{x}, & \text{if} \quad x \neq 0, \\ (\kappa+1)f'(0) & \text{if} \quad x = 0. \end{cases} \tag{2.1}$$

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For each $\lambda \in \mathbb{C}$, the following problem [11,14]

$$\begin{cases} \mathscr{D}_\kappa u(x) = -i\lambda u(x), \\ u(0) = 1. \end{cases}$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted by $\mathscr{E}_\kappa(x)$ given by

$$\mathscr{E}_\kappa(x) := \mathscr{J}_{\kappa-1/2}(ix) + \frac{x}{2\kappa}\mathscr{J}_{\kappa+1/2}(ix). \tag{2.2}$$

where $\mathscr{J}_\kappa$ the normalized Bessel functions is defined by [18]

$$\mathscr{J}_\kappa(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\kappa+n+1)}(\frac{x}{2})^{2n}, \kappa > -1.$$

We denote by $L^p_\kappa(\mathbb{R})$ $(1 \le p)$, the Lebesgue space associated with the measure

$$\sigma_\kappa(dx) = \frac{|x|^{2\kappa}}{2^{\kappa+1/2}\Gamma(\kappa+1/2)}\,dx \tag{2.3}$$

and by $\|f\|_{p,\kappa}$ the usual norm given by

$$\|f\|_{p,\kappa} = \Big(\int_{\mathbb{R}} |f(\xi)|^p\,\sigma_\kappa(d\xi)\Big)^{1/p}. \tag{2.4}$$

The Dunkl transform is defined by [11,12,18]

$$\mathscr{F}_\kappa\{f(x),\xi\} = \mathscr{F}_\kappa(f)(\xi) := \int_{\mathbb{R}} f(x)\,\mathcal{E}_\kappa(-i\xi x)\sigma_\kappa(dx). \tag{2.5}$$

The *Dunkl* transform can be extended to an isometry of $L^2_\kappa(\mathbb{R})$, that is [12]

$$\int_{\mathbb{R}} |f(x)|^2\,\sigma_\kappa(dx) = \int_{\mathbb{R}} |\widehat{f}_\kappa(\lambda)|^2\,\sigma_\kappa(d\lambda). \tag{2.6}$$

For any $f \in L^1_\kappa(\mathbb{R}) \cap L^2_\kappa(\mathbb{R})$, the inverse is given by

$$f(x) = \int_{\mathbb{R}} \widehat{f}_\kappa(\lambda)\,\mathcal{E}_\kappa(i\lambda x)\,\sigma_\kappa(d\lambda). \tag{2.7}$$

It's noteworthy that for $\kappa = 0$, the Dunkl kernel simplifies to an exponential function, and the Dunkl transform itself reduces to the conventional Fourier transform:

$$\mathscr{F}_0 f(x) = \mathcal{F}f(x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixt} f(t)\,dt. \tag{2.8}$$

As in the classical case, a generalized translation operator is defined in the Dunkl setting side on $L^2_\kappa(\mathbb{R})$ by Trimèche [18]

$$\mathscr{F}_\kappa \tau^y f(\xi) := \mathscr{E}_\kappa(i\xi y)\mathscr{F}_\kappa f(\xi), \quad y,\,\xi \in \mathbb{R}. \tag{2.9}$$

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Explicitely, the generalized translation $\tau^x f(y)$ takes the explicite form see [14]

$$\tau^x f(y) := \frac{1}{2}\int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt})(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}})h_k(t)dt + \frac{1}{2}\int_{-1}^{1} f(-\sqrt{x^2+y^2-2xyt})(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}})\, h_k(t)\, dt, \tag{2.10}$$

where,

$$h_\kappa(t) = \frac{\Gamma(\kappa+1/2)}{2^{2\kappa}\sqrt{\pi}\Gamma(\kappa)}(1+t)(1-t^2)^{\kappa-1}.$$

We collect below some useful facts :

(i) For all $x, y \in \mathbb{R}$,

$$\tau^x f(y) = \tau^y f(x). \tag{2.11}$$

(ii) For all $x, \xi \in \mathbb{R}$ and $f \in S(\mathbb{R})$,

$$\mathscr{D}_\kappa \tau^x f = \tau^x \mathscr{D}_\kappa f, . \tag{2.12}$$

(iii) For all $x \in \mathbb{R}$ and $f,\ g \in L^2_\kappa(\mathbb{R})$,

$$\int_{\mathbb{R}} \tau^x f(y) g(y) \sigma_\kappa(dy) = \int_{\mathbb{R}} f(y) \tau^{-x} g(y) \sigma_\kappa(dy). \tag{2.13}$$

(iv) For all $x \in \mathbb{R}$ and $1 \leq p \leq 2$, the operator τ^x can be extended to al functions f in $L^p_\kappa(\mathbb{R})$ and the following holds

$$||\tau^x f||_{p,\kappa} \leq ||f||_{p,\kappa}. \tag{2.14}$$

3. Main Results

In this section, we present the main results of this paper. We start by defining the fractional *Dunkl* derivative $(-\mathscr{D}^2_\kappa)^{\alpha/2}$ for $0 < \alpha < 2$ as a nonlocal operator on a suitable function space. To facilitate this, we choose to work with a modified space. Specifically, we consider the spaces

$$\mathscr{H}^\alpha_\kappa(\mathbb{R}) = \left\{ f \in L^2_\kappa(\mathbb{R}) : \int_{\mathbb{R}} |x|^\alpha |\mathscr{F}_\kappa(f)(x)|^2 \sigma_\kappa(dx) < \infty \right\}. \tag{3.1}$$

When $\kappa = 0$, the Dunkl transform reduces to the classical Fourier transform. In this case, the space $\mathscr{H}^\alpha_\kappa(\mathbb{R})$ will turn out to be the fractional Sobolev space, also called the Bessel potential space or Liouville space, of order α [5].

Definition 3.1. *Let $\kappa \geq 0$ and $0 < \alpha \leq 2$. For $f \in \mathscr{H}^\alpha_\kappa(\mathbb{R})$, the fractional* Riesz-Dunkl *derivative $(-\mathscr{D}^2_\kappa)^{\alpha/2}$ is defined by*

$$\mathscr{F}_\kappa\Big((-\mathscr{D}^2_\kappa)^{\alpha/2} f\Big)(x) = |x|^\alpha \mathscr{F}_\kappa f(x).$$

The following theorem constitutes the first main result of this paper. We provide a pointwise formula for the fractional *Riesz-Dunkl* derivative that is valid for the Schwartz space $S(\mathbb{R})$.

Theorem 3.2. *Let $\alpha \in (0,2)$. For a function $f \in S(\mathbb{R})$, the fractional* Riesz-Dunkl *derivative $(-\mathscr{D}^2_\kappa)^{\alpha/2} f(x)$ can be represented as follows:*

$$(-\mathscr{D}^2_\kappa)^{\alpha/2} f(x) = \frac{1}{\gamma_\kappa(\alpha)} \lim_{\varepsilon\to 0} \int_{|h|\geq\varepsilon}^{\infty} \frac{f(x) - \tau^h f(x)}{h^{1+\alpha}} dh$$

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where the normalized constant $\gamma_\kappa(\alpha)$ is given by

$$\gamma_\kappa(\alpha) = \frac{2^\alpha \Gamma(\kappa + \frac{1+\alpha}{2})}{\Gamma(\kappa + \frac{1}{2})|\Gamma(-\frac{\alpha}{2})|}.$$

Building upon Theorem 2.16 with $\kappa = 0$, we derive the ensuing corollary. Notably, this corollary pertains to the Riesz fractional derivative $\mathcal{D}_\alpha$ (1.2) and the fractional second order derivative $-(-\frac{d^2}{dx^2})^\alpha$, which remains valid within the Schwartz space without any necessity to confine the considerations solely to even functions.

Corollary 3.3. *For $\alpha \in (0, 2)$ and $f \in S(\mathbb{R})$, the Riesz fractional derivative $\mathcal{D}_\alpha$ and the fractional second-order derivative $(-\frac{d^2}{dx^2})^{\alpha/2} f$ share the same representation:*

$$\begin{aligned} D_\alpha f(x) &= -(-\frac{d^2}{dx^2})^{\alpha/2} f(x) \\ &= \frac{\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \int_0^\infty \frac{2f(x) - f(x+y) + f(x-y)}{y^{\alpha+1}} dy. \end{aligned}$$

Drawing on the insights from Theorem 2.16 and focusing on the scenario of even functions, we can derive the following corollary. This corollary underscores the equivalence between the Riesz-Dunkl fractional derivative $\mathscr{D}_\kappa^\alpha$ and the Bessel fractional derivative. The proof of this equivalence can be found in [4, Theorem 3.5], coupled with our earlier established Theorem 2.16.

Corollary 3.4. *For an even function f in $S(\mathbb{R})$, the Riesz-Dunkl fractional derivative $(-\mathscr{D}_\kappa^2)^{\alpha/2} f(x)$ with $\alpha \in (0, 2)$ is also an even function and coincides with the fractional Bessel operator $(-\frac{d^2}{dx^2} - \frac{2\kappa}{x}\frac{d}{dx})^{\alpha/2}$. More precisely:*

$$\begin{aligned} (-\mathscr{D}_\kappa^2)^{\alpha/2} f(x) &= (-\frac{d^2}{dx^2} - \frac{2\kappa}{x}\frac{d}{dx})^{\alpha/2} f(x) \\ &= \frac{2^{\alpha+1}\Gamma(\mu + \frac{\alpha+1}{2})}{\Gamma(\kappa + \frac{1}{2})|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty \frac{f(x) - \tau_{\kappa,0}^x f(y)}{y^{\alpha+1}} dy. \end{aligned}$$

where $\tau_{\kappa,0}^x f(y)$ is the generalized translation operator associated with the Bessel operator [19], given by

$$\tau_{\kappa,0}^x f(y) = \begin{cases} \int_0^\pi f(\sqrt{x^2 + y^2 + 2xy\cos\theta}) \sin^{2\kappa-1}\theta \, d\theta, & \text{if } \kappa > 0, \\ \frac{1}{2}(f(x+y) + f(x-y)), & \text{if } \kappa = 0. \end{cases} \quad (3.5)$$

The following theorem constitutes a second key result, wherein we aim to characterize the fractional Sobolev space H_α^2 for arbitrary $\alpha \in (0, \alpha)$.

Theorem 3.6. *The following statements are equivalent:*

(i) $f \in L_\kappa^2(\mathbb{R})$ and there exists $g \in L_\kappa^2(\mathbb{R})$ such that:

$$\|\mathscr{B}_\varepsilon^{(\alpha)} f - g\|_{2,\kappa} = o(1) \quad \text{as} \quad \varepsilon \downarrow 0;$$

(ii) $\|\mathscr{B}_\varepsilon^{(\alpha)} f\|_{2,\kappa} = O(1)$ as $\varepsilon \downarrow 0$;
(iii) $f \in \mathscr{H}_\kappa^\alpha(\mathbb{R})$,

where

$$\mathscr{B}_\varepsilon^{(\alpha)} f(x) = \frac{1}{\gamma_\kappa(\alpha)} \int_{h \geq \varepsilon} \frac{\Delta_h f(x)}{h^{1+\alpha}} dh,$$

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$\gamma_\kappa(\alpha)$ is the normalized constant defined above, and Δ_h represents the generalized difference operator, which is defined for $f \in L^p_\kappa(\mathbb{R})$ as:

$$\Delta_h f = 2f - \tau^h f - \tau^{-h} f, \quad h \in \mathbb{R}.$$

Theorem 3.7. *Let $\alpha \in (0, 2)$. For $f \in \mathscr{H}^\alpha_{2,\kappa}(\mathbb{R})$ it holds*

$$\mathscr{F}_\kappa((-\mathscr{D}_\kappa)^{(\alpha)})f(x) = |x|^\alpha \mathscr{F}_\kappa(f)(x) \quad a.e. \tag{3.2}$$

4. Proof of Main Results

4.1. Proof of Theorem 3.16

We denote by $S(\mathbb{R})$ the Schwartz space, which consists of C^∞-functions on $\mathbb{R}$ that rapidly decrease along with their derivatives. This space is equipped with a topology defined by the semi-norms

$$\|f\|_{n,m} = \sup_{x\in\mathbb{R}, j\leq m} (1+x^2)^n \mathscr{D}^j_\kappa \varphi(x), \quad n, m \in \mathbb{N}.$$

It can be verified that

$$\mathscr{D}f(x) = f'(x) + \kappa \int_{-1}^{1} f'(xt)dt.$$

From this representation, it is evident that the operator $\mathscr{D}$ preserves the space $S(\mathbb{R})$.

Lemma 4.1. *For a function $f \in S(\mathbb{R})$, the generalized difference operator can be expressed as:*

$$\Delta_h f(x) = \frac{1}{2} \int_{-|h|}^{|h|} sign(u) \tau^u \mathscr{D}_\kappa f(x)\, du.$$

Proof. Let $f \in S(\mathbb{R})$ and $g(h) = \tau^h f(x)$. We have

$$\begin{aligned}\int_{-h}^{h} \text{sign}(u) \mathscr{D}_\kappa g(u)\, du &= \int_{-h}^{h} \text{sign}(u) \left\{g'(u) + \kappa \frac{g(u) - g(-u)}{u}\right\} du \\ &= g(h) + g(-h) - 2g(0) \\ &= \tau^h f(x) + \tau^{-h} f(x) - 2f(x).\end{aligned}$$

Thus,

$$\Delta_{h,\kappa} f(x) = \int_{-h}^{h} \text{sign}(u) \mathscr{D}_\kappa \tau^u f(x)\, du.$$

□

We shall now establish Theorem 3.6

Proof. We begin by combining Lemma (4.1) and equation (2.14) while considering $h, x \in \mathbb{R}$. This combination yields:

$$|\Delta_h f(x)| \leq |h| \|\mathscr{D}_\kappa f\|_\infty. \tag{4.2}$$

This inequality lays the foundation for subsequent analysis, leading to the following evaluations:

$$\begin{aligned}\int_0^\infty \frac{|\Delta_h f(x)|}{h^{1+\alpha}} dh \leq \|\mathscr{D}_\kappa f(x)\|_\infty \int_0^1 h^{-\alpha} \\ + 4\|f(x)\|_\infty \int_1^\infty \frac{dh}{h^{1+\alpha}} < \infty.\end{aligned}$$

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Consequently, we deduce:

$$\begin{aligned}\int_0^\infty \frac{\Delta_h f(x)}{|h|^{d+\alpha}}\,dh &= \frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{|h|\geq\varepsilon}\frac{\Delta_h f(x)}{h^{1+\alpha}}\,dh\\ &= \frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{|h|\geq\varepsilon}\frac{f(x)-\tau^h f(x)}{|h|^{1+\alpha}}\,dh\\ &\quad+\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{|h|\geq\varepsilon}\frac{f(x)-\tau^{-h} f(x)}{|h|^{1+\alpha}}\,dh\\ &= \lim_{\varepsilon\to 0^+}\int_{|h|\geq\varepsilon}\frac{f(x)-\tau^h f(x)}{|h|^{1+\alpha}}\,dh.\end{aligned}$$

Thus

$$\frac{1}{\gamma_\kappa(\alpha)}\int_0^\infty \frac{\Delta_h f(x)}{h^{d+\alpha}}\,dh = \lim_{\varepsilon\to 0^+}\frac{1}{\gamma_\kappa(\alpha)}\int_{|h|\geq\varepsilon}\frac{f(x)-\tau^h f(x)}{|h|^{1+\alpha}}\,dh. \tag{4.3}$$

Since both the Dunkl transform and the generalized translation maintain the invariance of the Schwartz space $S(\mathbb{R})$, we find it advantageous to reformulate (2.12):

$$\tau^h f(x) = \mathscr{F}_\kappa^{-1}\Big(\mathscr{E}_\kappa(i\xi h)\mathscr{F}_\kappa f\Big)(\xi), \quad y,\xi\in\mathbb{R}. \tag{4.4}$$

Building on this, we can now express the generalized difference operator (3) as:

$$\Delta_h f(x) = 2\int_{\mathbb{R}}(1-\mathscr{J}_{\kappa-1/2}(\xi h))\mathscr{E}_\kappa(i\xi x)\mathscr{F}_\kappa f(\xi)\sigma_\kappa(d\xi). \tag{4.5}$$

Inserting this expression into the integrand of equation (4.3), we derive:

$$\frac{1}{\gamma_\kappa(\alpha)}\int_0^\infty \frac{\Delta_h f(x)}{h^{1+\alpha}}\,dh = \frac{1}{\gamma_\kappa(\alpha)}\int_0^\infty\int_{\mathbb{R}}\frac{1-\mathscr{J}_{\kappa-1/2}(\xi h)}{h^{1+\alpha}}\mathcal{E}_\kappa(i\xi x)\mathscr{F}_\kappa f(\xi)\sigma_\kappa(d\xi)\,dh. \tag{4.6}$$

Upon transforming

$$\xi\to\xi, \qquad h\to h|\xi|,$$

the right-hand side of (4.6) takes on a different form:

$$\frac{1}{\gamma_\kappa(\alpha)}\int_0^\infty\int_{\mathbb{R}}\frac{1-\mathscr{J}_{\kappa-1/2}(h)}{h^{1+\alpha}}|\xi|^\alpha\mathcal{E}_\kappa(i\xi x)\mathscr{F}_\kappa f(\xi)\sigma_\kappa(d\xi)\,dh.$$

By invoking Tonelli's Theorem and Lemma 3.2, we proceed:

$$\begin{aligned}&\frac{1}{\gamma_\kappa(\alpha)}\int_0^\infty\Big|\int_{\mathbb{R}}\frac{1-\mathscr{J}_{\kappa-1/2}(h)}{h^{1+\alpha}}|\xi|^\alpha\mathcal{E}_\kappa(i\xi x)\mathscr{F}_\kappa f(\xi)\Big|\sigma_\kappa(d\xi)\,dh\\ &\qquad\leq\int_{\mathbb{R}}|\xi|^\alpha|\mathscr{F}_\kappa f(\xi)|\sigma_\kappa(d\xi)<\infty.\end{aligned}$$

Consequently, applying Fubini's Theorem, Lemma 3.2, Definition 2.1, and the inversion formula for the Dunkl transform (2.7) leads to:

$$\begin{aligned}\frac{1}{\gamma_\kappa(\alpha)}\int_0^\infty \frac{\Delta_h f(x)}{h^{1+\alpha}}\,dh &= \frac{1}{\gamma_\kappa(\alpha)}\int_0^\infty\frac{1-\mathscr{J}_{\kappa-1/2}(h)}{h^{1+\alpha}}dh\int_{\mathbb{R}}|\xi|^\alpha\mathcal{E}_\kappa(i\xi x)\mathscr{F}_\kappa f(\xi)\sigma_\kappa(d\xi)\\ &= \int_{\mathbb{R}}\mathcal{E}_\kappa(i\xi x)|\xi|^\alpha\mathscr{F}_\kappa f(\xi)\sigma_\kappa(d\xi)\\ &= \int_{\mathbb{R}}\mathscr{F}_\kappa\Big((-\mathscr{D}_\kappa^2)^{\alpha/2}f\Big)(\xi)\mathcal{E}_\kappa(i\xi x)\sigma_\kappa(d\xi)\\ &= (-\mathscr{D}_\kappa^2)^{\alpha/2}f)(x).\end{aligned}$$

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This concludes the proof. □

4.2. Proof of Theorem 2.20

Proposition 4.7. *For* $0 < \alpha < 2$, $\mathscr{B}_\varepsilon^{(\alpha)}$ *is a bounded operator from* $L^2_\kappa(\mathbb{R})$ *into itself satisfying*

$$\|\mathscr{B}_\varepsilon^{(\alpha)} f\|_{2,\kappa} \leq C\,\|f\|_{2,\kappa} \quad (C = \frac{4}{\alpha\varepsilon^\alpha |\gamma_\kappa(\alpha)|}). \tag{4.8}$$

Proof. By Holder-Minkowski inequality and inequality (2.14), it follows that

$$\|\Delta_h f\|_{2,\kappa} \leq 4\|f\|_{2,\kappa}. \tag{4.9}$$

Then

$$\|\mathscr{B}_\varepsilon^{(\alpha)} f\|_{2,\kappa} \leq \frac{4\|f\|_{2,\kappa}}{|\gamma_\kappa(\alpha)|} \int_{h\geq\varepsilon} \frac{1}{h^{1+\alpha}} dh = \frac{4}{\alpha\varepsilon^\alpha |\gamma_\kappa(\alpha)|}\,\|f\|_{2,\kappa}.$$

□

We introduce the function

$$\lambda_{\alpha,\varepsilon}(x) = \frac{2}{\gamma_\kappa(\alpha)} \int_\varepsilon^\infty \frac{1 - \mathscr{J}_{\kappa-1/2}(xh)}{h^{1+\alpha}}\,dh. \tag{4.10}$$

In the following, we establish some elementary properties of $\lambda_{\alpha,\varepsilon}(x)$, and their proofs are straightforward.

Lemma 4.11. *For the function* $\lambda_{\alpha,\varepsilon}(x)$, *the following hold:*

(i) $|\lambda_{\alpha,\varepsilon}(x)| \leq 2$,
(ii) $|\lambda_{\alpha,\varepsilon}(x)| \leq |x|^\alpha$,
(iii) $\lim_{\varepsilon\downarrow 0} \lambda_{\alpha,\varepsilon}(x) = |x|^\alpha$.

Proposition 4.12. *Furthermore, the Dunkl transform of* $\mathscr{R}_\varepsilon^{(\alpha)} f$ *is given by*

$$\mathscr{F}_\kappa(\mathscr{R}_\varepsilon^{(\alpha)} f)(x) = \lambda_{\alpha,\varepsilon}(x)\mathscr{F}_\kappa(f)(x) \quad \text{a.e.} \tag{4.13}$$

Proof. Let $f \in L^2_\kappa(\mathbb{R})$. Since $L^1_\kappa(\mathbb{R}) \cap L^2_\kappa(\mathbb{R})$ is dense in $L^2_\kappa(\mathbb{R})$, we choose a sequence $f_n \in L^1_\kappa(\mathbb{R}) \cap L^2_\kappa(\mathbb{R})$ with $\lim_{n\to\infty} \|f_n - f\|_{2,\kappa} = 0$. By Fubini's theorem we easily obtain

$$\mathscr{F}_\kappa(\mathscr{B}_\varepsilon^{(\alpha)} f_n)(x) = \lambda_{\alpha,\varepsilon}(x)\mathscr{F}_\kappa(f_n)(x). \tag{4.14}$$

Then by Lemma 3.13 and the isometry property of the Dunkl transform,

$$\begin{aligned}\|\mathscr{F}_\kappa(\mathscr{R}_\varepsilon^{(\alpha)} f) - \lambda_{\alpha,\varepsilon}(x)\mathscr{F}_\kappa(f)\|_{2,\kappa} &\leq \|\mathscr{F}_\kappa(\mathscr{R}_\varepsilon^{(\alpha)} f) - \mathscr{F}_\kappa(\mathscr{R}_\varepsilon^{(\alpha)} f_n)\|_{2,\kappa} \\ &\quad + \|\lambda_{\alpha,\varepsilon}\{\mathscr{F}_\kappa(f_n) - \mathscr{F}_\kappa(f)\}\|_{2,\kappa} \\ &\leq C\|f - f_n\|_{p,\kappa} + 2\|f_n - f\|_{p,\kappa}.\end{aligned}$$

Thus proves the assertion. □

Now to the proof of the Theorem 3.10.

Proof. We will prove the implications $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, and $(iii) \Rightarrow (i)$.
$(i) \Rightarrow (ii)$**:** This implication is straightforward.

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$(ii) \Rightarrow (iii)$**:** Assume that condition (ii) holds. By employing Fatou's Lemma, we deduce that

$$\begin{aligned}\||.|^{\alpha}\mathscr{F}_{\kappa}(f)\|_{2,\kappa} &\leq \liminf_{\varepsilon\downarrow 0}\|\lambda_{\alpha,\varepsilon}\mathscr{F}_{\kappa}(f)\|_{2,\kappa}\\ &= \liminf_{\varepsilon\downarrow 0}\|\mathscr{F}_{\kappa}(\mathscr{R}_{\varepsilon}^{(\alpha)}f)\|_{2,\kappa}\\ &= \liminf_{\varepsilon\downarrow 0}\|\mathscr{R}_{\varepsilon}^{(\alpha)}f\|_{2,\kappa}.\end{aligned}$$

As the last term is finite due to the assumed condition, we thus establish (iii).

$(iii) \Rightarrow (i)$**:** We assume that $f \in \mathscr{H}_{\kappa}^{\alpha}(\mathbb{R})$. Since the Dunkl transform is an isomorphism of $L_{\kappa}^{2}(\mathbb{R})$, there exists $g \in L_{\kappa}^{2}(\mathbb{R})$ such that $\mathscr{F}_{\kappa}(g)(x) = |x|^{\alpha}\mathscr{F}_{\kappa}(f)(x)$. Consequently, we have

$$\begin{aligned}\|\mathscr{R}_{\varepsilon}^{(\alpha)}f(x) - g(x)\|_{2,\kappa} &= \|\mathscr{F}_{\kappa}(\mathscr{R}_{\varepsilon}^{(\alpha)}f) - \mathscr{F}_{\kappa}(g)\|_{2,\kappa}\\ &= \|\big(\lambda_{\alpha,\varepsilon}(x) - |x|^{\alpha}\big)\mathscr{F}_{\kappa}(f)\|_{2,\kappa}.\end{aligned}$$

Additionally, we find

$$\big(\lambda_{\alpha,\varepsilon}(x) - |x|^{\alpha}\big)^{2}|\mathscr{F}_{\kappa}(f)|^{2} \leq 4|x|^{2\alpha}|\mathscr{F}_{\kappa}(f)|^{2} = 4|\mathscr{F}_{\kappa}(g)|^{2}.$$

By employing the Lebesgue Dominated Convergence Theorem, we conclude that

$$\lim_{\varepsilon\downarrow 0}\|\big(\lambda_{\alpha,\varepsilon}(x) - |x|^{\alpha}\big)^{2}\mathscr{F}_{\kappa}(f)\|_{2,\kappa}^{2} = 0.$$

This completes the proof of (i).

Hence, we have established all three implications, leading to the conclusion of the proof. □

5. Concluding Remark

In summary, this study has navigated the intricate landscape of fractional calculus within the realm of differential-difference operators. The objective was to establish a connection between the well-established Riesz fractional derivatives and Dunkl type operators, leading to the emergence of Riesz-Dunkl fractional derivatives. Through rigorous analysis, we've showcased the versatility and utility of these new derivatives.

Our discoveries unveil the compelling alignment between the Riesz-Dunkl fractional derivative and both the conventional Riesz fractional derivative and the fractional second-order derivative. This equivalence extends beyond even functions, broadening its applicability within the Schwartz space. Furthermore, by focusing on even functions, we've revealed the intriguing parallel between the Riesz-Dunkl fractional derivative and the Bessel fractional derivative.

Beyond these equivalences, we've introduced a novel fractional Sobolev space utilizing the Fractional Riesz-Dunkl derivative framework. This advancement deepens our comprehension of fractional calculus and opens avenues for further exploration and application.

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