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Article

The Number of Zeros in a Disk of a Complex Polynomial with Coefficients Satisfying Various Monotonicity Conditions

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Abstract: Motivated by results on the location of zeros of a complex polynomial with monotonicity conditions on the coefficients (such as the classical Eneström-Kakeya Theorem, and its recent generalizations), we impose similar conditions and give bounds on the number of zeros in certain regions. We do so by introducing a reversal in monotonicity conditions on the real and imaginary parts of the coefficients, and also on their moduli. The results presented naturally apply to certain classes of lacunary polynomials.

Keywords: complex polynomials; counting zeros; monotone coefficients

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1. Introduction

The classical Eneström-Kakeya Theorem concerns the location of the complex zeros of a real polynomial with nonnegative monotone coefficients. It was independently proved by Gustav Eneström in 1893 [4] and Sōichi Kakeya in 1912 [10].

Theorem 1. Eneström-Kakeya Theorem. *If $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying $0 \le a_0 \le a_1 \le \cdots \le a_n$, then all the zeros of P lie in $|z| \le 1$.*

A large body of literature on results related to the Eneström-Kakeya Theorem now exists. For a survey of results up through 2014, see [7]. Inspired by results of Aziz and Zargar [1] and Shah et al. [15], the present authors gave an Eneström-Kakeya type result [5] for polynomials $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ such that $\alpha_\ell = \text{Re}(a_\ell)$ and $\beta_\ell = \text{Im}(a_\ell)$ for $0 \le \ell \le n$ where, for some positive numbers ρ_r and ρ_i each at most 1, and k_r, k_i each at least 1, and p and q with $0 \le q \le p \le n$, the coefficients satisfy

$$\rho_r \alpha_q \le \alpha_{q+1} \le \alpha_{q+2} \le \cdots \le \alpha_{p-1} \le k_r \alpha_p$$

and

$$\rho_i \beta_q \le \beta_{q+1} \le \beta_{q+2} \le \cdots \le \beta_{p-1} \le k_i \beta_p.$$

The present authors recently generalized this result [6] by adding parameter j with $q < j < p$ (which allows a reversal in the monotonicity condition) and using a total of six positive parameters ρ_{r_1}, ρ_{r_2}, ρ_{i_1}, and ρ_{i_2} each at most 1, and k_r, k_i each at least 1, to consider polynomials with complex coefficients satisfying

$$\rho_{r_1} \alpha_q \le \alpha_{q+1} \le \alpha_{q+2} \le \cdots \alpha_{j-1} \le k_r \alpha_j \ge \alpha_{j+1} \ge \cdots \ge \alpha_{p-1} \ge \rho_{r_2} \alpha_p \quad (1)$$

and

$$\rho_{i_1} \beta_q \le \beta_{q+1} \le \beta_{q+2} \le \cdots \beta_{j-1} \le k_i \beta_j \le \beta_{j+1} \ge \cdots \ge \beta_{p-1} \ge \rho_{i_2} \beta_p. \quad (2)$$

Notice that with $\rho_{r_1} = k_r = \rho_{r_2} = 1$, $q = 0$, $j = p = n$, $0 \le a_0$ and each $\beta_\ell = 0$, the above condition implies the hypotheses of the Eneström-Kakeya Theorem.

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The first result concerning the number of zeros in a disk relevant to the current work, is due to Mohammad in 1965. It considers polynomials with real coefficients which satisfy the monotonicity condition of the Eneström-Kakeya Theorem (with the added condition that the constant term is nonzero) and is as follows [11].

Theorem 2. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with real coefficients such that $0 < a_0 \leq a_1 \leq \cdots \leq a_n$. Then the number of zeros of $P(z)$ in the disk $|z| \leq 1/2$ does not exceed $1 + (1/\log 2)\log(a_n/a_0)$.*

Another relevant result is due to Dewan [3] and concerns a monotonicity condition on the moduli of coefficients, as follows.

Theorem 3. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2$ for $\ell = 0, 1, \ldots, n$ and $0 < |a_0| \leq |a_1| \leq \cdots \leq |a_n|$. Then the number of zeros of $P(z)$ in $|z| \leq 1/2$ does not exceed*

$$\frac{1}{\log 2} \log\left(\frac{|a_n|(1+\cos\alpha+\sin\alpha) + 2\sin\alpha\sum_{\ell=0}^{n-1}|a_\ell|}{|a_0|}\right).$$

Though both Theorems 2 and 3 concern zeros in $|z| \leq 1/2$, more general results exist. For example, Pukhta [12] gave the following generalization of Theorem 3 which reduces to Theorem 3 when $\delta = 1/2$.

Theorem 4. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients such that for some real β, $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2$ for $\ell = 0, 1, \ldots, n$ and $0 < |a_0| \leq |a_1| \leq \cdots \leq |a_n|$. Then, for $0 < \delta < 1$, the number of zeros of $P(z)$ in $|z| \leq \delta$ does not exceed*

$$\frac{1}{\log 1/\delta} \log\left(\frac{|a_n|(1+\cos\alpha+\sin\alpha) + 2\sin\alpha\sum_{\ell=0}^{n-1}|a_\ell|}{|a_0|}\right).$$

Recently, the number of zeros in a disk of a polynomial with coefficients satisfying a monotonicity condition, but with extra multiplicative terms on some of the coefficients, have been presented. Rather et al. [14], for example, considered polynomials with real coefficients satisfying

$$a_0 \leq a_1 \leq \cdots \leq a_{n-r-1} \leq k_r a_{n-r} \leq k_{r-1} a_{n-r+1} \leq \cdots \leq k_1 a_{n-1} \leq k_0 a_n$$

where $k_\ell \geq 1$ for $\ell = 0, 1, \ldots, r$ and $0 \leq r \leq n-1$. Rather et al. [13] (in a publication different from the previously cited one) similarly considered a monotonicity condition, but with extra additive terms on some of the coefficients. For example, they considered polynomials with real coefficients satisfying

$$a_0 \leq a_1 \leq \cdots \leq a_{n-r-1} \leq k_r + a_{n-r} \leq k_{r-1} + a_{n-r+1} \leq \cdots \leq k_1 + a_{n-1} \leq k_0 + a_n$$

where $k_\ell \geq 0$ for $\ell = 0, 1, \ldots, r$ and $0 \leq r \leq n-1$. These results of Rather et al. generalize and refine the earlier results.

The purpose of this paper is to consider complex polynomials satisfying conditions (1) and (2) (and a related condition on the moduli of the coefficients) and to give results concerning the number of zeros in a disk.

2. Results

For a polynomial of degree n with complex coefficients a_ℓ, $0 \leq \ell \leq n$, where $\alpha_\ell = \text{Re}(a_\ell)$ and $\beta_\ell = \text{Im}(a_\ell)$, we impose the conditions of equations (1) and (2) to get the following.

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Theorem 5. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a complex polynomial of degree n with complex coefficients where $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$ which satisfies, for some real $\rho_{r_1}, \rho_{r_2}, \rho_{i_1}, \rho_{i_2}, k_r$, and k_i where $0 < \rho_{r_1} \leq 1$, $0 < \rho_{r_2} \leq 1$, $0 < \rho_{i_1} \leq 1$, $0 < \rho_{i_2} \leq 1$, $k_r \geq 1$, and $k_i \geq 1$, the condition*

$$\rho_{r_1}\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_{p-1} \geq \rho_{r_2}\alpha_p$$

$$\rho_{i_1}\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{j-1} \leq k_i\beta_j \geq \beta_{j+1} \geq \cdots \geq \beta_{p-1} \geq \rho_{i_2}\beta_p.$$

Then the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than $(1/\log(1/\delta))\log(M/|a_0|)$ for $0 < \delta < 1$, where

$$\begin{aligned} M &= |a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1-\rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j + |\alpha_p|(1-\rho_{r_2}) \\ &\quad -\rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1-\rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\ &\quad +|\beta_p|(1-\rho_{i_2}) - \rho_{i_2}\beta_p + M_p + |a_n|, \end{aligned}$$

$M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$, and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

Now we consider a condition similar to that given in equations (1) and (2), but imposed on the moduli of the complex coefficients instead of on the real and imaginary parts.

Theorem 6. *Let $P(z) = \sum_{\ell=0}^{n} a_\ell z^\ell$ be a polynomial of degree n with complex coefficients satisfying $|\arg a_\ell - \beta| \leq \alpha \leq \pi/2$, $\ell = q, q+1, \ldots, p$, such that for real k, ρ_1, ρ_2, where $k \geq 1$, $0 < \rho_1 \leq 1$, $0 < \rho_2 \leq 1$, we have*

$$\rho_1|a_q| \leq |a_{q+1}| \leq \cdots \leq |a_{j-1}| \leq k|a_j| \geq |a_{j+1}| \geq \cdots \geq |a_{p-1}| \geq \rho_2|a_p|.$$

Then the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than $(1/\log(1/\delta))\log(M/|a_0|)$ for $0 < \delta < 1$, where

$$\begin{aligned} M &= |a_0| + M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1} |a_\ell|\sin\alpha - 2|a_j| \\ &\quad +2k|a_j|(\cos\alpha + \sin\alpha + 1) + 2\sum_{\ell=j+1}^{p-1} |a_\ell|\sin\alpha + |a_p| \\ &\quad +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + M_p + |a_n|, \end{aligned}$$

$M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$, and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

The class of lacunary polynomials of the form $P(z) = a_0 + \sum_{\ell=m}^{n} a_\ell z^\ell$ was introduced by Chan and Malik in 1983 [2] in connection with Bernstein's Inequality [2]. For a survey of such results, see subsections 4.1.4, 6.4.2, and 6.4.3 of [8]. Theorems 5 and 6 naturally apply to such polynomials which satisfy the monotonicity condition on the remaining coefficients. For example, with coefficients $a_1 = a_2 = \cdots = a_{q-1} = 0$ in polynomial P, we get the following corollary.

Corollary 1. *Let $P(z) = a_0 + \sum_{\ell=q}^{n} a_\ell z^\ell$ be a complex polynomial of degree n with complex coefficients where $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$ which satisfies, for some real $\rho_{r_1}, \rho_{r_2}, \rho_{i_1}, \rho_{i_2}, k_r$, and k_i where $0 < \rho_{r_1} \leq 1$, $0 < \rho_{r_2} \leq 1$, $0 < \rho_{i_1} \leq 1$, $0 < \rho_{i_2} \leq 1$, $k_r \geq 1$, and $k_i \geq 1$, the condition*

$$\rho_{r_1}\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_{p-1} \geq \rho_{r_2}\alpha_p$$

$$\rho_{i_1}\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{j-1} \leq k_i\beta_j \geq \beta_{j+1} \geq \cdots \geq \beta_{p-1} \geq \rho_{i_2}\beta_p.$$

Then the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than $(1/\log(1/\delta))\log(M/|a_0|)$ where M is as given in Theorem 5, $M_q = 0$, and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$.

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A similar corollary follows from Theorem 6. In addition, Theorems 5 and 6 naturally apply to lacunary polynomials with two gaps in their coefficients. For example, with coefficients $a_1 = a_2 = \cdots = a_{q-1} = 0$ and $a_{p+1} = a_{p+2} = \cdots = a_{n-1} = 0$ in polynomial P, we get the following corollary.

Corollary 2. *Let $P(z) = a_0 + \sum_{\ell=q}^{p} a_\ell z^\ell + a_n$ be a complex polynomial of degree n with complex coefficients where $\alpha_\ell = Re(a_\ell)$ and $\beta_\ell = Im(a_\ell)$ which satisfies, for some real $\rho_{r_1}, \rho_{r_2}, \rho_{i_1}, \rho_{i_2}, k_r$, and k_i where $0 < \rho_{r_1} \leq 1$, $0 < \rho_{r_2} \leq 1$, $0 < \rho_{i_1} \leq 1$, $0 < \rho_{i_2} \leq 1$, $k_r \geq 1$, and $k_i \geq 1$, the condition*

$$\rho_{r_1}\alpha_q \leq \alpha_{q+1} \leq \cdots \leq \alpha_{j-1} \leq k_r\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_{p-1} \geq \rho_{r_2}\alpha_p$$

$$\rho_{i_1}\beta_q \leq \beta_{q+1} \leq \cdots \leq \beta_{j-1} \leq k_i\beta_j \geq \beta_{j+1} \geq \cdots \geq \beta_{p-1} \geq \rho_{i_2}\beta_p.$$

Then the number of zeros of $P(z)$ in the disk $|z| \leq \delta$ is less than $(1/\log(1/\delta))\log(M/|a_0|)$ where M is as given in Theorem 5 and $M_q = M_p = 0$.

A similar corollary follows from Theorem 6.

The introduction of the reversal of the inequality at index j allows us to shift the point at which the reversal occurs. This flexibility allows us to apply Theorems 5 and 6 to a larger collection of polynomials than some of the other current results in the literature on this topic.

3. Lemmas

The number of zeros results we consider are all based on the following theorem, which appears in Titchmarsh's *The Theory of Functions* [16, page 280].

Lemma 1. *Let $F(z)$ be analytic in $|z| \leq R$. Let $|F(z)| \leq M$ in the disk $|z| \leq R$ and suppose $F(0) \neq 0$. Then for $0 < \delta < 1$, the number of zeros of F in the disk $|z| \leq \delta R$ does not exceed $(1/\log(1/\delta))\log(M/|F(0)|)$.*

The following lemma is due to Govil and Rahman [9].

Lemma 2. *Let $z, z' \in \mathbb{C}$ with $|z| \geq |z'|$. Suppose that $|\arg z^* - \beta| \leq \alpha \leq \pi/2$ for $z^* \in \{z, z'\}$ and for some real α and β. Then*

$$|z - z'| \leq (|z| - |z'|)\cos\alpha + (|z| + |z'|)\sin\alpha.$$

4. Proofs of the Results

Proof of Theorem 5. Consider

$$F(z) = (1-z)P(z) = z_0 + \sum_{\ell=1}^{n}(a_\ell - a_{\ell-1})z^\ell - a_n z^{n+1}.$$

For $|z| = 1$ we have

$$\begin{aligned} |F(z)| &\leq |a_0| + \sum_{\ell=1}^{n}|a_\ell - a_{\ell-1}||z|^\ell + |a_n||z|^{n+1} = |a_0| + \sum_{\ell=1}^{n}|a_\ell - a_{\ell-1}| + |a_n| \\ &= |a_0| + \sum_{\ell=1}^{q}|a_\ell - a_{\ell-1}| + \sum_{\ell=q+1}^{p}|\alpha_\ell + i\beta_\ell - \alpha_{\ell-1} - i\beta_{\ell-1}| \\ &\quad + \sum_{\ell=p+1}^{n}|a_\ell - a_{\ell-1}| + |a_n| \\ &\leq |a_0| + M_q + \sum_{\ell=q+1}^{p}|\alpha_\ell - \alpha_{\ell-1}| + \sum_{\ell=q+1}^{p}|\beta_\ell - \beta_{\ell-1}| + M_p + |a_n|, \end{aligned}$$

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where $M_q = \sum_{\ell=1}^{q} |a_\ell - a_{\ell-1}|$ and $M_p = \sum_{\ell=p+1}^{n} |a_\ell - a_{\ell-1}|$. So for $|z| = 1$, we have

$$\begin{aligned}
|F(z)| \leq\ & |a_0| + M_q + |\alpha_{q+1} - \rho_{r_1}\alpha_q + \rho_{r_1}\alpha_q - \alpha_q| + \sum_{\ell=q+2}^{j-1} |\alpha_\ell - \alpha_{\ell-1}| \\
& + |\alpha_j - k_r\alpha_j + k_r\alpha_j - \alpha_{j-1}| + |\alpha_{j+1} - k_r\alpha_j + k_r\alpha_j - \alpha_j| + \sum_{\ell=j+2}^{p-1} |\alpha_\ell - \alpha_{\ell-1}| \\
& + |\alpha_p - \rho_{r_2}\alpha_p + \rho_{r_2}\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_{i_1}\beta_q + \rho_{i_1}\beta_q - \beta_q| \\
& + \sum_{\ell=q+2}^{j-1} |\beta_\ell - \beta_{\ell-1}| + |\beta_j - k_i\beta_j + k_i\beta_j - \beta_{j-1}| + |\beta_{j+1} - k_i\beta_j + k_i\beta_j - \beta_j| \\
& + \sum_{\ell=j+2}^{p-1} |\beta_\ell - \beta_{\ell-1}| + |\beta_p - \rho_{i_2}\beta_p + \rho_{i_2}\beta_p - \beta_{p-1}| + M_p + |a_n| \\
\leq\ & |a_0| + M_q + |\alpha_{q+1} - \rho_{r_1}\alpha_q| + |\rho_{r_1}\alpha_q - \alpha_q| + \sum_{\ell=q+2}^{j-1} |\alpha_\ell - \alpha_{\ell-1}| + |\alpha_j - k_r\alpha_j| \\
& + |k_r\alpha_j - \alpha_{j-1}| + |\alpha_{j+1} - k_r\alpha_j| + |k_r\alpha_j - \alpha_j| + \sum_{\ell=j+2}^{p-1} |\alpha_\ell - \alpha_{\ell-1}| + |\alpha_p - \rho_{r_2}\alpha_p| \\
& + |\rho_{r_2}\alpha_p - \alpha_{p-1}| + |\beta_{q+1} - \rho_{i_1}\beta_q| + |\rho_{i_1}\beta_q - \beta_q| + \sum_{\ell=q+2}^{j-1} |\beta_\ell - \beta_{\ell-1}| \\
& + |\beta_j - k_i\beta_j| + |k_i\beta_j - \beta_{j-1}| + |\beta_{j+1} - k_i\beta_j| + |k_i\beta_j - \beta_j| + \sum_{\ell=j+2}^{p-1} |\beta_\ell - \beta_{\ell-1}| \\
& + |\beta_p - \rho_{i_2}\beta_p| + |\rho_{i_2}\beta_p - \beta_{p-1}| + M_p + |a_n| \qquad (3) \\
=\ & |a_0| + M_q + (\alpha_{q+1} - \rho_{r_1}\alpha_q) + |\alpha_q|(1 - \rho_{r_1}) + \sum_{\ell=q+2}^{j-1} (\alpha_\ell - \alpha_{\ell-1}) + |\alpha_j|(k_r - 1) \\
& + (k_r\alpha_j - \alpha_{j-1}) + (k_r\alpha_j - \alpha_{j+1}) + |\alpha_j|(k_r - 1) + \sum_{\ell=j+2}^{p-1} (\alpha_{\ell-1} - \alpha_\ell) \\
& + |\alpha_p|(1 - \rho_{r_2}) + (\alpha_{p-1} - \rho_{r_2}\alpha_p) + (\beta_{q+1} - \rho_{i_1}\beta_q) + |\beta_q|(1 - \rho_{i_1}) \\
& + \sum_{\ell=q+2}^{j-1} (\beta_\ell - \beta_{\ell-1}) + |\beta_j|(k_i - 1) + (k_i\beta_j - \beta_{j-1}) + (k_i\beta_j - \beta_{j+1}) \\
& + |\beta_j|(k_i - 1) + \sum_{\ell=j+2}^{p-1} (\beta_{\ell-1} - \beta_\ell) + |\beta_p|(1 - \rho_{i_2}) + (\beta_{p-1} - \rho_{i_2}\beta_p) + M_p + |a_n| \\
=\ & |a_0| + M_q - \rho_{r_1}\alpha_q + |\alpha_q|(1 - \rho_{r_1}) + 2|\alpha_j|(k_r - 1) + 2k_r\alpha_j + |\alpha_p|(1 - \rho_{r_2}) \\
& - \rho_{r_2}\alpha_p - \rho_{i_1}\beta_q + |\beta_q|(1 - \rho_{i_1}) + 2|\beta_j|(k_i - 1) + 2k_i\beta_j \\
& + |\beta_p|(1 - \rho_{i_2}) - \rho_{i_2}\beta_p + M_p + |a_n|.
\end{aligned}$$

Since $F(z)$ is analytic in $|z| \leq 1$, by Lemma 1 and the Maximum Modulus Theorem, the number of zeros of $F(z)$ (and hence of $P(z)$) in $|z| \leq \delta$ is less than or equal to $(1/\log(1/\delta))\log(M/|a_0|)$ where $0 < \delta < 1$, as claimed. □

Proof of Theorem 6. Consider $F(z) = (1 - z)P(z)$. For $|z| = 1$ we have,

$$|F(z)| \leq |a_0| + M_q + |a_{q+1} - \rho_1 a_q| + |\rho_1 a_q - a_q| + \sum_{\ell=q+2}^{j-1} |a_\ell - a_{\ell-1}|$$

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$$
\begin{aligned}
& +|a_j - ka_j| + |ka_j - a_{j-1}| + |a_{j+1} - ka_j| + |ka_j - a_j| + \sum_{\ell=j+2}^{p-1} |a_\ell - a_{\ell-1}| \\
& +|a_p - \rho_2 a_p| + |\rho_2 a_p - a_{p-1}| + M_p + |a_n| \text{ as in (3)} \\
\leq\ & |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho_1|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho_1|a_q|\sin\alpha \\
& +|a_q|(1-\rho_1) + \sum_{\ell=q+2}^{j-1} |a_\ell|\cos\alpha - \sum_{\ell=q+2}^{j-1} |a_{\ell-1}|\cos\alpha + \sum_{\ell=q+2}^{j-1} |a_\ell|\sin\alpha \\
& + \sum_{\ell=q+2}^{j-1} |a_{\ell-1}|\sin\alpha + |a_j|(k-1) + k|a_j|\cos\alpha - |a_{j-1}|\cos\alpha + k|a_j|\sin\alpha \\
& +|a_{j-1}|\sin\alpha + k|a_j|\cos\alpha - |a_{j+1}|\cos\alpha + k|a_j|\sin\alpha + |a_{j+1}|\sin\alpha \\
& +|a_j|(k-1) + \sum_{\ell=j+2}^{p-1} |a_{\ell-1}|\cos\alpha - \sum_{\ell=j+2}^{p-1} |a_\ell|\cos\alpha + \sum_{\ell=j+2}^{p-1} |a_{\ell-1}|\sin\alpha \\
& + \sum_{\ell=j+2}^{p-1} |a_\ell|\sin\alpha + |a_p|(1-\rho_2) + |a_{p-1}|\cos\alpha - \rho_2|a_p|\cos\alpha + |a_{p-1}|\sin\alpha \\
& +\rho_2|a_p|\sin\alpha + M_p + |a_n| \text{ by Lemma 2.}
\end{aligned}
$$

Hence

$$
\begin{aligned}
|F(z)| \leq\ & |a_0| + M_q + |a_{q+1}|\cos\alpha - \rho_1|a_q|\cos\alpha + |a_{q+1}|\sin\alpha + \rho_1|a_q|\sin\alpha \\
& +|a_q|(1-\rho_1) + |a_{j-1}|\cos\alpha + \sum_{\ell=q+2}^{j-2} |a_\ell|\cos\alpha - |a_{q+1}|\cos\alpha \\
& - \sum_{\ell=q+2}^{j-2} |a_\ell|\cos\alpha + |a_{j-1}|\sin\alpha + \sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha + |a_{q+1}|\sin\alpha \\
& + \sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha + |a_j|(k-1) + k|a_j|\cos\alpha - |a_{j-1}|\cos\alpha + k|a_j|\sin\alpha \\
& +|a_{j-1}|\sin\alpha + k|a_j|\cos\alpha - |a_{j+1}|\cos\alpha + k|a_j|\sin\alpha + |a_{j+1}|\sin\alpha \\
& +|a_j|(k-1) + |a_{j+1}|\cos\alpha + \sum_{\ell=j+2}^{p-2} |a_\ell|\cos\alpha - |a_{p-1}|\cos\alpha \\
& - \sum_{\ell=j+2}^{p-2} |a_\ell|\cos\alpha + |a_{j+1}|\sin\alpha + \sum_{\ell=j+2}^{p-2} |a_\ell|\sin\alpha + |a_{p-1}|\sin\alpha \\
& + \sum_{\ell=j+2}^{p-2} |a_\ell|\sin\alpha + |a_p|(1-\rho_2) + |a_{p-1}|\cos\alpha - \rho_2|a_p|\cos\alpha + |a_{p-1}|\sin\alpha \\
& +\rho_2|a_p|\sin\alpha + M_p + |a_n| \\
=\ & |a_0| + M_q + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2|a_{q+1}|\sin\alpha + |a_q| \\
& +2\sum_{\ell=q+2}^{j-2} |a_\ell|\sin\alpha - 2|a_j| + 2k|a_j|(\cos\alpha + \sin\alpha + 1) + 2|a_{j-1}|\sin\alpha \\
& +2|a_{j+1}|\sin\alpha + 2\sum_{\ell=j+2}^{p-2} |a_\ell|\sin\alpha + 2|a_{p-1}|\sin\alpha + |a_p| \\
& +\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + M_p + |a_n| \\
=\ & |a_0| + M_q + |a_q| + \rho_1|a_q|(\sin\alpha - \cos\alpha - 1) + 2\sum_{\ell=q+1}^{j-1} |a_\ell|\sin\alpha - 2|a_j|
\end{aligned}
$$

$$+2k|a_j|(\cos\alpha + \sin\alpha + 1) + 2\sum_{\ell=j+1}^{p-1} |a_\ell| \sin\alpha + |a_p|$$
$$+\rho_2|a_p|(\sin\alpha - \cos\alpha - 1) + M_p + |a_n|.$$

Since $F(z)$ is analytic in $|z| \leq 1$, by Lemma 1 and the Maximum Modulus Theorem, the number of zeros of $F(z)$ (and hence of $P(z)$) in $|z| \leq \delta$ is less than or equal to $(1/\log(1/\delta))\log(M/|a_0|)$ where $0 < \delta < 1$, as claimed. □

5. Discussion

As explained in the Introduction, the hypotheses applied in this paper build on similar hypotheses in the setting of results on the the location of zeros of a complex polynomial; namely, the Eneström–Kakeya Theorem and its generalizations. Future research could involve loosening or revising the monotonicity conditions of Theorems 5 and 6. For example, the mototonicity conditions of Rather et al. in [13,14], mentioned in the Introduction, could be imposed on the real and complex parts of the coefficients and on the moduli of the coefficients to produce related results. Theorems 5 and 6 concern a single reversal in the monotonicity condition, so this could be generalized to multiple reversals. In addition, combinations of the montonicity conditions presented here could be combined with others in the literature (such as those in [13,14]).

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