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Article

Optimal Sparse Control Formulation for Reconstruction of Noise-Affected Images

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Abstract: We discuss the optimal control formulation for enhancement and denoising of satellite multiband images and propose to take it in the form of an L^1-control problem for quasi-linear parabolic equation with non-local $p[u]$-Laplacian and with a cost functional of a tracking type. The main characteristic feature of the considered class of parabolic problems is the fact that the variable exponent $p(t,x)$ and the anisotropic diffusion tensor $D(t,x)$ are not well predefined a priori, but instead these characteristic non-locally depend on a solution of this problem, i.e., $p_u = p(t,x,u)$ and $D_u = D(t,x,u)$. We prove the existence of optimal pairs with sparse L^1-controls using for that the indirect approach and a special family of approximation problems.

Keywords: optimal control; parabolic equation; variable order of nonlinearity; noncoercive problem; compensated compactness technique

1. Introduction and Some Preliminaries

The main goal of any image denoising problems is to restore the noise-free gray-scale image $u : \Omega \to \mathbb{R}$ from the observed one $f : \Omega \to \mathbb{R}$. In this paper, we issue from the assumption that the observed image can be represented as

$$f = u + v + n,$$

where n is the white Gaussian noise following the Gaussian distribution $\mathcal{N}(0,\sigma^2)$, and v stands for the noise with a probably strong impulsive nature which the Gaussian model fails to describe. We assume that both noises occur simultaneously and independently in the entire domain.

To eliminate both Gaussian noise n and impulse noise v, we propose the following optimal control problem:

$$(\mathcal{R}) \quad \text{Minimize } J(v,u) = \|v\|^2_{L^2(0,T;L^1(\Omega))} + \frac{1}{2}\int_\Omega |u(T) - f_0|^2\,dx \tag{1}$$

subject to the constraints

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(|D_u(t,x)\nabla u|^{p_u(t,x)-2} D_u(t,x)\nabla u\right) = \kappa\,(f - u - v) \quad \text{in } Q_T := (0,T)\times\Omega, \tag{2}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{3}$$

$$u(0,\cdot) = f_0(\cdot) \quad \text{in } \Omega, \tag{4}$$

$$v_a(x) \le v(t,x) \le v_b(x), \quad \text{a.e. in } Q_T. \tag{5}$$

Here, $\Omega \subset \mathbb{R}^2$ is a bounded open set with a sufficiently smooth boundary $\partial\Omega$, $T > 0$ is a positive value, $\kappa \in \mathbb{R}$ is a given positive parameter, $f \in L^2(\Omega)$ is the original noise-corrupted image, $f_0 \in L^2(\Omega)$

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is the pre-denoised image by applying a median filter to f, $v_a, v_b \in L^2(\Omega)$, $v_a(x) \leq v_b(x)$ a.e. in Ω, are given distributions,

$$\|v\|^2_{L^2(0,T;L^1(\Omega))} = \int_0^T \left(\int_\Omega |v|\, dx \right)^2 dx \tag{6}$$

is the so-called directional sparsity term which, in fact, measures the L^1-norm in the space of the L^2-norm in time, $D_u = D(t,x,u)$ is the matrix of anisotropy, and the variable exponent $p_u : Q_T \to \mathbb{R}$ is defined by the rule

$$p_u(t,x) := 1 + g\left(\frac{1}{h} \int_{t-h}^t |(\nabla G_\sigma * \widetilde{u}(\tau,\cdot))(x)|\, d\tau \right), \quad \forall\, (t,x) \in Q_T, \tag{7}$$

where

$$g(s) = \delta + \frac{a^2(1-\delta)}{a^2+s^2}, \quad \forall\, s \in [0,+\infty), \tag{8}$$

$$G_\sigma(x) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^2} \exp\left(-\frac{|x|^2}{2\sigma^2} \right), \quad \sigma > 0, \tag{9}$$

$$(G_\sigma * \widetilde{u}(t,\cdot))(x) = \int_{\mathbb{R}^2} G_\sigma(x-y)\widetilde{u}(t,y)\, dy, \tag{10}$$

$\widetilde{u}$ stands for zero extension of u from Q_T to $\mathbb{R} \times \mathbb{R}^2$, and $h > 0$ and $0 < \delta \ll 1$ are given small positive values. As for the parameters $\lambda > 0$ and $a > 0$, they act as regularization and smoothing parameters.

It is clear now that, for each function $u \in L^2(0,T;W^{1,1}(\Omega))$, the inclusion $p_u(t,x) \in [p^-, p^+] \subset (1,2]$ holds almost everywhere in Q_T with $p^- = 1+\delta$ and $p^+ = 2$.

The study of optimal control problems for PDEs with variable nonlinearity is motivated by various applications in the image enhancement, where some special cases of the problem (2)–(5) appear in the natural way [1–4]. We also refer to [5], where the authors deal with a special case of the model (2)–(5) and show the given class of optimal control problems is well posed.

In recent years, many different techniques have been proposed for the reconstruction of noise-affected digital images. In particular, the following nonlinear hybrid diffusion model, which is a symboisis of the mean curvature diffusion with the Gaussian heat diffusion, has been proposed for image denoising (see [6])

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{(|\nabla u|^2+1)^{(2-p(|\nabla u|^2))/2}} \right), \quad \text{in } (0,T)\times\Omega, \tag{11}$$

$$u(0,x) = f, \quad \text{in } \Omega, \tag{12}$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } (0,T)\times\partial\Omega, \tag{13}$$

where

$$p(|\nabla u|^2) = 1 + \frac{1}{1+k|\nabla u|^2}, \tag{14}$$

f is an input image, $k > 0$ and $T > 0$ are fixed constants, Ω is a bounded open domain of $\mathbb{R}^2$ with the sufficiently smooth boundary, and ν is the unit outward normal to the boundary $\partial\Omega$.

The important characteristic of this model is the fact that it has a hybrid diffusion type which combines the mean curvature diffusion with the heat diffusion such that inside those regions, where the gradient of u is small enough, the new model acts like the heat equation and resulting in isotropic smoothing, whereas near the region's contours where the magnitude of the gradient is large, this model acts like the mean curvature equation. Although the authors of [6] believe that this model generalizes the well-known ones for nowadays (in particular, Perona-Malik model [7] or the models with $p(x)$-Laplacian operator that has been proposed in [4]), in fact, it is far from to be true because none of the mentioned models can be obtained as a particular case of (11)–(13).

It is worth to notice that because of the variable character of exponent p, we have a gap between the monotonicity and coercivity conditions. Because of this, the problem (1)–(4) can be specified as an optimal control problem for the quasi-linear parabolic equations with nonstandard growth conditions, and it can be interpreted as a generalization of the evolutional version of $p(t,x)$-Laplacian equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p(t,x)-2}\nabla u\right) \tag{15}$$

with an exponent that depends only on t and x. Equation (15) was intensively studied by many authors. There is extensive literature devoted to equation (15). We restrict ourselves by referring to the following sources [8–13] which provide an excellent insight to the theory of evolutional $p(t,x)$-Laplacian equations.

However, to our best knowledge, the above mentioned results on the solvability of parabolic equations of the type (2) mainly concern the equations with variable exponent depending on (t,x) only, whereas hardly any attention has been paid to IBVP of the form (2) with $D_u(t,x)$ and the exponent $p_u(t,x)$ given by the rule (7). Moreover, in contrast to the majority of the existing results, in this paper we do not predefine the exponent p a priori. Instead we associate this characteristic with a particular solution of IBVP (2)–(5). So, the rate of nonlinearity of p and the tensor D can be affected by the values of the unknown solution u. It is also worth to emphasize that we do not assume in this paper that the dependency of p_u and D_u on u is local whereas it is the crucial assumption in the most existing publications (see for instance [8,14]). In fact, all weak solution to this problem live in the corresponding 'personal' functional space, and, in view of the appropriate assumptions on the structure of $D_u(t,x)$ and $p_u(t,x)$, the problem (2)–(5) can admit the weak solutions that may not possess the standard properties of solutions to parabolic equations. In particular, it is unknown whether a weak solution to the above is unique and satisfies the standard energy equality.

In spite of the fact that a number of different regularizations have been suggested in the literature for optimal control problem related to the degenerate elliptic equations (see, for instance, [15–17]), the question about solvability of the proposed optimal control problem remains perhaps open for nowadays.

In view of this, our primary goal is to study the solvability issues for the OCP (1)–(5). In particular, a couple of characteristic features of the proposed problem can be emphasized here. The first one is that the tensor of anisotropy D and the exponent p depend not only on (t,x) but also on $u(t,x)$. The second feature is that the optimal control problem is formulated with $L^1(\Omega; L^2(0,T))$ control cost (together with additional pointwise control constraints). As a result, the optimal control may have directional sparsity, i.e., its support is constant in time whereas the control v can be identically zero on some parts of the domain Ω.

The paper is organized as follows: In Section 2 we introduce some preliminaries and give the main assumptions on the structure of anisotropic diffusion tensor $D_u(t,x)$ and variable exponent $p_u(t,x)$. We also discuss here the basic auxiliary results concerning the Orlicz spaces and Sobolev-Orlicz spaces with variable exponent. In Section 3 we focus mainly on the solvability issues for IBVP (2)–(5). To this end, we follow the indirect approach using a special technique of passing to the limit in the sequences od variational problems. Precise statement of the optimal control problems for quasi-linear elliptic equation with the sparse control is discussed in Section 4. We also discuss in this section the main topological properties of feasible solutions to the given OCP and, as a consequence, we derive the conditions when the set of optimal solutions is nonempty. As for the optimality conditions, their substantiation, and the results of numerical simulations, these issues are the subject of a forthcoming paper.

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2. Main Assumptions and Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with a sufficiently smooth boundary $\partial\Omega$, and let $T > 0$ be a given value. For the simplicity, we suppose that the unit outward normal $\nu = \nu(x)$ is well-defined for a.e. $x \in \partial\Omega$. We also set $Q_T = (0, T) \times \Omega$. For any measurable subset $D \subset \Omega$, we denote by $|D|$ its 2-dimensional Lebesgue measure $\mathcal{L}^2(D)$. We denote its closure by $\overline{D}$ and its boundary by ∂D. We also make use of the following notation $\operatorname{diam}\Omega = \sup_{x,y\in\Omega} |x - y|$.

For vectors $\xi \in \mathbb{R}^2$ and $\eta \in \mathbb{R}^2$, $(\xi, \eta) = \xi^t \eta$ stands for the standard vector inner product in $\mathbb{R}^2$, where ^{t} stands for the transpose operator. As for the norm $|\xi|$, we take it as the Euclidean norm given by the rule $|\xi| = \sqrt{(\xi, \xi)}$.

2.1. Functional Spaces

Let X be a real Banach space with norm $\|\cdot\|_X$, and let X' be its dual. Let $\langle \cdot, \cdot \rangle_{X';X}$ be the duality bilinear form on $X' \times X$. By $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$, we denote the weak and weak-* convergence in the spaces X and X', respectively.

For given $1 \le p \le +\infty$, the Lebesgue space $L^p(\Omega; \mathbb{R}^2)$ is defined by the standard rule

$$L^p(\Omega; \mathbb{R}^2) = \left\{ f : \Omega \to \mathbb{R}^2 \ : \ \|f\|_{L^p(\Omega;\mathbb{R}^2)} < +\infty \right\},$$

where $\|f\|_{L^p(\Omega;\mathbb{R}^2)} = \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}$ for $1 \le p < +\infty$. The inner product of two functions f and g in $L^p(\Omega; \mathbb{R}^2)$ with $p \in [1, \infty)$ is given by

$$(f, g)_{L^p(\Omega;\mathbb{R}^2)} = \int_\Omega (f(x), g(x)) \, dx = \int_\Omega \sum_{k=1}^{2} f_k(x) g_k(x) \, dx.$$

Let $C_c^\infty(\mathbb{R}^2)$ be the locally convex space of all infinitely differentiable functions with compact support in $\mathbb{R}^2$. We also define the Banach space $W^{1,p^-}(\Omega)$ with $p^- > 1$ as the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|y\|_{W^{1,p^-}(\Omega)} = \left(\int_\Omega \left(|y|^{p^-} + |\nabla y|^{p^-} \right) dx \right)^{1/p^-}.$$

We denote by $\left(W^{1,p^-}(\Omega)\right)'$ the dual space of $W^{1,p^-}(\Omega)$. Let us remark that in this case the embedding $L^2(\Omega) \hookrightarrow \left(W^{1,p^-}(\Omega)\right)'$ is continuous.

Given a real separable Banach space X, we denote by $C([0, T]; X)$ the space of all continuous functions from $[0, T]$ into X.

We recall that a function $u : [0, T] \to X$ is said to be Lebesgue measurable if there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ of step functions (i.e., $u_k = \sum_{j=1}^{n_k} a_j^k \chi_{A_j^k}$ for a finite number n_k of Borel subsets $A_j^k \subset [0, T]$ and with $a_j^k \in X$) converging to u almost everywhere with respect to the Lebesgue measure in $[0, T]$. Then for $1 \le p < \infty$, $L^p(0, T; X)$ is the space of all measurable functions $u : [0, T] \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty,$$

while $L^\infty(0, T; X)$ is the space of measurable functions such that

$$\|u\|_{L^\infty(0,T;X)} = \sup_{t\in[0,T]} \|u(t)\|_X < \infty.$$

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This choice makes $L^p(0,T;X)$ a Banach space and guarantees that its dual can be identified with $L^{p'}(0,T;X')$, where $p' = p/(p-1)$ and X' is the dual space to X. In particular, for functions $f \in L^2(0,T;L^1(\Omega))$ the continuous Minkowski inequality yields $f \in L^1(0,T;L^2(\Omega))$ and moreover

$$\|f\|_{L^2(0,T;L^1(\Omega))} := \left(\int_0^T \left(\int_\Omega |f|\,dx\right)^2 dx\right)^{1/2} \le \int_\Omega \left(\int_0^T |f|^2\,dt\right)^{1/2} dx =: \|f\|_{L^1(0,T;L^2(\Omega))}.$$

Hence, we have $L^2(0,T;L^1(\Omega)) \hookrightarrow L^1(0,T;L^2(\Omega))$. The full presentation of this topic can be found in [18].

2.2. Variable Exponent

Let $u \in L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a given function. We associate with $u : Q_T \mapsto \mathbb{R}$ the variable exponent $p_u : Q_T \to \mathbb{R}$ defined by the rule (7).

Since $G_\sigma \in C^\infty(\mathbb{R}^2)$, it follows from (7) and absolute continuity of the Lebesgue integral that $1 < p_u(t,x) \le 2$ in Q_T and $p_u \in C^1([0,T];C^\infty(\mathbb{R}^2))$ even if u is just an absolutely integrable function in Q_T. Then, for each $t \in [0,T]$, $p_u(t,x) \approx 1$ in those points of Ω where some discontinuities are present in $u(t,\cdot)$, and $p_u(t,x) \approx 2$ if $u(t,x)$ is smooth or contains homogeneous features. So, $p_u(t,x)$ can be interpreted as a characteristic of the sparse texture of the function u.

For our further analysis, we make use of the following result (for comparison, we refer to [19, Lemma 2.1]).

Lemma 1. *Let $\{u_k\}_{k\in\mathbb{N}} \subset L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a given sequence of measurable functions. We assume that each element of this sequence is extended by zero outside of Q_T and*

$$\begin{gathered}\sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(0,T;L^2(\Omega))} < +\infty, \\ u_k \to u \ \text{ weakly in } L^1(0,T;L^1(\Omega)) \text{ for some } \ u \in L^1(0,T;L^1(\Omega)).\end{gathered} \tag{16}$$

Let

$$\left\{p_{u_k} = 1 + g\left(\frac{1}{h}\int_{t-h}^t |(\nabla G_\sigma * u_k(\tau,\cdot))|\,d\tau\right)\right\}_{k\in\mathbb{N}}$$

be the corresponding sequence of exponents. Then there exists a constant $C > 0$ depending on Ω, G, and $\sup_{k\in\mathbb{N}} \|u_k\|_{L^1(0,T;L^1(\Omega))}$ such that

$$p^- := 1 + \delta \le p_{u_k}(t,x) \le p^+ := 2, \quad \forall\,(t,x) \in Q_T,\ \forall\,k \in \mathbb{N}, \tag{17}$$

$$\{p_{u_k}(\cdot)\} \subset \mathfrak{S} = \left\{ q \in C^{0,1}(Q_T) \;\middle|\; \begin{array}{c} |q(t,x) - q(s,y)| \le C\left(|x-y| + |t-s|\right), \\ \forall\,(t,x),(s,y) \in \overline{Q_T}, \\ 1 < p^- \le q(\cdot,\cdot) \le p^+ \text{ in } \overline{Q_T}. \end{array} \right\} \tag{18}$$

$$\begin{gathered} p_{u_k} \to p_u = 1 + g\left(\frac{1}{h}\int_{t-h}^t |(\nabla G_\sigma * u(\tau,\cdot))(\cdot)|\,d\tau\right) \\ \text{uniformly in } \overline{Q_T} \text{ as } k \to \infty. \end{gathered} \tag{19}$$

Proof. Since the sequence $\{u_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^1(0,T;L^1(\Omega))$ and the Gaussian filter kernel G_σ is smooth, it follows that

$$\begin{aligned}\frac{1}{h}\int_{t-h}^{t}\left|(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot))(x)\right|d\tau &\le h^{-1}\int_{t-h}^{t}\left(\int_\Omega |\nabla G_\sigma(x-y)|\,|\widetilde{u}_k(\tau,y)|\,dy\right)d\tau\\ &\le \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}h^{-1}\|u_k\|_{L^1(0,T;L^1(\Omega))},\\ 2\ge p_{u_k}(t,x) = 1+\delta+&\frac{a^2(1-\delta)h^2}{a^2h^2+\left(\int_{t-h}^{t}|(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot))\,(x)|\,d\tau\right)^2}\\ &\ge 1+\delta+\frac{a^2h^2(1-\delta)}{a^2h^2+\|u_k\|^2_{L^1(0,T;L^1(\Omega))}\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}},\\ &\qquad\forall\,(t,x)\in Q_T,\end{aligned}$$

where

$$\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} = \max_{\substack{z=x-y\\ x\in\overline{\Omega},y\in\overline{\Omega}}}\left[|\nabla G_\sigma(z)|+|G_\sigma(z)|\right] = \frac{e^{-1}}{\left(\sqrt{2\pi}\sigma\right)^2}\left[1+\frac{1}{\sigma^2}\operatorname{diam}\Omega\right]. \tag{20}$$

Then L^1-boundedness of $\{u_k\}_{k\in\mathbb{N}}$ guarantees the existence of a value $\widehat{\delta}\in(0,1)$ such that $\widehat{\delta}>\delta$ and $p_{u_k}(t,x)\ge 1+\widehat{\delta}$. Hence, the estimate (17) holds true for all $k\in\mathbb{N}$.

Moreover, as immediately follows from the relations

$$\begin{aligned}&\left|p_{u_k}(t,x)-p_{u_k}(t,y)\right|\\ &\le \frac{a^2h^2(1-\delta)}{a^4h^4}\left|\left(\int_{t-h}^{t}|(\nabla G_\sigma * u_k(\tau,\cdot))\,(x)|\,d\tau\right)^2-\left(\int_{t-h}^{t}|(\nabla G_\sigma * u_k(\tau,\cdot))\,(y)|\,d\tau\right)^2\right|\\ &\le \frac{1-\delta}{a^2h^2}\int_0^T\left(|(\nabla G_\sigma * u_k(\tau,\cdot))\,(x)|+|(\nabla G_\sigma * u_k(\tau,\cdot))\,(y)|\right)d\tau\\ &\quad\times\int_0^T|(\nabla G_\sigma * u_k(\tau,\cdot))\,(x)-(\nabla G_\sigma * u_k(\tau,\cdot))\,(y)|\,d\tau\\ &\le \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}(1-\delta)\|u_k\|_{L^1(0,T;L^1(\Omega))}}{a^2h^2}\\ &\quad\times\int_0^T\int_\Omega|u(\tau,z)|\,dz\,d\tau\max_{z\in\Omega}|\nabla G_\sigma(x-z)-\nabla G_\sigma(y-z)|\\ &= \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}(1-\delta)\gamma_1^2}{a^2h^2}\max_{z\in\Omega}|\nabla G_\sigma(x-z)-\nabla G_\sigma(y-z)|,\quad\forall\,x,y\in\overline{\Omega}\end{aligned} \tag{21}$$

with $\gamma_1^2=\left(\sup_{k\in\mathbb{N}}\|u_k\|_{L^1(0,T;L^1(\Omega))}\right)^2$, and from the smoothness of the function $\nabla G_\sigma(\cdot)$, there exists a constant $C_G>0$ independent of k such that, for each $t\in[0,T]$, the following estimate

$$\left|p_{u_k}(t,x)-p_{u_k}(t,y)\right|\le\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}(1-\delta)\gamma_1^2C_G}{a^2h^2}|x-y|,\ \forall\,x,y\in\overline{\Omega}$$

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holds true. Arguing in a similar manner, we see that

$$\begin{aligned}
&\left|p_{u_k}(s,y) - p_{u_k}(t,y)\right| \\
&\le \frac{1-\delta}{a^2h^2}\left|\left(\int_{t-h}^{t} \left|\left(\nabla G_\sigma * u_k(\tau,\cdot)\right)(y)\right|\,d\tau\right)^2 - \left(\int_{s-h}^{s} \left|\left(\nabla G_\sigma * u_k(\tau,\cdot)\right)(y)\right|\,d\tau\right)^2\right| \\
&\le \frac{2(1-\delta)\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1}{a^2h^2} \\
&\quad\times \left|\int_{s}^{t} \left|\left(\nabla G_\sigma * u_k(\tau,\cdot)\right)(y)\right|\,d\tau - \int_{s-h}^{t-h} \left|\left(\nabla G_\sigma * u_k(\tau,\cdot)\right)(y)\right|\,d\tau\right| \\
&\le \frac{4(1-\delta)\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}\sqrt{|\Omega|}\gamma_1\gamma_2}{a^2h^2}|s-t|, \quad \forall\, t,s \in [0,T]
\end{aligned} \tag{22}$$

with $\gamma_2 = \sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(0,T;L^2(\Omega))}$.

Utilizing the estimates (21)–(22), and setting

$$C := \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}(1-\delta)\gamma_1}{a^2h^2}\left(\gamma_1 C_G + 2\gamma_2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\sqrt{|\Omega|}\right), \tag{23}$$

we obtain

$$\begin{aligned}
\left|p_{u_k}(s,x) - p_{u_k}(t,y)\right| &\le \left|p_{u_k}(t,x) - p_{u_k}(t,y)\right| + \left|p_{u_k}(t,y) - p_{u_k}(s,y)\right| \\
&\le C\left[|y-x| + |t-s|\right], \\
&\forall\,(t,x),(s,y) \in \overline{Q_T} := [0,T]\times\overline{\Omega}.
\end{aligned} \tag{24}$$

Thus, we see that $\{p_{u_k}\} \subset \mathfrak{S}$. Since each element of the sequence $\{p_{u_k}\}_{k\in\mathbb{N}}$ has the same modulus of continuity and $\max_{(t,x)\in\overline{Q_T}} |p_{u_k}(t,x)| \le p^+$, it follows that this sequence is uniformly bounded and equi-continuous. Hence, by Arzelà–Ascoli Theorem the sequence $\{p_{u_k}\}_{k\in\mathbb{N}}$ is relatively compact with respect to the strong topology of $C(\overline{Q_T})$. Then, in view of the estimate (24) and the fact that the set $\mathfrak{S}$ is closed with respect to the uniform convergence and

$$\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * u_k(\tau,\cdot)\right)(x)\right|\,d\tau \to \frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * u(\tau,\cdot)\right)(x)\right|\,d\tau \\ \text{as } k\to\infty,\ \forall\,(t,x)\in Q_T$$

by definition of the weak convergence in $L^1(0,T;L^1(\Omega))$, we finally deduce: $p_{u_k} \to p_u$ uniformly in $\overline{Q_T}$ as $k\to\infty$, where

$$p_u(t,x) = 1 + g\left(\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * u(\tau,\cdot)\right)(\cdot)\right|\,d\tau\right)$$

in Q_T. □

2.3. Anisotropic Diffusion Tensor

Let $\mathbb{S}^2$ be the set of all symmetric matrices $A = [a_{ij}]_{i,j=1}^2$, ($a_{ij} = a_{ji} \in \mathbb{R}$). We endow $\mathbb{S}^2$ with the Euclidian scalar product $A\cdot B = \operatorname{tr}(A\,B) := \sum_{i,j=1}^2 a_{ij}b_{ij}$ and with the corresponding Euclidian norm $\|A\|_{\mathbb{S}^2} = (A\cdot A)^{1/2} = \sqrt{\operatorname{tr}(A^2)}$. We also introduce the spectral norm $\|A\|_2 := \sup\{|A\xi| : \xi\in\mathbb{R}^2 \text{ with } |\xi| = 1\}$ of matrices $A \in \mathbb{S}^2$. Note that in this case we have the following relation $\|A\|_2 \le \|A\|_{\mathbb{S}^2} \le \sqrt{2}\|A\|_2$ for all $A\in\mathbb{S}^2$.

Let $u \in L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be a given function. We assume that this function is zero-extended outside of Q_T. Let $u_\sigma(t,x)$ be its convolution with 2-dimensional Gaussian kernel of width (standard deviation) $\sigma > 0$ (see (9)–(10)).

Being mainly motivated by the practical implementations in image processing [20,21], we define the structure tensor $J_\rho(u_\sigma)$ associated with the function $u : Q_T \mapsto \mathbb{R}$ as follows

$$J_\rho(u_\sigma) := \frac{1}{h}\int_{t-h}^{t} G_\rho * (\nabla u_\sigma \otimes \nabla u_\sigma)\, d\tau = \frac{1}{h}\int_{t-h}^{t} G_\rho * \left(\nabla u_\sigma (\nabla u_\sigma)^t\right) d\tau, \tag{25}$$

where G_ρ stands for the Gaussian convolution kernel, and

$$\nabla u_\sigma(t,x) = (\nabla G_\sigma * \widetilde{u}(t,\cdot))\,(x).$$

It is easy to check that the symmetric matrix $J_\rho(u_\sigma) = \begin{bmatrix} j_{11} & j_{12} \\ j_{12} & j_{22} \end{bmatrix}$ is positively semi-definite and uniformly bounded in Ω. Indeed, for any $\xi \in \mathbb{R}^2$, we have

$$\begin{aligned}
\xi^t J_\rho(u_\sigma)\xi &\le 2\frac{1}{h}\int_{t-h}^{t}\int_\Omega G_\rho(x-y)|\nabla u_\sigma(\tau,\cdot)|^2|\xi|^2\, dx d\tau \\
&\le \frac{2e^{-1}h^{-1}}{\left(\sqrt{2\pi}\rho\right)^2}|\Omega|\int_{t-h}^{t}\|u(\tau,\cdot)\|^2_{L^1(\Omega)}\, d\tau \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\xi|^2, \\
&\le \frac{2e^{-1}}{\left(\sqrt{2\pi}\rho\right)^2}\|u\|^2_{L^\infty(0,T;L^1(\Omega))}\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\Omega||\xi|^2, \quad \forall\,(t,x)\in Q_T, \qquad (26)\\
\xi^t J_\rho(u_\sigma)\xi &= \frac{1}{h}\int_{t-h}^{t}\int_\Omega G_\rho(x-y)\left(\nabla u_\sigma(\tau,y),\xi\right)^2_{\mathbb{R}^2}\, dy d\tau \ge 0, \quad \forall\,(t,x)\in Q_T. \qquad (27)
\end{aligned}$$

Taking this into account, we introduce the relaxed version of the anisotropic diffusion tensor $D_u(t,x)$ and define it as follows (for comparison we refer to [1,2])

$$D_u(t,x) := \gamma I + J_\rho(u_\sigma), \tag{28}$$

where $0 < \gamma \ll 1$ is a small parameter, and $I \in (\mathbb{R}^2, \mathbb{R}^2)$ is the unit matrix.

Then it easily follows from (26)–(27) and (28) that, for any $u \in C([0,T]; L^1(\Omega))$, the two-side estimate

$$d_1^2|\xi|^2 \le \xi^t\,[D_u(t,x)]^2\,\xi \le d_2^2|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^2, \quad \forall\,(t,x)\in Q_T. \tag{29}$$

holds true with

$$d_1 = \gamma, \quad d_2 = d_1 + \frac{2e^{-1}}{\left(\sqrt{2\pi}\rho\right)^2}\|u\|^2_{L^\infty(0,T;L^1(\Omega))}|\Omega|\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}.$$

For the further convenience, we always suppose that $d_2 \ge 1$.

Then arguing as in the proof of Lemma 1, it is easy to establish the following result.

Lemma 2. *Let $\{u_k\}_{k\in\mathbb{N}}, u \subset L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be measurable and extended by zero outside of Q_T functions with properties (16). Let $\left\{D_{u_k}(t,x)\right\}_{k\in\mathbb{N}}$ be a collection of the corresponding anisotropic diffusion tensors. Then*

$$d_1^2|\xi|^2 \le \xi^t\left[D_{u_k}(t,x)\right]^2\xi \le d_2^2|\xi|^2, \quad \forall\,(t,x)\in Q_T,\ \forall\,\xi\in\mathbb{R}^2, \quad \forall\,k\in\mathbb{N}, \tag{30}$$

$$D_{u_k}(t,x) \to D_u(t,x) \text{ uniformly in } \overline{Q_T} \text{ as } k\to\infty, \tag{31}$$

$$\left\{D_{u_k}\right\} \subset \mathfrak{D}, \tag{32}$$

where

$$d_1 = \gamma, \quad , d_2 = d_1 + \frac{2e^{-1}}{\left(\sqrt{2\pi}\rho\right)^2} \sup_{k \in \not\mathbb{F}} \|u_k\|^2_{L^\infty(0,T;L^1(\Omega))} \|G_\sigma\|^2_{C^1(\overline{\Omega - \Omega})} |\Omega|,$$

$$\mathfrak{D} = \left\{ \Lambda \in C^{0,1}(Q_T; \mathbb{R}^{2\times 2}) \;\middle|\; \begin{array}{c} \|\Lambda(s,x) - \Lambda(t,y)\|_2 \le C\left(|x-y| + |t-s|\right), \\ \forall\,(t,x),(s,y) \in \overline{Q_T}. \end{array} \right\}$$

2.4. On Orlicz Spaces

Let $w \in L^1(0,T;L^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ be given. Let $p_w : Q_T \to \mathbb{R}$ be the corresponding variable exponent defining by the rule (7). Then

$$1 < p^- \le p_w(t,x) \le p^+ < \infty \text{ a.e. in } Q_T \tag{33}$$

(see Lemma 1), where p^- and p^+ are the constants given by (17). Let $p'_w(t,x) = \frac{p_w(t,x)}{p_w(t,x)-1}$ be the corresponding conjugate exponent. Then

$$2 = \underbrace{\frac{p^+}{p^+ - 1}}_{(p^+)'} \le p'_w(t,x) \le \underbrace{\frac{p^-}{p^- - 1}}_{(p^-)'} = \frac{p^-}{\delta} \;\; a.e. \text{ in } Q_T, \tag{34}$$

where $(p^+)'$ and $(p^-)'$ stand for the conjugates of constant exponents. Denote by $L^{p_w(\cdot)}(Q_T)$ the set of all measurable functions $f : Q_T \to \mathbb{R}$ such that their modular is finite, i.e.

$$\rho_{p_w(t,x)}(f) := \int_{Q_T} |f(t,x)|^{p_w(t,x)}\,dxdt < \infty. \tag{35}$$

Equipped with the Luxembourg norm

$$\|f\|_{L^{p_w(\cdot)}(Q_T)} = \inf\left\{\lambda > 0 \;:\; \int_{Q_T} |\lambda^{-1} f(t,x)|^{p_w(t,x)}\,dxdt \le 1\right\}. \tag{36}$$

$L^{p_w(\cdot)}(Q_T)$ becomes a Banach space (see [22,23] for the details). The space $L^{p_w(\cdot)}(Q_T)$ is a sort of Musielak-Orlicz space. In fact, it can be denoted by generalised Lebesgue space because its main properties are inherited from the classical Lebesgue spaces. In particular, the two-sides inequality (33) implies that $L^{p_w(\cdot)}(Q_T)$ is reflexive, separable, and the set $C_0^\infty(Q_T)$ is dense in $L^{p_w(\cdot)}(Q_T)$. Moreover, under condition (33), $L^\infty(Q_T) \cap L^{p_w(\cdot)}(Q_T)$ is also dense in $L^{p_w(\cdot)}(Q_T)$.

Its dual can be identified with $L^{p'_w(\cdot)}(Q_T)$ and, therefore, any continuous functional $F = F(f)$ on $L^{p_w(\cdot)}(Q_T)$ has the representation (see [13, Lemma 13.2])

$$F(f) = \int_{Q_T} fg\,dxdt, \quad \text{with } g \in L^{p'_w(\cdot)}(Q_T).$$

Since the relation between the modular (35) and the norm (36) is not so direct as in the classical Lebesgue spaces, it can be proven, from its definitions in (35) and (36), that

$$\begin{aligned} \min\left\{\|f\|^{p^-}_{L^{p_w(\cdot)}(Q_T)}, \|f\|^{p^+}_{L^{p_w(\cdot)}(Q_T)}\right\} &\le \rho_{p_w(t,x)}(f) \le \max\left\{\|f\|^{p^-}_{L^{p_w(\cdot)}(Q_T)}, \|f\|^{p^+}_{L^{p_w(\cdot)}(Q_T)}\right\}, \\ \min\left\{\rho^{\frac{1}{p^-}}_{p_w(t,x)}(f), \rho^{\frac{1}{p^+}}_{p_w(t,x)}(f)\right\} &\le \|f\|_{L^{p_w(\cdot)}(Q_T)} \le \max\left\{\rho^{\frac{1}{p^-}}_{p_w(t,x)}(f), \rho^{\frac{1}{p^+}}_{p_w(t,x)}(f)\right\}. \end{aligned} \tag{37}$$

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The following consequence of (37) is very useful,

$$\|f\|^{p^-}_{L^{p_w(\cdot)}(Q_T)} - 1 \le \int_{Q_T} |f(t,x)|^{p_w(t,x)}\, dxdt \le \|f\|^{p^+}_{L^{p_w(\cdot)}(Q_T)} + 1, \quad \forall f \in L^{p_w(\cdot)}(Q_T), \tag{38}$$

$$\|f_k - f\|_{L^{p_w(\cdot)}(Q_T)} \to 0 \iff \int_{Q_T} |f_k(t,x) - f(t,x)|^{p_w(t,x)}\, dxdt \to 0 \quad \text{as } k \to \infty. \tag{39}$$

Moreover, if $f \in L^{p_w(\cdot)}(Q_T)$ then

$$\|f\|_{L^{p^-}(Q_T)} \le (1 + T|\Omega|)^{1/p^-} \|f\|_{L^{p_w(\cdot)}(Q_T)}, \tag{40}$$

$$\|f\|_{L^{p_w(\cdot)}(Q_T)} \le (1 + T|\Omega|)^{1/(p^+)'} \|f\|_{L^{p^+}(Q_T)}, \quad (p^+)' = \frac{p^+}{p^+ - 1}, \; \forall f \in L^{p^+}(Q_T), \tag{41}$$

(see, for instance, [22–24]).

In generalised Lebesgue spaces, there holds a version of Young's inequality,

$$|fg| \le \varepsilon \frac{|f|^{p_w(\cdot)}}{p_w(\cdot)} + C(\varepsilon) \frac{|g|^{p'_w(\cdot)}}{p'_w(\cdot)},$$

with some positive constant $C(\varepsilon)$ and arbitrary $\varepsilon > 0$.

The following result can be interpreted as an analogous of the Hölder inequality in variable Lebesgue spaces (for the details, we refer to [22,23]).

Proposition 1. *If $f \in L^{p_w(\cdot)}(Q_T; \mathbb{R}^2)$ and $g \in L^{p'_w(\cdot)}(Q_T; \mathbb{R}^2)$, then $(f, g) \in L^1(Q_T)$ and*

$$\int_{Q_T} (f, g)\, dxdt \le 2\|f\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^2)} \|g\|_{L^{p'_w(\cdot)}(Q_T;\mathbb{R}^2)}. \tag{42}$$

As a bi-product of (42), we have that, for a bounded domain $Q_T = (0, T) \times \Omega$ and for $p_w(\cdot)$ satisfying (33), the following imbedding

$$L^{p_w(\cdot)}(Q_T) \hookrightarrow L^{r(\cdot)}(Q_T) \text{ whenever } p_w(t,x) \ge r(t,x) \text{ for a.e. } (t,x) \in Q_T \tag{43}$$

is continuous.

Let $\widehat{\delta} \in (0, 1]$ and let $\{p_k\}_{k\in\mathbb{N}} \subset C^{0,\widehat{\delta}}(\overline{Q_T})$ be a given sequence of exponents. Assume that

$$p, p_k \in C^{0,\widehat{\delta}}(\overline{Q_T}) \text{ for } k = 1, 2, \dots, \text{ and} \quad p_k(\cdot) \to p(\cdot) \text{ uniformly in } \overline{Q_T} \text{ as } k \to \infty. \tag{44}$$

We associate with these exponents the another sequence $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$. We see that each element f_k lives in the corresponding Orlicz space $L^{p_k(\cdot)}(Q_T)$. So, in fact, we have a sequence in the scale of spaces $\left\{L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$. We say that the sequence $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ is bounded if

$$\limsup_{k\to\infty} \int_{Q_T} |f_k(t,x)|^{p_k(t,x)}\, dxdt < +\infty. \tag{45}$$

Definition 1. *A bounded sequence* $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ *is called to be weakly convergent in the variable Orlicz space* $L^{p_k(\cdot)}(Q_T)$ *to a function* $f \in L^{p(\cdot)}(Q_T)$*, where* $p \in C^{0,\delta}(\overline{Q_T})$ *is the limit of* $\{p_k\}_{k\in\mathbb{N}} \subset C^{0,\widehat{\delta}}(\overline{Q_T})$ *in the uniform topology of* $C(\overline{Q_T})$*, if*

$$\lim_{k\to\infty} \int_{Q_T} f_k\varphi\, dxdt = \int_{Q_T} f\varphi\, dxdt, \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^2). \tag{46}$$

For our further analysis, we make use of the following assertion concerning the lower semicontinuity property of the variable $L^{p_k(\cdot)}$-norm with respect to the weak convergence in $L^{p_k(\cdot)}(Q_T)$ (for the proof, we refer to [25, Lemma 3.1], see also [13, Lemma 13.3] and [19, Lemma 2.1] for comparison).

Proposition 2. *Assume that a sequence of exponents* $\{p_k\}_{k\in\mathbb{N}}$ *satisfies condition (33),* $p_k \to p$ *as* $k\to\infty$ *a.e.in* Q_T*, and a bounded sequence* $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ *converges weakly in* $L^{p^-}(Q_T)$ *to* f*. Then* $f\in L^{p(\cdot)}(Q_T)$*,* $f_k \rightharpoonup f$ *in variable* $L^{p_k(\cdot)}(Q_T)$*, and*

$$\liminf_{k\to\infty} \int_{Q_T} |f_k(t,x)|^{p_k(t,x)}\, dxdt \ge \int_{Q_T} |f(t,x)|^{p(t,x)}\, dxdt. \tag{47}$$

We recall also the inequality which is well-known in the theory of p-Laplace equations: if $1 < p \le 2$ then, for all $\xi,\eta \in \mathbb{R}^N$, the following estimate holds true

$$(p-1)|\xi-\eta|^2 \le \left(\left[|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right], \xi-\eta\right)\left(|\xi|^p + |\eta|^p\right)^{\frac{2-p}{p}}.$$

2.5. On Weighted Energy Space with Variable Exponent

Let $D_w(t,x)$ be a diffusion tensor associated with some function $w \in C([0,T];L^2(\Omega))$ and given by the rule (28). We define the weighted energy space $W_w(Q_T)$ as the set of all functions $u(t,x)$ such that

$$\begin{aligned} &u\in L^2(Q_T), \quad u(t,\cdot)\in W^{1,1}(\Omega) \text{ for a.e. } t\in[0,T],\\ &\int_{Q_T} |D_w(t,x)\nabla u|^{p_w(t,x)}\, dxdt < +\infty. \end{aligned} \tag{48}$$

We equip the space $W_w(Q_T)$ with the norm

$$\|u\|_{W_w(Q_T)} = \|u\|_{L^2(Q_T)} + \|D_w\nabla u\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^2)}, \tag{49}$$

where the second term in (49) is the norm of the vector-valued function $D_w(t,x)\nabla u(t,x)$ in the Orlicz space $L^{p_w(\cdot)}(Q_T;\mathbb{R}^2)$. Since (see (29))

$$d_1^2|\xi|^2 \le \xi^t\,[D_w(t,x)]^2\,\xi \le d_2^2|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^2,\ \forall\,(t,x)\in Q_T, \tag{50}$$

it follows that $W_w(Q_T)$, equiped with the norm (49), is a reflexive Banach space. Since the exponent $p_w : Q_T \to \mathbb{R}$ is Lipschitz continuous, it follows that the smooth functions are dense in the weighted Sobolev-Orlicz space $W_w(Q_T)$ (see [26]). So, $W_w(Q_T)$ can be represented as the closure of the set $\{\varphi\in C^\infty(\overline{Q}_T)\}$ with respect to the norm $\|\cdot\|_{W_w(Q_T)}$.

2.6. On the weak convergence of fluxes to flux

Let us consider the following set of parabolic equations of monotone type

$$\frac{\partial u_k}{\partial t} - \operatorname{div} A_k(t,x,\nabla u_k) = f, \quad (t,x)\in Q_T, \tag{51}$$

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where $f \in L^2(\Omega)$ and $k = 1, 2, \ldots$. Let, for a given $k \in \mathbb{N}$, u_k be a solution of (51) in the sense of distributions. Assume that $A_k(\cdot,\cdot,\xi) \to A(\cdot,\cdot,\xi)$ as $k \to \infty$ pointwise a.e. with respect to the first two arguments and for all $\xi \in \mathbb{R}^N$.

A typical situation in the study of optimization problems can be described as follows: a solution $u_k \in L^2(0,T;W^{1,p^-}(\Omega))$ of (51) and the corresponding flow $w_k = A_k(\cdot,\cdot,\nabla u_k) \in L^{(p^+)'}(Q_T;\mathbb{R}^N)$ converge weakly, namely,

$$u_k \rightharpoonup u \ \text{ in } \ L^2(0,T;W^{1,p^-}(\Omega)), \quad w_k \rightharpoonup w \ \text{ in } \ L^{(p^+)'}(Q_T;\mathbb{R}^N),$$
$$1 < p^- < p^+, \ (p^+)' = \frac{p^+}{p^+ - 1}.$$

The main question is whether the equality for the limit elements $A(t,x,\nabla u) = w$ holds, i.e., whether a flux converges to a flux. The situation, in general, is not trivial because the function $A(\cdot,\cdot,v)$ is nonlinear in v and the weak convergence $v_k \rightharpoonup v$ is rather far from sufficient to derive the limit relation $A_k(\cdot,\cdot,v_k) \rightharpoonup A(\cdot,\cdot,v)$. So, the important problem is to show that $w = A(\cdot,\cdot,\nabla u)$. The conditions (first of all, on the exponents p^- and p^+) under which the answer to the above question is affirmative, have been obtained by Zhikov and Pastukhova in their celebrated paper [27].

Theorem 1. *Assume that the following assumptions are satisfied:*

- **(C1)** $A_k(t,x,\xi)$ *and* $A(t,x,\xi)$ *are* $\mathbb{R}^N$*-valued Carathéodory functions, i.e., these functions are continuous in* $\xi \in \mathbb{R}^N$ *for a.e.* $(t,x) \in Q_T$ *and measurable with respect to* $(t,x) \in Q_T$ *for each* $\xi \in \mathbb{R}^N$;
- **(C2)** $\Big(A_k(t,x,\xi) - A_k(t,x,\zeta), \xi - \zeta\Big) \geq 0$, $A_k(t,x,0) = 0 \ \forall \xi,\zeta \in \mathbb{R}^N$ *and for a.e.* $(t,x) \in Q_T$;
- **(C3)** $|A_k(t,x,\xi)| \leq c(|\xi|) < \infty$ *and* $\lim_{k\to\infty} A_k(t,x,\xi) = A(t,x,\xi)$ *for all* $\xi \in \mathbb{R}^N$ *and for a.e.* $(t,x) \in Q_T$;
- **(C4)** $u_k \rightharpoonup u$ *in* $L^{p^-}(0,T;W^{1,p^-}(\Omega))$, $p^- > 1$, *and* $\{u_k\}_{k\in\mathbb{N}}$ *are bounded in* $L^\infty(0,T;L^2(\Omega))$;
- **(C5)** $w_k = A_k(t,x,\nabla u_k) \rightharpoonup w$ *in* $L^{(p^+)'}(Q_T;\mathbb{R}^N)$, $p^+ > 1$;
- **(C6)** $u_k \in L^{p^+}(0,T;W^{1,p^+}(\Omega))$ *for all* $k \in \mathbb{N}$, *and* $\sup_{k\in\mathbb{N}} \| (w_k, \nabla u_k) \|_{L^1(Q_T)} < \infty$;
- **(C7)** $1 < p^- < p^+ < 2p^-$.

Then the flux $A_k(t,x,\nabla u_k)$ *weakly converges in the Lebesgue space* $L^{(p^+)'}(Q_T;\mathbb{R}^N)$ *to the flux* $A(t,x,\nabla u)$.

Further we make use of the following well-known results.

Lemma 3 ([24]). *Let* Ψ *be a set of integrands* $F(t,x,\xi)$ *such that they are convex with respect to* $\xi \in \mathbb{R}^N$, *measurable with respect to* $(t,x) \in Q_T$, *and satisfy the estimate*

$$c_1|\xi|^{p^-} \leq F(t,x,\xi) \leq c_2|\xi|^{p^+}, \quad 1 < p^- \leq p^+ < \infty, \ c_1, c_2 > 0.$$

Assume that F_k *and* F *belong to the set* Ψ *and the following condition holds:*

$$\lim_{k\to\infty} F_k(t,x,\xi) = F(t,x,\xi) \quad \text{for a.e. } (t,x) \in Q_T \text{ and any } \xi \in \mathbb{R}^N.$$

Then the following lower semicontinuity property is valid: if $v_k \rightharpoonup v$ *in* $L^1(Q_T;\mathbb{R}^N)$ *then*

$$\liminf_{k\to\infty} \int_{Q_T} F_k(t,x,v_k)\, dxdt \geq \int_{Q_T} F(t,x,v)\, dxdt. \tag{52}$$

Lemma 4 ([28]). *Let* $A_k(t,x,\xi)$ *and* $A(t,x,\xi)$ *be* $\mathbb{R}^N$*-valued Carathéodory functions with properties (C1)–(C3). Assume that*

$$v_k \rightharpoonup v, \ \ w_k = A_k(t,x,v_k) \rightharpoonup w \ \text{ in } L^1(Q_T;\mathbb{R}^N) \text{ as } k \to \infty,$$

and $(w,v) \in L^1(Q_T)$. *Then*

$$\liminf_{k\to\infty} \int_{Q_T} (A_k(t,x,v_k), v_k)\, dxdt \geq \int_{Q_T} (w,v)\, dxdt. \tag{53}$$

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3. Existence Result for a Class of Parabolic Equations with Variable Nonlocal Exponent

Let $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$ be given distributions. Let us consider the following initial-boundary value problem (IBVP)

$$\frac{\partial u}{\partial t} - \operatorname{div} A_u(t,x,\nabla u) + \kappa u = \kappa(f - v) \quad \text{in } Q_T, \tag{54}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{55}$$

$$u(0,\cdot) = f_0 \quad \text{in } \Omega. \tag{56}$$

Here,

$$A_w(t,x,\nabla u) := |D_w(t,x)\nabla u|^{p_w(t,x)-2} D_w(t,x)\nabla u, \tag{57}$$

the exponent $p_w : Q_T \to (1,2]$ is given by (7), the matrix $D_w(t,x)$ is defined in (28), ∂_ν is the outward normal derivative, $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$ are given distributions, $v \in \mathcal{V}_{ad}$ stands for the control, and the class of admissible controls $\mathcal{V}_{ad}$ is defined as

$$\mathcal{V}_{ad} = \left\{ v \in L^2(Q_T) \ : \ v_a(x) \le v(t,x) \le v_b(x), \text{ a.e. in } Q_T \right\}. \tag{58}$$

As follows from (57), (28), and Lemma 1, for each function $w \in C([0,T]; L^2(\Omega))$, the mapping $(t,x,\xi) \mapsto A_w(t,x,\xi)$ is a Carathéodory vector function, i.e., $A_w(t,x,\xi)$ is continuous in $\xi \in \mathbb{R}^2$ and is measurable with respect to (t,x) for each $\xi \in \mathbb{R}^2$. Moreover, the following monotonicity, coerciveness and boundedness conditions hold for a.e. $(t,x) \in Q_T$ [13]:

$$\Big(A_w(t,x,\xi) - A_w(t,x,\zeta), \xi - \zeta\Big) \ge 0, \quad \forall\, \xi,\zeta \in \mathbb{R}^2, \tag{59}$$

$$\begin{aligned} (A_w(t,x,\xi),\xi) &= |D_w(t,x)\xi|^{p_w(t,x)-2}\Big(D_w(t,x)\xi, D_w^{-1}(t,x)D_w(t,x)\xi\Big) \\ &\overset{\text{by } (50)}{\ge} d_2^{-1} d_1^{p_w(t,x)} |\xi|^{p_w(t,x)} \ge d_2^{-1} d_1^2 |\xi|^{p_w(t,x)}, \quad \forall\, \xi \in \mathbb{R}^2, \end{aligned} \tag{60}$$

$$|A_w(t,x,\xi)|^{p'_w(t,x)} \le d_2^{p_w(t,x)} |\xi|^{p_w(t,x)} \le d_2^2 |\xi|^{p_w(t,x)}, \quad \forall\, \xi \in \mathbb{R}^2, \tag{61}$$

However, in general, the operator $-\operatorname{div} A_u(t,x,\nabla u) + \kappa u$ provides an example of a strongly non-linear, non-monotone, and non-coercive operator in divergence form. Mainly because of this, the existence of the strong solutions (see Definition 2.1 in [29]) to IBVP (54)–(56) and the issue of their uniqueness remain, apparently, open questions for nowadays [30, Chapter III]. In view of this, we make use of the following concept:

Definition 2. *We say that, for given $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$, a function u is a weak solution to the problem (54)–(56) if $u \in W_u(Q_T)$, i.e.,*

$$\begin{aligned} &u \in L^2(Q_T),\ u(t,\cdot) \in W^{1,1}(\Omega) \ \text{for a.e. } t \in [0,T], \\ &\int_{Q_T} |D_u(t,x)\nabla u|^{p_u(t,x)}\, dxdt < +\infty, \end{aligned} \tag{62}$$

and the integral identity

$$\int_{Q_T} \left(-u\frac{\partial \varphi}{\partial t} + (A_u(t,x,\nabla u), \nabla\varphi) + \kappa u\varphi \right) dxdt = \kappa \int_{Q_T} (f - v)\varphi\, dxdt + \int_\Omega f_0 \varphi|_{t=0}\, dx \tag{63}$$

holds true for any $\varphi \in \Phi$, where $\Phi = \left\{\varphi \in C^\infty(\overline{Q}_T) \ : \ \varphi|_{t=T} = 0\right\}$.

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To clarify the sense in which the initial value $u(0,\cdot) = f_0$ is assumed for the weak solutions, we give the following assertion (for the proof, we refer to [19, Proposition 2.2]).

Proposition 3. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$ be given distributions. Let $u \in W_u(Q_T)$ be a weak solution to the problem (54)–(56) in the sense of Definition 2. Then, for any $\eta \in C^\infty(\overline{\Omega})$, the scalar function $h(t) = \int_\Omega u(t,x)\eta(x)\,dx$ belongs to $W^{1,1}(0,T)$ and $h(0) = \int_\Omega f_0(x)\eta(x)\,dx$.*

We next recall some known results that have been recently proven basing on the Schauder fixed-point theorem and using the perturbation technique (see [19, Theorem 3.2]).

Theorem 2. *For given $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$, the problem (54)–(56) admits at least one weak solution $u \in W_u(Q_T)$ for which the following energy inequality*

$$\frac{1}{2}\int_\Omega u^2\,dx + \int_0^t \int_\Omega \Big((A_u(t,x,\nabla u), \nabla u) + \kappa u^2 \Big)\,dxdt \\ \le \kappa \int_{Q_T} (f+v)u\,dxdt + \int_\Omega f_0^2\,dx \quad (64)$$

holds true for all $t \in [0,T]$.

Using (64), we can derive the following estimates

$$\|u\|^2_{L^2(Q_T)} \le C_1^2 = \kappa T \left(\|f\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(Q_T)} \right) + \kappa^{-1}\|f_0\|^2_{L^2(\Omega)}, \quad (65)$$

$$\|\nabla u\|_{L^{p_u(\cdot)}(Q_T;\mathbb{R}^2)} \overset{\text{by (38)}}{\le} \left(\int_{Q_T} |\nabla u|^{p_u(t,x)}\,dxdt + 1 \right)^{1/p^-} \\ \overset{\text{by (60)}}{\le} \left(\frac{d_2}{d_1^2} \left(\frac{3}{2}\|f_0\|^2_{L^2(\Omega)} + \frac{2\kappa + \kappa^2 T}{2} \left(\|f\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(Q_T)} \right) \right) + 1 \right)^{1/p^-} \\ =: C_2, \quad (66)$$

$$\|\nabla u\|_{L^{p^-}(Q_T;\mathbb{R}^2)} \le (1 + T|\Omega|)^{1/p^-} C_2, \quad (67)$$

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{2}\sqrt{\kappa \left(\|f\|^2_{L^2(Q_T)} + \|v\|^2_{L^2(Q_T)} \right) + \|f_0\|^2_{L^2(\Omega)}}. \quad (68)$$

Since the uniqueness issue for the weak solutions of the initial-boundary value problem (54)–(56) seems to be an open question, we adopt the following concept.

Definition 3. *We say that a weak solution $u \in W_u(Q_T)$ to the problem (54)–(56), for given distributions $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$, is W_0-attainable if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero as $n \to \infty$ and such that*

$$\left.\begin{array}{c} u_n \rightharpoonup u \ \text{ in } \ L^{p^-}(0,T;W^{1,p^-}(\Omega)), \\ A_{u_{n-1}}(t,x,\nabla u_n) \rightharpoonup A_u(t,x,\nabla u) \ \text{ in } \ L^{(p^+)'}(Q_T;\mathbb{R}^2) \end{array}\right\} \text{ as } n \to \infty, \quad (69)$$

where

$$u_n \in W(0,T) = \left\{ w \in L^2(0,T;W^{1,2}(\Omega)),\ \frac{dw}{dt} \in L^2(0,T;\left[W^{1,2}(\Omega)\right]') \right\}, \quad \forall n \in \mathbb{N}, \\ A_w(t,x,\nabla u) := |D_w(t,x)\nabla u|^{p_w(t,x)-2} D_w(t,x)\nabla u, \quad (70)$$

and, for each $n \in \mathbb{N}$, u_n is the weak solutions to the following perturbed problem

$$\frac{\partial u}{\partial t} - \varepsilon_n \Delta u - \operatorname{div} A_{u_{n-1}}(t,x,\nabla u) + \kappa u = \kappa(f - v) \quad in \ Q_T, \tag{71}$$

$$\partial_\nu u = 0 \quad on \ (0,T) \times \partial\Omega, \tag{72}$$

$$u(0,\cdot) = f_0 \quad in \ \Omega. \tag{73}$$

Remark 1. *It is worth to emphasize that (see the recent results in [19]) Theorem 3.3 can be now specified as follows: For given $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v \in \mathcal{V}_{ad}$, the initial-boundary value problem (54)–(56) admits at least one W_0-attainable weak solution $u \in W_u(Q_T)$ for which the energy inequality (64) holds true for all $t \in [0,T]$. Moreover, as follows from estimates (65)–(68), this solution in bounded in $L^{p^-}(0,T;W^{1,p^-}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.*

4. Setting of the Optimal Control Problem and Existence Result

As was pointed out in the previous sections, the operator $-\operatorname{div} A_u(t,x,\nabla u) + \kappa u$ provides an example of a non-linear operator in divergence form with is neither monotone nor coercive. In this case (see Theorem 2) a weak solution of the initial-boundary value problem (54)–(56) under some admissible control $v \in \mathcal{V}_{ad}$ may be not unique. Moreover, it is unknown whether all weak solutions to (54)–(56) satisfy energy inequality (64) that plays a crucial role for derivation of a priori estimates (65)–(68).

Our prime interest in this section is to consider the following optimal control problem of the tracking type

$$\begin{gathered}\text{Minimize } J(v,u) = \|v\|^2_{L^2(0,T;L^1(\Omega))} + \frac{\mu}{2}\int_\Omega |u(T) - f_0|^2\,dx \\ \text{subject to the constraints (2)–(4), (58),}\end{gathered} \tag{74}$$

where $f \in L^2(\Omega)$ is the original noise-corrupted image, $f_0 \in L^2(\Omega)$ is the pre-denoised image by applying a median filter to f, $v_a, v_b \in L^2(\Omega)$, $v_a(x) \le v_b(x)$ a.e. in Ω, are given distributions.

We say that $(v.u)$ is a feasible pair to this problem if:

$$\begin{gathered} v \in \mathcal{V}_{ad}, \quad u \in W_u(Q_T), \quad J(v,u) < +\infty, \\ (v,u) \text{ are related by integral identity (63) and inequality (64),} \\ \text{and } u \text{ is a } W_0\text{-attainable weak solution to (54)–(56) for the given } v. \end{gathered} \tag{75}$$

Let $\Xi \subset L^2(Q_T) \times W_u(Q_T)$ be the set of feasible solutions to the problem (74). Then Theorem 2 implies that $\Xi \neq \emptyset$. Since main topological properties of the set Ξ are unknown, in general, we begin with the following observation.

Theorem 3. *For given $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$, the set Ξ is sequentially closed with respect to the weak topology of $L^2(Q_T) \times L^{p^-}(0,T;W^{1,p^-}(\Omega))$.*

Proof. Let $\{(v_k,u_k)\}_{k\in\mathbb{N}} \subset \Xi$ be a sequence such that

$$v_k \rightharpoonup v \ \text{ in } \ L^2(Q_T), \quad u_k \rightharpoonup u \ \text{ in } \ L^{p^-}(0,T;W^{1,p^-}(\Omega)). \tag{76}$$

Since the set $\mathcal{V}_{ad}$ is convex and closed, it follows by Mazur's theorem that $\mathcal{V}_{ad}$ is sequentially closed with respect to the weak topology of $L^2(Q_T)$. Therefore, $v \in \mathcal{V}_{ad}$. Let us show that $(v,u) \in \Xi$. We will do it in several steps.

Step 1. By the initial assumptions, for each $k \in \mathbb{N}$, the pair (v_k, u_k) satisfies the energy inequality (64), and u_k is a W_0-attainable weak solution of (54)–(56). Hence, in view of Definition 3, we may always suppose that there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset W(0,T)$ such that $\{u_{k,n}\}_{n\in\mathbb{N}}$ are the weak solutions (in the sense of distributions) of (71)–(73) with $\varepsilon_n = 1/n$ and $v = v_k$. and

$$u_{k,n} \rightharpoonup u_k \text{ in } L^{p^-}(0,T;W^{1,p^-}(\Omega)), \quad \text{as } n \to \infty, \tag{77}$$

$$A_{u_{k,n-1}}(t,x,\nabla u_{k,n}) \rightharpoonup A_{u_k}(t,x,\nabla u_k) \text{ in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \quad \text{as } n \to \infty, \tag{78}$$

Moreover, the fact that the energy equality

$$\frac{1}{2}\int_\Omega u_{k,n}^2\,dx + \int_0^t\int_\Omega \left(\frac{1}{n}|\nabla u_{k,n}|^2 + \left(A_{u_{k,n-1}}(t,x,\nabla u_{k,n}),\nabla u_{k,n}\right) + \kappa u_{k,n}^2\right) dxdt$$
$$= \kappa\int_{Q_T}(f - v_k)u_{k,n}\,dxdt + \int_\Omega f_0^2\,dx, \quad \forall t \in [0,T] \tag{79}$$

is valid for all $n,k \in \mathbb{N}$, implies the boundedness of the sequence $\{u_{k,k}\}_{k\in\mathbb{N}}$ in the space $L^{p^-}(0,T;W^{1,p^-}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Hence, combining this fact with (77) and (76), we deduce

$$u_{k,k} \rightharpoonup u \text{ in } L^{p^-}(0,T;W^{1,p^-}(\Omega)), \quad \text{as } k \to \infty, \tag{80}$$

$$u_{k,k} \rightharpoonup u \text{ in } L^2(0,T;L^2(\Omega)), \quad \text{as } k \to \infty. \tag{81}$$

Step 2. Utilizing the energy equality (79) and arguing as in (65)–(68), we can derive the following a priori estimates

$$\|u_{k,k}\|^2_{L^2(Q_T)} \le \kappa T\left(\|f\|^2_{L^2(Q_T)} + \sup_{k\in\mathbb{N}}\|v_k\|^2_{L^2(Q_T)}\right) + \kappa^{-1}\|f_0\|^2_{L^2(\Omega)} =: S_1^2, \tag{82}$$

$$\|\nabla u_{k,k}\|^{p^-}_{L^{p_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)}$$
$$\le \frac{d_2}{d_1^2}\left(\frac{3}{2}\|f_0\|^2_{L^2(\Omega)} + \frac{2\kappa + \kappa^2 T}{2}\left(\|f\|^2_{L^2(Q_T)} + \sup_{k\in\mathbb{N}}\|v_k\|^2_{L^2(Q_T)}\right)\right) + 1 =: S_2, \tag{83}$$

$$\|\nabla u_{k,k}\|_{L^{p^-}(Q_T;\mathbb{R}^2)} \le (1 + T|\Omega|)^{1/p^-} S_2, \tag{84}$$

$$\|u_{k,k}\|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{2}\sqrt{\kappa\left(\|f\|^2_{L^2(Q_T)} + \sup_{k\in\mathbb{N}}\|v_k\|^2_{L^2(Q_T)}\right) + \|f_0\|^2_{L^2(\Omega)}}, \tag{85}$$

$$\|\nabla u_{k,k}\|_{L^2(Q_T;\mathbb{R}^N)} \le \sqrt{k}\left(\|f_0\|^2_{L^2(\Omega)} + \kappa\|f + v_k\|_{L^2(Q_T)}\|u_{k,k}\|_{L^2(Q_T)}\right) \overset{\text{by } (82)}{\le} \sqrt{k}S_3. \tag{86}$$

for all $k \in \mathbb{N}$, where

$$\sup_{k\in\mathbb{N}}\|v_k\|_{L^2(Q_T)} \le \sqrt{T}\|v_b\|_{L^2(\Omega)} < +\infty. \tag{87}$$

Our main intention at this step is to establish the following asymptotic property:

$$\frac{1}{k}\nabla u_{k,k} \rightharpoonup 0 \quad \text{in } L^2(Q_T;\mathbb{R}^2). \tag{88}$$

Indeed, for any vector-valued test function $\varphi \in C_0^\infty(Q_T)$, we have

$$\left|\int_{Q_T}\left(\frac{1}{k}\nabla u_{k,k},\varphi\right) dxdt\right| \le \frac{1}{\sqrt{k}}\left(\int_{Q_T}\frac{1}{k}|\nabla u_{k,k}|^2\,dxdt\right)^{1/2}\left(\int_{Q_T}|\varphi|^2\,dxdt\right)^{1/2}.$$

Hence, the sequence $\left\{\frac{1}{k}\nabla u_{k,k}\right\}_{k\in\mathbb{N}}$ is bounded in $L^2(Q_T;\mathbb{R}^2)$. As a result, we obtain

$$\left|\int_{Q_T}\left(\frac{1}{k}\nabla u_{k,k},\varphi\right)\,dxdt\right| \overset{\text{by (86)}}{\leq} S_3\frac{1}{\sqrt{k}}\left(\int_{Q_T}|\varphi|^2\,dxdt\right)^{1/2} \to 0 \quad \text{as } k\to\infty.$$

Step 3. Let us show that in this case the flux $\frac{1}{k}\nabla u_{k,k} + A_{u_{k,k-1}}(t,x,\nabla u_{k,k})$ weakly converges in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$ to the flux $A_u(t,x,\nabla u)$ as $k\to\infty$. To do so, it is enough to show that all preconditions (C1)–(C7) of Theorem 1 are fulfilled.

To begin with, we notice that the conclusion, similar to (80), can be also made with respect to the sequence $\{u_{k,k-1}\}_{k\in\mathbb{N}}$. Then Lemmas 1 and 2 imply that

$$\begin{gathered} D_{u_{k,k-1}}(t,x)\to D_u(t,x) \text{ and } p_{u_{k,k-1}}(t,x)\to p_u(t,x) \\ \text{uniformly in } \overline{Q_T} \text{ as } j\to\infty. \end{gathered} \tag{89}$$

Moreover, we deduce from (61) and (83) that the sequence

$$\left\{\frac{1}{k}\nabla u_{k,k} + A_{u_{k,k-1}}(t,x,\nabla u_{k,k})\right\}_{k\in\mathbb{R}}$$

is bounded in $L^{(p^+)'}(Q_T;\mathbb{R}^2)$. Hence, there exists an element $z\in L^{(p^+)'}(Q_T;\mathbb{R}^2)$ such that

$$\frac{1}{k}\nabla u_{k,k} + A_{u_{k,k-1}}(t,x,\nabla u_{k,k}) \rightharpoonup z \quad \text{weakly in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \text{ as } k\to\infty. \tag{90}$$

We also make use of the following observation: the sequence

$$\left\{\frac{1}{k}|\nabla u_{k,k}|^2 + \left(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla u_{k,k}\right)\right\}_{k\in\mathbb{N}} \tag{91}$$

is uniformly bounded in $L^1(Q_T)$. Indeed, this inference is a direct consequence of estimates (86), (83), and the following one

$$\begin{aligned} |A_{u_{k,k-1}}(t,x,\nabla u_{k,k})||\nabla\varphi| &\leq \frac{1}{p'_{u_{k,k-1}}(t,x)}|A_{u_{k,k-1}}(t,x,\nabla u_{k,k})|^{p'_{u_{k,k-1}}(t,x)} \\ &\quad + \frac{1}{p_{u_{k,k-1}}(t,x)}|\nabla\varphi|^{p_{u_{k,k-1}}(t,x)} \\ &\leq \frac{d_2^2}{2}|\nabla u_{k,k}|^{p_{u_{k,k-1}}(t,x)} + \frac{1}{p^-}|\nabla\varphi|^{p_{u_{k,k-1}}(t,x)}. \end{aligned} \tag{92}$$

Utilizing this fact together with the properties (89), (90), (59), (80) and

$$u_{k,k}\in L^{p^+}(0,T;W^{1,p^+}(\Omega))\ \forall k\in\mathbb{N} \text{ by (82),(86),}$$

and taking into account that $1 < 1+\delta = p^- < p^+ = 2 < 2p^-$, we see that all preconditions of Theorem 1 hold true. Hence, in view of the property (88), the assertion (90) can be rewritten as follows

$$\frac{1}{k}\nabla u_{k,k} + A_{u_{k,k-1}}(t,x,\nabla u_{k,k}) \rightharpoonup A_u(t,x,\nabla u) \quad \text{weakly in } L^{(p^+)'}(Q_T;\mathbb{R}^2) \text{ as } k\to\infty. \tag{93}$$

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Step 4. At this stage we show that the limit pair (v,u) is related by integral identity (63). First we notice that $u_{k,k}$ is a weak solution (in the sense of distributions) of (71)–(73) with $n=k$, $\varepsilon_n=1/k$ and $v=v_k$. Hence, $u_{k,k}$ satisfies the integral identity

$$\int_{Q_T}\left(-u_{k,k}\frac{\partial\varphi}{\partial t}+\frac{1}{k}(\nabla u_{k,k},\nabla\varphi)+\left(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla\varphi\right)+\kappa u_{k,k}\varphi\right)dxdt$$
$$=\kappa\int_{Q_T}(f-v_k)\varphi\,dxdt+\int_\Omega f_0\varphi|_{t=0}\,dx\quad\forall\,\varphi\in\Phi. \tag{94}$$

Then utilizing the properties (93), (80), and (76), and passing to the limit in (94) as $k\to\infty$, we immediately arrive at the announced identity (63).

Step 5. In order to show that the limit pair (v,u) satisfies the energy inequality (64), we have to realize the limit passage as $k\to\infty$ in the relation (see [19])

$$\frac{1}{2}\int_\Omega u_{k,k}^2\,dx+\int_0^t\int_\Omega\left(\frac{1}{k}|\nabla u_{k,k}|^2+\left(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla u_{k,k}\right)+\kappa u_{k,k}^2\right)dxdt$$
$$=\kappa\int_{Q_T}(f-v_k)u_{k,k}\,dxdt+\int_\Omega f_0^2\,dx\quad\forall\,t\in[0,T]. \tag{95}$$

that can be viewed as the energy equality for the weak solutions of the problem (71)–(73) with $n=k$, $\varepsilon_n=1/k$ and $v=v_k$. With that in mind, we notice that the weak convergence in (80), by the Sobolev embedding Theorem, implies the pointwise convergence

$$u_{k,k}^2(t,\cdot)\to u^2(t,\cdot)\quad\text{a.e. in }\Omega\text{ for a.a. }t\in(0,T).$$

Then, in view of estimate (85), we have the strong convergence $u_{k,k}^2(t,\cdot)\to u^2(t,\cdot)$ in $L^1(\Omega)$ for a.a. $t\in(0,T)$ (by the Lebesgue dominated Theorem), and, therefore,

$$\frac{1}{2}\lim_{k\to\infty}\int_\Omega u_{k,k}^2(t,x)\,dx=\frac{1}{2}\int_\Omega u^2(t,x)\,dx\quad\text{for a.a. }t\in(0,T). \tag{96}$$

Moreover, taking into account that the $L^2(Q_T)$-norm is lower semi-continuous with respect to the weak convergence (81), we see that

$$\lim_{k\to\infty}\int_0^t\int_\Omega u_{k,k}^2\,dxdt\geq\int_0^t\int_\Omega u^2\,dxdt. \tag{97}$$

We also notice that due to the properties (88), (93), and (80), we have

$$\nabla u_{k,k}\rightharpoonup\nabla u\ \text{ and }\ A_{u_{k,k-1}}(t,x,\nabla u_{k,k})\rightharpoonup A_u(t,x,\nabla u)\ \text{ in }L^1(Q_T;\mathbb{R}^2)\text{ as }k\to\infty.$$

Since $(A_u(t,x,\nabla u),\nabla u)\in L^1(Q_T)$ (see (92)), it follows from Lemma 4 (see also Proposition 2) that

$$\lim_{k\to\infty}\int_0^t\int_\Omega\left[\frac{1}{k}|\nabla u_{k,k}|^2+\left(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla u_{k,k}\right)\right]dxdt$$
$$\geq\lim_{k\to\infty}\int_0^t\int_\Omega\left[\frac{1}{k}|\nabla u_{k,k}|^2\right]dxdt$$
$$+\liminf_{k\to\infty}\int_0^t\int_\Omega\left(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}),\nabla u_{k,k}\right)dxdt$$
$$\overset{\text{by (86)}}{\geq}\int_0^t\int_\Omega(A_u(t,x,\nabla u),\nabla u)\,dxdt. \tag{98}$$

So, in order to pass to the limit in (95), it remains to find out the asymptotic behaviour of the term $\int_{Q_T}(f - v_k)u_{k,k}\,dxdt$ as $k \to \infty$. We prove it at the next step using the well-known Aubin-Lions lemma.

Step 6. We recall that the Aubin-Lions lemma states criteria when a set of functions is relatively compact in $L^p(0,T;B)$, where $p \in [1,\infty)$, $T > 0$, and B is a Banach space. The standard formulation of the Aubin-Lions lemma states that if U is a bounded set in $L^p(0,T;X)$ and $\partial U/\partial t = \{\partial u/\partial t \ : \ u \in U\}$ is bounded in $L^r(0,T;Y)$, $r \geq 1$, then U is relatively compact in $L^p(0,T;B)$, under the conditions that

$$X \hookrightarrow B \text{ compactly}, \quad B \hookrightarrow Y \text{ continuously}.$$

Setting $U = \{u_{k,k}\}_{k\in\mathbb{N}}$, we deduce from (82)–(85) that

$$\{u_{k,k}\}_{k\in\mathbb{N}} \text{ is bounded in } L^{p^-}(0,T;W^{1,p^-}(\Omega)\cap L^2(\Omega)). \tag{99}$$

Since, by the Sobolev embedding Theorem, $W^{1,p^-}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ compactly, it follows from the Lebesgue dominated Theorem that the following embeddings are compact as well

$$W^{1,p^-}(\Omega)\cap L^2(\Omega) \hookrightarrow L^2(\Omega), \quad L^2(\Omega) \hookrightarrow \left(W^{1,2}(\Omega)\right)' \text{ (by the duality arguments)}. \tag{100}$$

Further, having in mind the fact that for each $k \in \mathbb{N}$, the functions $u_{k,k}$ are the solutions in $W(0,T)$ of the variational problem

$$\left\langle \frac{\partial u_{k,k}(t)}{\partial t}, \varphi \right\rangle_{(W^{1,2}(\Omega))';W^{1,2}(\Omega)} + \int_\Omega \left[\frac{1}{k}\left(\nabla u_{k,k}(t), \nabla\varphi\right)\right] dx$$

$$+ \int_\Omega \left[\left(A_{u_{k,k-1}}(t,x,\nabla u_{k,k}(t)), \nabla\varphi\right) + \kappa u_{k,k}(t)\varphi\right] dx \tag{101}$$

$$= \kappa\int_\Omega (f(t) - v_k(t))\varphi\,dx, \quad \forall\,\varphi \in W^{1,2}(\Omega) \quad \text{a.e. in } [0,T], \tag{102}$$

$$u_{k,k}(0) = f_0. \tag{103}$$

we derive from this the following estimate

$$\begin{aligned}
\left|\left\langle \frac{\partial u_{k,k}}{\partial t}, \varphi\right\rangle\right| &\leq \frac{1}{\sqrt{k}}\|\nabla u_{k,k}\|_{L^2(Q_T;\mathbb{R}^2)}\|\nabla\varphi\|_{L^2(Q_T;\mathbb{R}^2)} \\
&+ 2\|A_{u_{k,k-1}}(t,x,\nabla u_{k,k})\|_{L^{p'_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)}\|\nabla\varphi\|_{L^{p_{u_{k,k-1}}(\cdot)}(Q_T;\mathbb{R}^2)} \\
&+ \kappa\|u_{k,k}\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)} + \kappa\|f - v_k\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)} \\
&\leq \text{(by (82)–(86))} \\
&\leq \left[S_3 + \kappa S_1 + \kappa\|f\|_{L^2(Q_T)} + \kappa\sup_{k\in\mathbb{N}}\|v_k\|_{L^2(Q_T)}\right]\|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))} \\
&+ \left(1 + \int_{Q_T}|A_{u_{k,k-1}}(t,x,\nabla u_{k,k})|^{p'_{u_{k,k-1}}(t,x)}\,dxdt\right)^{1/2}(1 + T|\Omega|)^{1/2}\|\varphi\|_{L^2(Q_T)} \\
&\overset{\text{by (83),(61)}}{\leq} \text{const}\|\varphi\|_{L^2(0,T;W^{1,2}(\Omega))}, \quad \forall\, v \in L^2(0,T;W^{1,2}(\Omega)).
\end{aligned}$$

Hence,

$$\left\|\frac{\partial u_{k,k}}{\partial t}\right\|_{L^2(0,T;(W^{1,2}(\Omega))')} < +\infty. \tag{104}$$

Utilizing this fact together with (99) and (100), we deduce from the the Aubin-Lions lemma that the set $U = \{u_{k,k}\}_{k\in\mathbb{N}}$ is relatively compact in $L^{p^-}(0,T;L^2(\Omega))$. Hence, we can complement properties by the following one: $u_{k,k} \to u$ strongly in $L^{p^-}(0,T;L^2(\Omega))$ as $k \to \infty$. Since, U is bounded in $L^\infty(0,T;L^2(\Omega))$, it leads to the conclusion

$$u_{k,k} \to u \text{ strongly in } L^2(0,T;L^2(\Omega)), \quad \text{as } k \to \infty. \tag{105}$$

Hence, the term $\int_{Q_T}(f - v_k)u_{k,k}\,dxdt$ is the product of weakly and strongly convergent sequences in $L^2(0,T;L^2(\Omega))$. As a result, wee have

$$\lim_{k\to\infty} \int_{Q_T} (f - v_k)u_{k,k}\,dxdt = \int_{Q_T} (f - v)u\,dxdt. \tag{106}$$

Thus, in view of the obtained collection of properties (see (96), (97), (98), and (106)), the limit passage in (95) as $k \to \infty$ finally leads us to the energy inequality (64).

Step 7. To end the proof, it remains to notice that, due to the properties (62), that were established at the previous steps, we have: $J(v,u) < +\infty$ and $u \in W_u(Q_T)$. Moreover, it has been proven that in this case the sequence $\{u_{k,k}\}_{k\in\mathbb{N}}$ satisfies all requirements that were mentioned in Definition 3. Hence, $u \in W_u(Q_T)$ is a W_0-attainable weak solution to the problem (54)–(56). The proof is complete. □

Taking this result into account, let us show that the original optimal control problem (74) has a solution. In fact, this issue immediately follows from Theorem 3 and the facts that the set of feasible solutions Ξ is bounded in $L^2(Q_T) \times L^{p^-}(0,T;W^{1,p^-}(\Omega))$ (see estimates (65)–(68) and (87)), and the objective functional $J(v,u)$ is lower semicontinuous with respect to the weak topology of $L^2(Q_T) \times \left(L^{p^-}(0,T;W^{1,p^-}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))\right)$. So, as a direct consequence, we can finalize this inference as follows:

Corollary 1. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $v_a, v_b \in L^2(\Omega)$, $v_a(x) \le v_b(x)$ a.e. in Ω, be given distributions, and let $\kappa > 0$, $\sigma > 0$, $\varepsilon > 0$, and $\mu > 0$ be some constants. Then the optimal control problem (74) admits at least one solution $(v^0, u^0) \in \Xi$.*

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