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Article

Thermal Effects in Ising Cosmology

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Abstract: We consider a real scalar field in de Sitter background and compute its thermal propagators. We propose that in a dS/CFT context, non-trivial thermal effects as seen by an 'out' observer can be encoded in the anomalous dimensions of the $d = 3$ Ising model. One of these anomalous dimensions, the critical exponent η, fixes completely a number of cosmological observables, which we compute.

Keywords: field theory; thermal field theory; cosmology; de Sitter space

1. Introduction

The rapidly expanding phase of the universe can be modelled by de Sitter (dS) space and the simplest form of matter by a real scalar. It is believed that basic effects that left an imprint on the Cosmic Microwave Background (CMB) were of thermal nature. Therefore a simple model that could explain some of the observed features of the CMB is a real scalar field ϕ in fixed dS background [1–3], formulated in the context of thermal quantum field theory [4]. The action is

$$\mathcal{S} = \int d^4x \, \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (m^2 + \xi \mathcal{R}) \phi^2 \right] , \tag{1.1}$$

which we will quantize taking into account finite temperature effects. After fixing the parametric freedom by an RG flow argument that has its origins in the $d = 3$ Ising model, we will extract several simple cosmological observables.

2. Propagators and temperature

Consider a $d + 1$ dimensional FRW spacetime with metric

$$ds^2 = a^2 \left(d\tau^2 - d\mathbf{x}^2 \right) \tag{2.1}$$

with τ the conformal time and $a(\tau)$ the scale factor. de Sitter space corresponds to $a = -\frac{1}{H\tau}$. The expanding Poincare patch of dS space is parametrized by $\tau \in (-\infty, 0]$. The scalar field mode in d-dimensional momentum space $\phi_{|\mathbf{k}|} = \frac{\chi_{|\mathbf{k}|}}{a}$ in this background yields the classical Klein-Gordon equation of motion ($k = (k^0, \mathbf{k})$ is the four-momentum and the dot is derivative with respect to τ)

$$\ddot{\chi}_{\mathbf{k}} + \omega^2_{|\mathbf{k}|} \chi_{\mathbf{k}} = 0 \, , \tag{2.2}$$

with $\omega^2_{|\mathbf{k}|} = |\mathbf{k}|^2 + m^2_{\mathrm{dS}}$ and a time-dependent mass given by $m^2_{\mathrm{dS}} = \frac{1}{\tau^2}(M^2 - \frac{d^2-1}{4})$. The dS mass parameter is $M^2 = \mu^2_H + 12\xi$ with $\mu^2_H = \frac{m^2}{H^2}$ and H the inverse curvature parameter of dS space, satisfying $\mathcal{R} = 12H^2$. The solutions to Eq. (2.2) are linear combinations of the Hankel function $H_{\nu_{\mathrm{cl}}}(\tau, |\mathbf{k}|)$ and its complex conjugate, of weight ν_{cl}, with

$$\nu_{\mathrm{cl}} = \frac{d}{2}\sqrt{1 - \frac{4M^2}{d^2}}\,. \tag{2.3}$$

Quantization of this system results in the notion of a time-dependent vacuum state and a doubled Hilbert space. Regarding the vacua, we will be concerned with the so called "in" vacuum defined at $\tau = -\infty$ and the "out" vacuum defined at the boundary (i.e. the horizon) of the expanding patch, at $\tau = 0$. These are empty vacua from the perspective of corresponding local (in conformal time) observers. The $|\mathrm{in}\rangle$ will be chosen to be the maximally symmetric Bunch-Davies vacuum [5,6]. The two vacua are related via the Bogolyubov Transformation (BT) $\langle J|\, \mathbf{\Phi}^I = \langle I|\, \mathbf{\Phi}^J$ where $I, J = \mathrm{in}, \mathrm{out}$ is a label of the vacuum and $\mathbf{\Phi}^I$ is the field operator with mode function $\chi^I_{|\mathbf{k}|}$. Note that the field is the same in both vacua, with the mode functions and the creation and annihilation operators inside it being the vacuum dependent quantities. Common notation is $\chi^{\mathrm{in}}_{|\mathbf{k}|} = u_{|\mathbf{k}|}$ and $\chi^{\mathrm{out}}_{|\mathbf{k}|} = v_{|\mathbf{k}|}$.

The doubled Hilbert space can be understood in the context of the Schwinger-Keldysh (SK) path integral as being related to a $+$ (or forward) branch and a $-$ (or backward) branch in conformal time evolution. The field propagator $\mathcal{D}$ in such a basis has a 2×2 matrix structure and is ($\mathcal{T}$ ($\mathcal{T}^*$) denoting time (anti-time) ordering and $\langle 0|$ is a generic vacuum):

$$\begin{aligned} \langle 0|\, \Phi^+(\tau_2)\Phi^-(\tau_1)\,|0\rangle &= \mathcal{D}_{-+}(\tau_1;\tau_2) \\ \langle 0|\, \Phi^-(\tau_1)\Phi^+(\tau_2)\,|0\rangle &= \mathcal{D}_{+-}(\tau_1;\tau_2) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \langle 0|\, \mathcal{T}[\Phi^+(\tau_1)\Phi^+(\tau_2)]\,|0\rangle &= \mathcal{D}_{++}(\tau_1;\tau_2) \\ \langle 0|\, \mathcal{T}^*[\Phi^-(\tau_1)\Phi^-(\tau_2)]\,|0\rangle &= \mathcal{D}_{--}(\tau_1;\tau_2) \end{aligned} \tag{2.5}$$

where $\mathcal{D}_{+-}(\tau_1;\tau_2) = \mathcal{D}^*_{-+}(\tau_1;\tau_2)$, $\mathcal{D}_{--}(\tau_1;\tau_2) = \mathcal{D}^*_{++}(\tau_1;\tau_2)$ and $\mathcal{D}_{-+}(\tau_1;\tau_2) = \chi_{|\mathbf{k}|}(\tau_1)\chi^*_{|\mathbf{k}|}(\tau_2)$, $\mathcal{D}_{++}(\tau_1;\tau_2) = \theta(\tau_1 - \tau_2)\mathcal{D}_{-+}(\tau_1;\tau_2) + \theta(\tau_2 - \tau_1)\mathcal{D}_{+-}(\tau_1;\tau_2)$. The above matrix elements satisfy the relation

$$\mathcal{D}_{++} + \mathcal{D}_{--} - \mathcal{D}_{+-} - \mathcal{D}_{-+} = 0. \tag{2.6}$$

Hidden in these expressions is the $i\varepsilon$ shift, implementing the projection on the vacuum at $\tau = -\infty$. It can be chosen so that in the flat limit the propagator becomes diagonal with $\mathcal{D}_{++} = \frac{-i}{k^2 - m^2 + i\varepsilon}$. The above construction of the propagator at zero temperature in dS spacetime has been recently studied in [7].

The thermal generalization of the propagator components in Eq. (2.4) and Eq. (2.5) is our next goal. If the Hamiltonian of the system was time-independent, one could just follow the process described in Appendix A and show that the propagator satisfies the KMS condition [8], which ensures that it is a good thermal propagator. Here however we are dealing with a time-dependent Hamiltonian and this is not straightforward. Instead, we will use the method introduced in [9] that takes advantage of the SK contour, by adding an extra,"thermal" leg to it. In particular, if $\mathcal{C}_+$ is the forward branch where time evolution follows the path $\tau_{\mathrm{in}} \to \tau_{\mathrm{out}}$, $\mathcal{C}_-$ is the backward branch where $\tau_{\mathrm{out}} \to \tau_{\mathrm{in}}$, we attach an extra part to the contour $\mathcal{C}_3$, where $\tau_{\mathrm{in}} \to \tau_{\mathrm{in}} - i\frac{\beta}{2}$ and $\beta = 1/T$ is the inverse temperature parameter:

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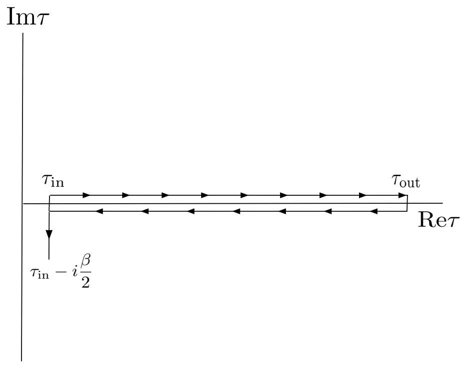

Furthermore, we introduce the propagators

$$\begin{aligned}\langle 0| \mathcal{T}[\Phi^3(\tau_1)\Phi^3(\tau_2)]\,|0\rangle &= \mathcal{D}_{33}(\tau_1;\tau_2)\\ \langle 0| \Phi^+(\tau_1)\Phi^3(\tau_2)\,|0\rangle &= \mathcal{D}_{3+}(\tau_1;\tau_2)\\ \langle 0| \Phi^-(\tau_1)\Phi^3(\tau_2)\,|0\rangle &= \mathcal{D}_{3-}(\tau_1;\tau_2)\end{aligned} \tag{2.7}$$

where $\tau_1, \tau_2 \in \mathbb{C}$ and we demand that the junction conditions for $a \in \{+,-,3\}$:

$$\mathcal{D}_{a+}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm out}} = \mathcal{D}_{a-}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm out}} \qquad \frac{\partial}{\partial\tau_2}\mathcal{D}_{a+}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm out}} = \frac{\partial}{\partial\tau_2}\mathcal{D}_{a-}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm out}} \tag{2.8}$$

are satisfied at the time instance $\tau = \tau_{\rm out}$ where the $\mathcal{C}_+$ and $\mathcal{C}_-$ contours meet, while the conditions

$$\mathcal{D}_{a-}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm in}} = \mathcal{D}_{a3}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm in}} \qquad \frac{\partial}{\partial\tau_2}\mathcal{D}_{a-}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm in}} = \frac{\partial}{\partial\tau_2}\mathcal{D}_{a3}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm in}} \tag{2.9}$$

need to be satisfied at $\tau = \tau_{\rm in}$ where $\mathcal{C}_-$ and $\mathcal{C}_3$ meet. Finally for the SK analogue of the KMS condition to hold, we need to sew together $\mathcal{C}_+$ and $\mathcal{C}_3$ which results in the conditions

$$\mathcal{D}_{a+}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm in}} = \mathcal{D}_{a3}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{in}-i\beta/2} \qquad \frac{\partial}{\partial\tau_2}\mathcal{D}_{a+}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm in}} = \frac{\partial}{\partial\tau_2}\mathcal{D}_{a3}(\tau_1;\tau_2)\Big|_{\tau_2=\tau_{\rm in}-i\beta/2} \tag{2.10}$$

that ensure the consistency of the deformed contour and yield a good thermal propagator.

The above conditions will introduce corrections of thermal nature into the propagators Eq. (2.4) and (2.5), which we compute by making two assumptions. Since the chosen contour allows for an imaginary time flow, we assume that there is no inflation in that direction. This means that the mode functions living on the $\mathcal{C}_3$ leg of the contour can be taken to have a plane wave form. In addition, at $\tau = \tau_{in}$ we assume the BD vacuum so that the mode functions are expressed in terms of the Hankel functions of $\nu = 3/2$ order. According to these assumptions, the solution to the conditions results in the in-in thermal propagator components [9]:

$$\begin{aligned}\mathcal{D}^{\beta/2}_{++} &= \mathcal{D}_{++} + n_B(\beta/2)\,(\mathcal{D}_{++} + \mathcal{D}_{--})\\ \mathcal{D}^{\beta/2}_{--} &= \mathcal{D}_{--} + n_B(\beta/2)\,(\mathcal{D}_{++} + \mathcal{D}_{--})\\ \mathcal{D}^{\beta/2}_{+-} &= \mathcal{D}_{+-} + n_B(\beta/2)\,(\mathcal{D}_{++} + \mathcal{D}_{--})\\ \mathcal{D}^{\beta/2}_{-+} &= \mathcal{D}_{-+} + n_B(\beta/2)\,(\mathcal{D}_{++} + \mathcal{D}_{--})\end{aligned} \tag{2.11}$$

with n_B the Bose-Einstein distribution parameter

$$n_B(\beta) = \frac{e^{-\beta\omega_{|\mathbf{k}|}}}{1 - e^{-\beta\omega_{|\mathbf{k}|}}} \,. \tag{2.12}$$

We can express conveniently this propagator collectively in a matrix notation as:

$$\mathcal{D}_{\beta/2} = \mathcal{D} + s^2(\beta/2)\,(\mathcal{D}_{++} + \mathcal{D}^*_{++}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.13}$$

with

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_{++} & \mathcal{D}_{+-} \\ \mathcal{D}_{-+} & \mathcal{D}_{--} \end{pmatrix},$$

and the parametrization $s(\beta/2) \equiv \sinh\theta_{|\mathbf{k}|}(\beta/2) = \sqrt{n_B(\beta/2)}$ and $c(\beta/2) \equiv \cosh\theta_{|\mathbf{k}|}(\beta/2)$.[1] It is easy to see that this thermal propagator satisfies a condition like Eq. (2.6).

Here we are actually interested in the out-out thermal propagator. We will first derive the result using a novel shortcut and then we will show that it indeed yields the correct result. The shortcut uses the Thermofield Dynamics (TFD) formalism, where the doubled Hilbert space is seen as the tensor product of the Hilbert spaces of positive and negative momenta $\mathcal{H}$ and $\tilde{\mathcal{H}}$. The fields living in these Hilbert spaces are Φ and $\tilde{\Phi}$ correspondingly. The validity of this strategy is based on the fact that the SK structure can be read also as a TFD structure, in which case the passage to finite temperature is via the transformation $\mathcal{D}_{\beta'} = U_{\beta'}\,\mathcal{D}\,U^T_{\beta'}$ and [10]

$$U_{\beta'} \equiv \begin{pmatrix} \cosh\theta_{|\mathbf{k}|}(\beta') & \sinh\theta_{|\mathbf{k}|}(\beta') \\ \sinh\theta_{|\mathbf{k}|}(\beta') & \cosh\theta_{|\mathbf{k}|}(\beta') \end{pmatrix} . \tag{2.14}$$

That this is an allowed operation on dS propagators is supported by the fact that a transformation by the matrix $U_{\beta'}$ is a BT with coefficients $\sinh\theta_{|\mathbf{k}|}(\beta') = \frac{e^{-\frac{\beta'}{2}\omega_{|\mathbf{k}|}}}{\sqrt{1-e^{-\beta'\omega_{|\mathbf{k}|}}}}$ and $\cosh\theta_{|\mathbf{k}|}(\beta') = \frac{1}{\sqrt{1-e^{-\beta'\omega_{|\mathbf{k}|}}}}$. Hence, we essentially calculate the thermal corrections that the BT has on the propagator via the TFD formalism. The result of the rotation gives the out-out thermal propagator

$$\mathcal{D}_{\beta'} = \mathcal{D} + (s^2(\beta') + s(\beta')c(\beta'))\,(\mathcal{D}_{++} + \mathcal{D}^*_{++}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} . \tag{2.15}$$

One immediately notices that the two expressions in Eq. (2.13) and Eq. (2.15) disagree in the thermal correction, as the latter has an extra term along $\sinh\theta_{|\mathbf{k}|}\cosh\theta_{|\mathbf{k}|}$. This might seem troublesome at first, however they both contain the same physical information. Taking advantage of the trivial identity

$$\frac{e^{-\beta\omega_{|\mathbf{k}|}}}{1 - e^{-\beta\omega_{|\mathbf{k}|}}} + \frac{1}{1 - e^{-\beta\omega_{|\mathbf{k}|}}} = \frac{e^{-\frac{\beta}{2}\omega_{|\mathbf{k}|}}}{1 - e^{-\frac{\beta}{2}\omega_{|\mathbf{k}|}}} \,, \tag{2.16}$$

the propagators in Eq. (2.13) and Eq. (2.15) are seen to be equal for $\beta' = \beta$. Note that the above identity does hold in the $\sinh\theta_{|\mathbf{k}|}$ and $\cosh\theta_{|\mathbf{k}|}$ parametrization, where it reads $s^2(\beta) + s(\beta)c(\beta) = s^2(\beta/2)$. We have therefore proved that the known form of the dS thermal propagator of [9] can be equivalently

[1] The flat limit of this propagator is diagonal and its $++$ component is such that the $i\epsilon$ shift of the zero temperature propagator denominator becomes $iE = i\epsilon\coth(\beta\omega_{|\mathbf{k}|}/2)$ in the thermal state.

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obtained via a TFD rotation of the zero temperature SK propagator of the half thermal parameter. The equivalence of the two expressions reflects of course the universal nature of the dS temperature as measured at an arbitrary time instance by the in and out observers. The advantage of the TFD rotation operation is that it is very simple and can be easily generalized to any background. Thus, we will use this point of view in the following.

The result of all allowed thermal transformations of $\mathcal{D}$ are correlators of the form

$$\mathcal{D}^I_{J,\gamma} = \langle J;\gamma|\,\mathcal{T}[\mathbf{\Phi}^I(\mathbf{\Phi}^I)^T]\,|J;\gamma\rangle\ . \tag{2.17}$$

The doublet field, now in the language of TFD, is $(\mathbf{\Phi}^I)^T = (\Phi^I, \tilde{\Phi}^I)$ and γ is a thermal index, associated with any combination of thermal transformations of the form Eq. (2.14). The label (not index) I on the field is a reminder of the vacuum state to which the mode functions belong. The two types of thermal transformations that are relevant to us are the insertion of an explicit density matrix, resulting in a transformation by a unitary operator U, as $|I;\beta\rangle = U\,|I\rangle$, where the eigenvalue of U is $U_\beta(\theta)$ and the Gibbons-Hawking (GH) effect [11] (for which we will momentarily use the parameter δ to distinguish it from β) that is expressed as $|I\rangle = |J;\delta\rangle$ with $I \neq J$. But the only temperature that dS space can sustain is the GH temperature which means that $1/\beta_{\text{dS}} = T_{\text{dS}} = H/2\pi = 1/\delta$. It is then sufficient to know the form of the thermal dS-scalar propagator for some generic temperature and then set $\beta = \beta_{\text{dS}}$.

3. The spectral index with thermal effects

The propagators in Eq. (2.13) and Eq. (2.15) determine several important observables. At equal space-time points and at the time of horizon exit, defined as $|\tau|H = 1$ and concentrating on horizon exiting modes specified by $|\mathbf{k}\tau| \lesssim 1$, they determine various cosmological indices derived from the scalar power spectrum (here $\mathbf{1}$ is the 2×2 matrix with unit elements)

$$P_{S,\beta}\mathbf{1} = \mathcal{D}_\beta\mathbf{1}|_{\tau_1=\tau_2}\,, \tag{3.1}$$

in terms of a single parameter (when the temperature takes its natural value $T = T_{\text{dS}}$):

$$\kappa \equiv \omega_{|\mathbf{k}|}|\tau|\Big|_{|\mathbf{k}\tau|=1} = \sqrt{\frac{5-d^2}{4} + M^2}\,. \tag{3.2}$$

This parameter can be traded for the weight of the Hankel function, as determined by the Klein-Gordon equation, in Eq. (2.3). Of special importance in $d = 3$ is the choice $M = 0$, or $\kappa = i$, which is known to generate a scale invariant CMB spectrum. This corresponds to $\nu_{\text{cl}} = \frac{3}{2}$ and decaying modes at the time of exit.

A particularly useful point of view [12] is to recognize the system at $\tau = -\infty$ as related to a UV Conformal Field Theory (CFT) labeled by the weight ν_{cl} and associated with the Gaussian fixed point of the $d = 3$ real scalar theory, that flows towards an interacting IR fixed point and the corresponding CFT at $\tau = 0$. It is clear that in the present context, exact scale invariance is realized in the $|\text{in}\rangle$ vacuum, with the deviations generated by a spontaneous shift in M that, according to Eq. (2.15), should have a finite temperature origin. Deviations can be encoded in general in a shift of the weight $\nu_{\text{cl}} \to \nu = \nu_{\text{cl}} + \nu_{\text{q}}$ that can be interpreted as a shift in the scaling dimension of a dS scalar field

$$\Delta_- = \frac{d}{2} - \nu = \frac{d}{2} - \nu_{\text{cl}} - \nu_{\text{q}} = \Delta_{\text{cl},-} - \nu_{\text{q}}\,. \tag{3.3}$$

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There is a corresponding shadow partner solution to this with $\Delta_+ = \frac{d}{2} + \nu$. In this letter, we will be concerned with $(\Delta_-, \Delta_+)_{\text{cl}} = (0, 3)$.

In order to understand ν_{q} (which will turn out to be a non-trivial zero) we first point out that the $|\text{out}; \text{fi}\rangle$ $(\beta > \beta_{\text{dS}})$ state is a BT of the Bunch-Davies vacuum. The mode functions before and after the transformation solve the same Bessel equation with frequency $\omega_{|\mathbf{k}|}$. Upon a time-dependent BT however, the frequency that an observer sees for a time other than his own, is [13]:

$$\Omega_{|\mathbf{k}|} = \omega_{|\mathbf{k}|}(|c|^2 + |s|^2)\,. \tag{3.4}$$

As a result, the horizon exit parameter is transformed as

$$\kappa \to \Lambda = \kappa\left(1 + 2\frac{e^{-2x\kappa}}{1 - e^{-2x\kappa}}\right) = \kappa \coth(x\kappa)\,, \tag{3.5}$$

where we have defined the dimensionless temperature parameter $x = \frac{\pi H}{2\pi T}$, that takes values in $[\pi, \infty]$. The transformed state in general has a reduced isometry with respect to the Bunch-Davies state. This can be seen by the fact that the BT introduces a non-zero mass term $(\mu_H^2 + \xi\frac{\mathcal{R}}{H^2})a^2H^2\phi^2$ in the Lagrangian with exit parameter $\Lambda^2 = |k\tau|^2 + a^2\left[\mu_H^2 + (\xi - \frac{1}{6})\frac{\mathcal{R}}{H^2})\right]$ and that the late time equations of motion

$$\ddot{\phi} + 2aH\dot{\phi} + \left(\mu_H^2 + \xi\frac{\mathcal{R}}{H^2}\right)a^2H^2\phi = 0\,, \qquad \dot{H} = -\frac{1}{2a}\dot{\phi}^2 \tag{3.6}$$

have no non-trivial solution with $H = \text{const.}$ and a non-zero, finite mass term.

The two limiting values of x are interesting. Its natural value $x = \pi$ where $T = T_{\text{dS}}$ gives $\Lambda = \infty$ for $\kappa = i$. This is a special case where we recover a dS solution of maximal isometry that corresponds to $|\text{out}; \text{fi}_{\text{dS}}\rangle$. As in the BD vacuum, no modes are seen to exit the horizon, this time due to their ultra-short wavelength. In the limit $x \to \infty$ on the other hand, the out observer sees modes of any wavelength as exiting modes, since in this limit the time of exit approaches the horizon. This means that if he calls his frequencies $\Omega_{|\mathbf{k}|}$, then his horizon exit parameter will be forced to $\Lambda_0 \equiv \lim_{\tau\to 0}(\Omega_{|\mathbf{k}|}\tau) \to 0$.[2] This suggests to construct a trajectory from $(\Lambda, x) \sim (\infty, \pi)$ to $(0, \infty)$ along which the value of some yet to be defined thermal effect is kept non-zero and constant, starting from a position a bit shifted away from the scale invariant limit (∞, π). Deviations from exact dS isometry due to finite temperature effects can be encoded in the shift of the spectral index of scalar curvature fluctuations

$$n_{S,\beta} = 1 + \frac{d\ln\left(|\mathbf{k}|^3 P_{S,\beta}\right)}{d\ln|\mathbf{k}|}\,. \tag{3.7}$$

In the previous section, we showed that the SK and TFD formalisms result to equivalent propagators. Consequently, from Eq. (3.1) they both determine the same thermal deviation

$$\delta n_S \equiv n_{S,\beta} - 1 = -\frac{2x}{\Lambda}\left[\frac{e^{-x\Lambda}}{1 - e^{-2x\Lambda}}\right]\,, \tag{3.8}$$

[2] In this limit x becomes an odd multiple of $\pi/2$.

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of n_S away from unity. Observe that in $|\text{out};\text{fi}_{\text{dS}}\rangle$ where $x = \pi$ and $\Lambda = \infty$, δn_S vanishes and we see a scale invariant spectrum. Moving a bit away from it, $x \gtrsim \pi$,[3] the state is $|\text{out};\text{fi}\rangle$ and δn_S becomes a one-parameter expression of Λ. We can fix this freedom by determining the value $n_{S,\beta}$ by interpreting its deviation from unity as an anomalous dimension in the dual field theory in the spirit of the dS/CFT correspondence. Then we can reach $x = \infty$ along a trajectory which keeps this value constant for all temperatures.

In [14] it is proposed that within the dual field theory that lives on the horizon, the anomalous dimension that shifts the spectral index is the critical exponent η, whose non-perturbative value is around 0.036. Thus, near the horizon

$$n_S \simeq 1 - \eta = 0.964\,. \tag{3.9}$$

This is a constraining statement that leaves no free parameters. In [14] it is also shown that the quantity by which $\Delta_{+,\text{cl}}$ shifts is the operator anomalous dimension of the trace of the Ising stress energy tensor Θ, which is an exact zero. This is however realized on the fixed point as the cancellation $\Gamma_\Theta = \eta - \eta$ and it is the term η that ends up shifting the spectral index. We therefore see that it is in this sense that ν_q is a non-trivial zero. Outside the fixed point, when for example the Ising field is massive, M deviates from zero in the bulk, the solution to Eq. (3.6) is not dS and ν_q becomes non-zero. It is important to understand that the main effect comes from the critical value η and the breaking effects that a non-zero ν_q represents are small as long as the system sits near the fixed point. For this reason the leading order results are independent of the source of the breaking. In a sense the only assumption here is that there is a mechanism of spontaneous breaking of scale invariance. From the point of view of the boundary this could be for example justified as some sort of a Coleman-Weinberg mechanism.

4. Line of constant physics and other observables

What we will demonstrate now is that in the bulk, there is a line of constant physics (LCP), labelled by the value $\delta n_S = -\eta$, along which the system is heated up from zero temperature where $\Lambda_0 = 0$ and $x = \infty$, up to the dS temperature. A few points on this line and a picture of the LCP can be found in Figure 1.

Λ	x
$\to 0$	$\to \infty$
10^{-6}	$3.5 \cdot 10^7$
0.01	1600
0.5	14.8
$\to 1.5117$	$\to \pi$

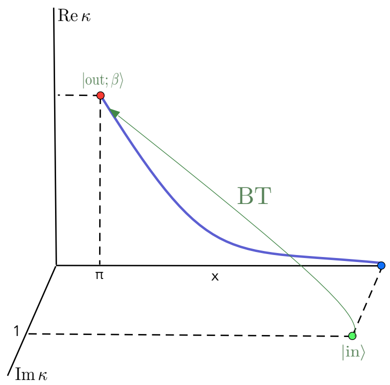

Figure 1. Left: A few points of the nearly conformal LCP defined by $\delta n_S = -\eta$. Right: The Bogolyubov Transformation $|\text{in}\rangle \to |\text{out};\text{fi}\rangle$ and the LCP, on the complex plane where $\kappa = \Lambda + i\,\mathrm{Im}\,\check{}$.

[3] It is implicitly assumed here that moving away from T_{dS} is a result of spontaneous breaking of scale invariance, which is expected to lower the temperature.

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We stress that for a given x the corresponding value of Λ is fixed by the label of the LCP. Thus near the endpoint of the LCP where $x \simeq \pi$, the value $\Lambda_\pi \simeq 1.5117$ is a fixed output. It is important to emphasize that the LCP is really meaningful up to just outside its two limiting points. Up to around $x \simeq \pi$ it is characterized by a non-zero δn_S which however at exactly $x = \pi$ becomes equal to zero, since the trace of the boundary stress-energy tensor to which the bulk scalar couples, vanishes. Analogously, the interpretation of each point on it as a dS space of the same $T_{\rm dS}$ is possible everywhere except at $x = \infty$, where the intrinsic temperature must become abruptly unobservable.

Since there are no free parameters, several other observables that are determined by $P_{S,\beta}$ are expected to be also fixed. Define for example the moment

$$n^{(1)}_{S,\beta} = \frac{dn_{S,\beta}}{d\ln|\mathbf{k}|} \tag{4.1}$$

and let us compute it using that $n^{(1)}_S = 0$. The result, evaluated under the same conditions as $n_{S,\beta}$, is

$$n^{(1)}_{S,\beta} = \delta n_S \left[2 - \frac{1}{\Lambda^2} - \frac{x}{\Lambda}\left(1 + \frac{2e^{-2x\Lambda}}{1 - e^{-2x\Lambda}}\right)\right] \tag{4.2}$$

which, substituting $x \simeq \pi$ and $\Lambda = \Lambda_\pi \simeq 1.5117$, gives

$$n^{(1)}_{S,\beta} = 0.0186 \tag{4.3}$$

for the running of the index.

Finally, the universal contribution to the non-Gaussianity parameter f_{NL} [15], can be expressed in terms of $N = \int_{t_i}^{t_f} dtH$ and its derivatives in the in-vacuum, as [16]

$$f_{NL} = \frac{5}{6}\frac{N_{\rho\rho}}{N_\rho^2} \tag{4.4}$$

with $N_\rho = \frac{\partial N}{\partial \rho}$, $N_{\rho\rho} = \frac{\partial^2 N}{\partial \rho^2}$ and $\rho \equiv P_{S,\beta}$. It is computed to be

$$f_{NL} = -\frac{5\left[x(-1+\Lambda^2)^2\left(1 + x\Lambda\cot(\frac{x\Lambda}{2})\right) + 2\Lambda^3\sinh(x\Lambda)\right]}{6\Lambda^2\left[x(-1+\Lambda^2) + \Lambda\sinh(x\Lambda)\right]}. \tag{4.5}$$

For $x \simeq \pi$ and $\Lambda = \Lambda_\pi \simeq 1.5117$ this gives

$$f_{NL} = -1.7138. \tag{4.6}$$

5. Conclusion

We considered a thermal scalar in de Sitter background. Starting from the Bunch-Davies $|\text{in}\rangle$ vacuum, a Bogolyubov Transformation placed us in the interior of the finite temperature phase diagram in a thermal state $|\text{out; fi}\rangle$. This state can be connected through holography to the vicinity of an interacting IR fixed point, in the universality class of the 3d Ising model. The system in this state is rather special, in the sense that the boundary operator that couples to the scalar curvature perturbations in the bulk has a classical scaling dimension. The critical exponent η is the order parameter of the breaking of the scale invariant spectrum of curvature fluctuations and a simple argument from the dS/CFT correspondence

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fixes the parametric freedom in the dS scalar theory, yielding the prediction $n_S = 0.964$. We also computed in the same context additional cosmological observables such as the first moment of the scalar spectral index and the non-Gaussianity bispectrum parameter f_{NL} and evaluated them numerically. Our predicted values of n_S, $n^{(1)}_{S,\beta}$ and f_{NL} are well within current experimental bounds [17,18].

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Appendix A

In this Appendix we discuss the real time construction in the Hamiltonian formulation. First we give a shortcut derivation of the thermal propagator Eq. (2.13) that starts from flat space and the definitions

$$\begin{aligned} \mathcal{D}^{\beta}_{+-}(\tau_1,\tau_2) &= W_2(\tau_1,\tau_2) + W_1(\tau_1,\tau_2) \\ \mathcal{D}^{\beta}_{-+}(\tau_1,\tau_2) &= W_1(\tau_2,\tau_1) + W_2(\tau_2,\tau_1) \end{aligned} \tag{A1}$$

with the Wightman functions defined as

$$\begin{aligned} W_1(\tau_1,\tau_2) &\equiv \frac{\mathrm{Tr}\{a^\dagger(\tau_1)a(\tau_2)\,\rho\}}{\mathrm{Tr}\{\rho\}} = n_B e^{i\omega(\tau_1-\tau_2)} \\ W_2(\tau_1,\tau_2) &\equiv \frac{\mathrm{Tr}\{a(\tau_1)a^\dagger(\tau_2)\,\rho\}}{\mathrm{Tr}\{\rho\}} = (1+n_B)e^{-i\omega(\tau_1-\tau_2)} \end{aligned} \tag{A2}$$

where $\rho = e^{-\beta\mathcal{H}}$ is the thermal density matrix, $\mathcal{H}$ is the (harmonic oscillator) Hamiltonian and the second equalities show the result of the trace computations. Now since the time dependent part of the mode function in flat space is $u(\tau) = e^{i\omega\tau}$ we can write $e^{i\omega(\tau_1-\tau_2)} = u(\tau_1)u^*(\tau_2)$ and pass to dS space via the substitution $u(\tau) \to \chi_{|\mathbf{k}|}(\tau)$. Then indeed

$$\begin{aligned} \mathcal{D}^{\beta}_{+-}(\tau_1,\tau_2) &= \chi^*_{|\mathbf{k}|}(\tau_1)\chi_{|\mathbf{k}|}(\tau_2) + n_B(\beta)\left(\chi_{|\mathbf{k}|}(\tau_1)\chi^*_{|\mathbf{k}|}(\tau_2) + \chi^*_{|\mathbf{k}|}(\tau_1)\chi_{|\mathbf{k}|}(\tau_2)\right) \\ \mathcal{D}^{\beta}_{-+}(\tau_1,\tau_2) &= \chi_{|\mathbf{k}|}(\tau_1)\chi^*_{|\mathbf{k}|}(\tau_2) + n_B(\beta)\left(\chi_{|\mathbf{k}|}(\tau_1)\chi^*_{|\mathbf{k}|}(\tau_2) + \chi^*_{|\mathbf{k}|}(\tau_1)\chi_{|\mathbf{k}|}(\tau_2)\right) \end{aligned} \tag{A3}$$

and by imposing $\mathcal{D}^{\beta}_{++}(\tau_1;\tau_2) = \theta(\tau_1-\tau_2)\mathcal{D}^{\beta}_{-+}(\tau_1;\tau_2) + \theta(\tau_2-\tau_1)\mathcal{D}^{\beta}_{+-}(\tau_1;\tau_2)$, $\mathcal{D}^{\beta}_{--}(\tau_1;\tau_2) = \mathcal{D}^{\beta*}_{++}(\tau_1;\tau_2)$ and applying for $\beta/2$, we arrive again at Eq. (2.13).

A thermal propagator has to satisfy a variant of the KMS condition. The KMS condition originates from the definition

$$\langle\phi(t_1,x_1)\phi(t_2,x_2)\rangle_\beta = \frac{\mathrm{Tr}\,\{\phi(t_1,x_1)\phi(t_2,x_2)\rho\}}{\mathrm{Tr}\{\rho\}} \tag{A4}$$

that leads, in principle, to the thermally corrected dS propagator. However for time dependent Hamiltonians the direct computation of the trace is not obvious.

The condition takes a simple form though near $\tau \to -\infty$, which we can show explicitly. In Thermofield Dynamics, the form of the KMS condition depends on a gauge parametrized by a real number, say α. It is a well known fact that TFD propagators satisfy such a condition in any of these α-gauges [19,20]. The condition holds due to the relation [10]

$$a^-_{\mathbf{k}}\,|0;\beta\rangle = e^{-\alpha\beta\omega_{|\mathbf{k}|}}\tilde{a}^+_{\mathbf{k}}\,|0;\beta\rangle\,, \qquad \langle 0;\beta|\,a^+_{\mathbf{k}} = \langle 0;\beta|\,\tilde{a}^-_{\mathbf{k}}\,e^{-(1-\alpha)\beta\omega_{|\mathbf{k}|}} \tag{A5}$$

between the standard annihilation operator acting on the vacuum of the Hilbert space $\mathcal{H}$ and the tilded creation operator acting on the vacuum of $\tilde{\mathcal{H}}$. The thermal vacuum $|0;\beta\rangle$ is defined by the action of a unitary operator on the tensor product of the vacuum states in $\mathcal{H}\times\tilde{\mathcal{H}}$. Using the above relations, one can straightforwardly show that the Wightman function between the fields $\Phi, \tilde{\Phi}$ for $\alpha = 1/2$ satisfies the condition (the mode functions of the scalar field near $\tau \to -\infty$ reduce to plane waves):

$$\langle 0;\beta|\Phi(\tau_1,\mathbf{x})\tilde{\Phi}(\tau_2,\mathbf{y})|0;\beta\rangle = \langle 0;\beta|\Phi(\tau_2+i\frac{\beta}{2},\mathbf{y})\tilde{\Phi}(\tau_1-i\frac{\beta}{2},\mathbf{x})|0;\beta\rangle\,, \tag{A6}$$

which is the KMS condition in the $\alpha = 1/2$ gauge. This is a relevant for us case, since the transformation matrix Eq. (2.14) is in this gauge [19]. A different gauge choice is to take $\alpha = 1$, where

$$a^-_{\mathbf{k}}\,|0;\beta\rangle = e^{-\beta\omega_{|\mathbf{k}|}}\tilde{a}^+_{\mathbf{k}}\,|0;\beta\rangle\,, \qquad \langle 0;\beta|\,a^+_{\mathbf{k}} = \langle 0;\beta|\,\tilde{a}^-_{\mathbf{k}}\,. \tag{A7}$$

Then, the same Wightman function as above needs to satisfy

$$\langle 0;\beta|\Phi(\tau_1,\mathbf{x})\tilde{\Phi}(\tau_2,\mathbf{y})|0;\beta\rangle = \langle 0;\beta|\Phi(\tau_2,\mathbf{y})\tilde{\Phi}(\tau_1-i\beta,\mathbf{x})|0;\beta\rangle\,, \tag{A8}$$

the KMS condition in the $\alpha = 1$ gauge. Note that this is a relation where the usual form of the KMS condition of thermal field theory can be recognised. These two gauges however can be readily seen to correspond to equivalent Wightman functions, as they can be related by a shift in the imaginary time, by $\tau_{1,2} \longrightarrow \tau_{1,2} - i\frac{\beta}{2}$. By this shift freedom, one can also see that the diagonal elements of the propagator do not satisfy any non-trivial constraint. In conclusion, to the extent that Eq. (A4) applies to dS space and the trace is computable, the thermal propagator it defines satisfies a KMS condition.

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