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The Robustness of the $c\mu$-Rule for an Unreliable Single-Server Two-Class Queueing System with Constant Retrial Rates

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Abstract: We study the robustness of the $c\mu$-rule for the optimal allocation of a resource consisting of one unreliable server to parallel queues with two different classes of customers. The customers in queues can be served with respect to a FCFS retrial discipline, when the customers at the head of queues repeatedly tries to occupy the server in a random time. It is proved that for the scheduling problem in the system without arrivals the $c\mu$-rule minimizes the total average cost. For the system with arrivals it is difficult directly to prove the optimality of the same policy with explicit relations. We derived for an infinite-buffer model a static control policy that also prescribes for the certain values of system parameters the service exclusively for the class-i customers if both of queues are not empty with the aim to minimize the average cost per unit of time. It is also shown that in a finite-buffer case the $c\mu$-rule fails.

Keywords: Queueing system; $c\mu$-rule; scheduling problem; static policy; average cost

1. Introduction

The modelling and analysis of telecommunications and computer systems is now inconceivable without the various tasks associated with resource allocation and formulated in the framework of the queueing theory. One of such classic problems is to allocate a server to multiple parallel queues with the mostly studied objective to minimize the average cost per unit of time. It was shown that for many systems with such kind of resource allocation problem the allocation policy in form of the $c\mu$-rule is optimal. In literature this rule is also known as Smith's rule or Weighted Shortest Processing Time. According to this rule the waiting class-i customer from a non-empty queue is allocated to the server if it has a minimal weight of the form $c_i\mu_i$, where c_i is a holding cost per unit of time the customer is in the queue or at server and μ_i is a overall service rate of the class-i customer. The $c\mu$-rule is a very attractive policy, since it is a static one and requires only the information whether a certain queue is empty or not. Obviously, to apply such a policy the values c_i and μ_i must be known, which unfortunately is not always the case, especially for the overall service intensity.

The optimality of the $c\mu$-rule for ordinary multi-class single-server queues in different settings was proved, see e.g. [3,4]. In [16], the authors analyzed the $c\mu$-rule for queueing models with non-linear costs. The optimality of this rule for discrete-time queueing models with a general distributed inter-arrival time and geometrically distributed service time was established in [2]. A concept of a generalized $c\mu$-rule was proposed in [16], where it was shown that this rule is asymptotically optimal for non-decreasing convex delay costs. A classic scheduling problem with a single resource shared by two competing queues was considered in [10] in the context of the stochastic flow model where it was shown that the $c\mu$-rule is optimal. The non-preemptive assignment of a single server to two infinite-capacity retrial queues was analyzed in [18] where $c\mu$-rule was optimal for a scheduling problem in a system without arrivals. The authors in [11] considered the learning-based variants of the $c\mu$-rule where the service rates μ are unknown.

The constant retrial policy was introduced in [8], where it was assumed that only the customer at the head of the queue can request for service after an exponentially distributed retrial time. The single class queueing systems with constant retrial rate and different options were analysed intensively, e.g. in [6,7,17] and other papers. The uncontrolled analogue of a two-class queueing system has also been investigated by a number of authors. For example, in [1] the authors studied a two-class system with a single exponential service requirements and constant retrial policy. The regenerative approach was used there to derive a set of necessary stability conditions of such a system. The generating function of the stationary distribution of the number of orbiting customers at service completion epochs was obtained in [5].

This paper deals with a controllable unreliable single-server two-class queueing system with constant retrial rates. The optimal allocation problem for this queueing model with retrial and reliability attributes is a new one. The emphases of the paper is on answering the question how robust is the $c\mu$-rule as an optimal allocation policy in the queueing system under study. The queueing system is studied without and with arrivals. In first case explicit relations of the $c\mu$-rule can be derived. In second case, the relations for the optimality of the $c\mu$-rule were obtained for the model with a certain constraint on arrival process.

In Section 2 the queueing model is described. The main results including the analysis of the optimal allocation policy and some numerical experiments are presented in Section 3.

2. Model description

We analyze the markovian single-server queueing system servicing two classes of customers as illustrated in Figure 1. The customers of each class $i = 1, 2$ arrive to the system according to a Poisson stream with a rate λ_i. Independently of the state of server the class-i customers join upon arrival the corresponding waiting line or queue with infinite capacity $N_i = \infty$.

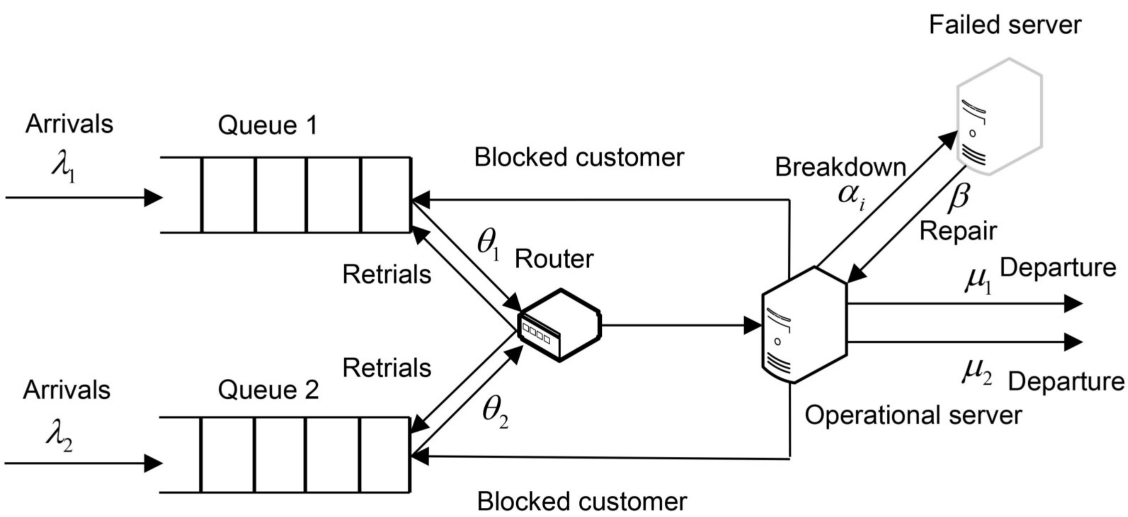

Figure 1. Schema of the single server two-class controllable queueing system

The service of customers from the queue occurs according to a FIFO retrial discipline, i.e. the customer waiting at the head of the queue retries to occupy the server in exponentially distributed time with a rate θ_i. The server during a service process of the ith-class customer can fail in exponentially distributed time with a rate α_i. In this case the customer leaves the service area and joins again the head of its queue. In failed state the server can be repaired in exponentially distributed time with a rate β. All type of time intervals are assumed to be mutually independent. At moments of retrial arrival the

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idle server may accept the customer of a certain class, who attempts to occupy the server, or can deny the service. The system performance is described by the steady state average cost which is of the form

$$g^f = c_1 L_1^f + c_2 L_2^f, \tag{1}$$

where L_i^f is the average number of the ith-class customers present in the system given the allocation policy is f and c_i the holding cost per unit of time the ith-class customer spends in the system. The objective of the present analysis is to provide how robust is a static policy f defined as a $c\mu$-rule for the system under study by minimizing the average cost per unit of time (1).

Denote by $Q_i(t)$ the number of ith-class customers in the system at time t and by $D(t)$ the state of the server at time t which is defined as follows,

$$D(t) = \begin{cases} 0 & \text{the server is idle,} \\ 1 & \text{the server services the 1st class customer,} \\ 2 & \text{the server services the 2nd class customer,} \\ 3 & \text{the server is failed.} \end{cases}$$

Consider the three-dimensional continuous-time Markov chain

$$\{X(t)\}_{t\geq 0} = \{Q_1(t), Q_2(t), D(t)\}_{t\geq 0}, \tag{2}$$

with a state space

$$E = \{x = (q_1, q_2, d) : q_1, q_2 \in \mathbb{N}_0,\ d \in \{0, 1, 2, 3\}\}$$

and policy-dependent infinitesimal matrix $\Lambda^f = [\lambda^f_{xy}]$ with components for $x \neq y$,

$$\lambda^f_{xy} = \begin{cases} \lambda_i & y = x + \mathbf{e}_i, \\ \mu_i & y = x - i\mathbf{e}_3,\ d(x) = i, \\ \theta_i & y = x - \mathbf{e}_i + i\mathbf{e}_3,\ d(x) = 0,\ f(x) = i, \\ \alpha_i & y = x + \mathbf{e}_i - i\mathbf{e}_3,\ d(x) = i, \\ \beta & y = x - 3\mathbf{e}_3, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where $\mathbf{e}_i$ stands for a vector of dimension 3 with 1 in the ith position, $i = 1, 2, 3$, and 0 elsewhere. Here f is a control policy which prescribes at the moments of retrial arrival the allocation of the ith-class customer to the server. The set of admissible control actions is defined by $A = \{1, 2\}$, where $f(x) = i \in A$ means that upon retrial arrival in state x the ith-class customer must be accepted for service. The sets of admissible control actions in state x is denoted by $A(x) \subseteq A$, where $A(x) = A$ for $x = (q_1, q_2, 0), q_1, q_2 \geq 1, A(x) = 1$ for $x = (q_1, 0, 0)$ and $A(x) = 2$ for $x = (0, q_2, 0)$. It is assumed that the stability condition is fulfilled. According to a general result for the $M/G/1$ system with parallel queues, see e.g. [4], the system is stable if the total load $\rho = \rho_1 + \rho_2 < 1$, where ρ_i the load of the class-i queue. The value ρ_i can be obtained if we treat the class-i queue as independent single-server queueing

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system with parameters $\lambda_i, \mu_i, \theta_i, \alpha_i$ and β. The corresponding continuous-time Markov-chain is then a quasi-birth-and-death (QBD) process with a three-diagonal block infinitesimal matrix

$$Q = \begin{pmatrix} B_0 & B_1 & 0 & 0 & \\ A_2 & A_1 & A_0 & 0 & \\ 0 & A_2 & A_1 & A_0 & \dots \\ 0 & 0 & A_2 & A_1 & \dots \\ & & \vdots & \vdots & \end{pmatrix}, \text{ where}$$

$$B_0 = \begin{pmatrix} -\lambda_i & 0 & 0 \\ \mu_i & -(\lambda_i + \mu_i + \alpha_i) & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & \alpha_i \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -(\lambda_i + \theta_i) & 0 & 0 \\ \mu_i & -(\lambda_i + \mu_i + \alpha_i) & 0 \\ \beta & 0 & -(\lambda_i + \beta), \end{pmatrix},$$

$$A_0 = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & \alpha_i \\ 0 & 0 & \lambda_i \end{pmatrix}, A_2 = \begin{pmatrix} 0 & \theta_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

According to a matrix-analytic approach for QBD-processes [12], the queue i is stable if the following inequality holds,

$$\mathbf{p}A_2\mathbf{e} > \mathbf{p}A_0\mathbf{e},\ \mathbf{e} = (1,1,1)'. \tag{4}$$

where $\mathbf{p} = (p_1, p_2, p_3)$ is a stationary distribution for the infinitesimal transition matrix

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} -\theta_i & \theta_i & 0 \\ \mu_i & -(\alpha_i + \mu_i) & \alpha_i \\ \beta & 0 & -\beta \end{pmatrix}$$

which satisfies the system $\mathbf{p}A = \mathbf{0}$ and $\mathbf{pe} = 1$. The solution of the system is given by

$$p_1 = \frac{(\alpha_i + \mu_i)\beta}{C},\ p_2 = \frac{\beta\theta_i}{C},\ p_3 = \frac{\alpha\theta_i}{C}, \tag{5}$$

where $C = \alpha_i(\beta + \theta_i) + \beta(\mu_i + \theta_i)$. By substituting the solution (5) into (4) we get

$$\rho_i = \frac{\lambda_i C}{\beta_i \mu_i \theta_i} = \lambda_i \frac{\alpha_i + \mu_i}{\mu_i \theta_i}\Big(\frac{\theta_i}{\beta}\frac{\alpha_i + \beta}{\alpha_i + \mu_i} + 1\Big) < 1.$$

and the stability condition is then defined as

$$\sum_{i=1}^{2} \lambda_i \frac{\alpha_i + \mu_i}{\mu_i \theta_i}\Big(\frac{\theta_i}{\beta}\frac{\alpha_i + \beta}{\alpha_i + \mu_i} + 1\Big) < 1. \tag{6}$$

Below we present the main results of this paper obtained for a system with two classes of customers. However, these results can also be generalized to the case of an arbitrary number of classes.

3. Optimal allocation problem

Consider first a classic scheduling problem for the allocation of customers in the system without arrivals, i.e. when $\lambda_1 = \lambda_2 = 0$, in which the customers are queued. The waiting customers must be

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served in such a way that the overall average cost is minimized. It is assumed that the allocation policy $f(q_1, 0, 0) = 1$ and $f(0, q_2, 0) = 2$ for $q_1, q_2 \geq 1$. Here we have a classical scheduling problem

Proposition 1. *In state $x = (q_1, q_2, 0)$ the optimal allocation policy can be defined in form of a $c\mu$-rule,*

$$f(x) = \begin{cases} 1, & \text{if } c_2\tilde{\mu}_1 \leq c_1\tilde{\mu}_2, \\ 2, & \text{if } c_2\tilde{\mu}_1 \geq c_1\tilde{\mu}_2 \end{cases}, \tag{7}$$

where $\tilde{\mu}_i = m_i\Big(\frac{\alpha_i+\beta}{m_i\mu_i\beta} + 1\Big)$, $m_i = \frac{\alpha_i+\mu_i}{\mu_i\theta_i}$, $i = 1, 2$. In case of equality $c_2\tilde{\mu}_1 = c_1\tilde{\mu}_2$, the control actions 1 and 2 are equivalent.

Proof. Denote by $V(x)$ the total minimal average cost given the initial state is $x \in E$. This function is given by

$$\begin{aligned} V(q_1, q_2, 0) &= \min\Big\{\frac{1}{\theta_1}(q_1c_1 + q_2c_2) + V(q_1 - 1, q_2, 1), \frac{1}{\theta_2}(q_1c_1 + q_2c_2) + V(q_1, q_2 - 1, 2)\Big\}, \\ &\quad q_1,\ q_2 \geq 1, \qquad (8) \\ V(q_1, q_2, 1) &= \frac{1}{\mu_1+\alpha_1}((q_1+1)c_1 + q_2c_2) + \frac{\mu_1}{\mu_1+\alpha_1}V(q_1, q_2, 0) + \frac{\alpha_1}{\mu_1+\alpha_1}V(q_1+1, q_2, 3), \\ &\quad q_1, q_2 \geq 0, \qquad (9) \\ V(q_1, q_2, 2) &= \frac{1}{\mu_2+\alpha_2}(q_1c_1 + (q_2+1)c_2) + \frac{\mu_2}{\mu_2+\alpha_2}V(q_1, q_2, 0) + \frac{\alpha_2}{\mu_2+\alpha_2}V(q_1, q_2+1, 3), \\ &\quad q_1, q_2 \geq 0, \qquad (10) \\ V(q_1, q_2, 3) &= \frac{1}{\beta}(q_1c_1 + q_2c_2) + V(q_1, q_2, 0),\ q_1, q_2 \geq 0. \qquad (11) \end{aligned}$$

We need to prove that $f(q_1, q_2, 0) = 1$ if $c_2\tilde{\mu}_1 < c_1\tilde{\mu}_2$ as defined in a proposition. First we show that $f(q_1, q_2 - 1, 0) = 1$ implies $f(q_1, q_2, 0) = 1$. In case $f(q_1, q_2, 0) = 2$ and $f(q_1, q_2 - 1, 0) = 1$ we get from (8), (10) and (11) after some simple algebra the relation for $V(q_1, q_2, 0)$,

$$\begin{aligned} V(q_1, q_2, 0) = &\Big(\frac{1}{\mu_2} + \frac{\alpha_2}{\beta\mu_2} + \frac{\alpha_2+\mu_2}{\mu_2\theta_2}\Big)(q_1c_1 + q_2c_2) + \Big(\frac{1}{\mu_1} + \frac{\alpha_1}{\beta\mu_1} + \frac{\alpha_1+\mu_1}{\mu_1\theta_1}\Big)(q_1c_1 + (q_2-1)c_2) \\ &+ V(q_1 - 1, q_2 - 1, 0). \end{aligned} \tag{12}$$

In case $f(q_1, q_2, 0) = 1$, if in state $(q_1, q_2 - 1, 0)$ the action 2 is chosen instead of the action 1, we get the following inequality,

$$\begin{aligned} V(q_1, q_2, 0) \geq &\Big(\frac{1}{\mu_1} + \frac{\alpha_1}{\beta\mu_1} + \frac{\alpha_1+\mu_1}{\mu_1\theta_1}\Big)(q_1c_1 + q_2c_2) + \Big(\frac{1}{\mu_2} + \frac{\alpha_2}{\mu_2\beta} + \frac{\alpha_2+\mu_2}{\mu_2\theta_2}\Big)((q_1-1)c_1 + q_2c_2) \\ &+ V(q_1 - 1, q_2 - 1, 0). \end{aligned} \tag{13}$$

Now, if $f(q_1, q_2, 0) = 1$ is optimal, then the difference of expressions (12) and (13) must be positive, i.e.

$$\begin{aligned} &c_1\Big(\frac{1}{\mu_2} + \frac{\alpha_2}{\beta\mu_2} + \frac{\alpha_2+\mu_2}{\mu_2\theta_2}\Big) - c_2\Big(\frac{1}{\mu_1} + \frac{\alpha_1}{\beta\mu_1} + \frac{\alpha_1+\mu_1}{\mu_1\theta_1}\Big) \\ &= c_1\frac{\alpha_2+\mu_2}{\mu_2\theta_2}\Big(\frac{\theta_2}{\beta}\frac{\alpha_2+\beta}{\alpha_2+\mu_2} + 1\Big) - c_2\frac{\alpha_1+\mu_1}{\mu_1\theta_1}\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1} + 1\Big) \\ &= c_1m_2\Big(\frac{\alpha_2+\beta}{m_2\mu_2\beta} + 1\Big) - c_2m_1\Big(\frac{\alpha_1+\beta}{m_1\mu_1\beta} + 1\Big) \geq 0. \end{aligned} \tag{14}$$

what is coincide with a statement.

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In case of equality $c_2\tilde{\mu}_1 = c_1\tilde{\mu}_2$ the control actions 1 and 2 are equivalent, i.e. $f(x) = 1 \equiv 2$ for $x = (q_1, q_2, 0)$. We show that the optimal policy $f(q_1 - 1, q_2, 0) = 1 \equiv 2$ and $f(q_1, q_2 - 1, 0) = 1 \equiv 2$ implies that $f(q_1, q_2, 0) = 1 \equiv 2$. Note that $f(0, q_2, 0) = 2$ and $f(q_1, 0, 0) = 1$ for $q_1, q_2 \geq 1$. For the minimum in (8) in case $f(q_1, q_2 - 1, 0) = 1 \equiv 2$ we get then for the control action $f(q_1, q_2, 0) = 2$ the relation

$$\frac{1}{\theta_2}(q_1c_1 + q_2c_2) + V(q_1, q_2 - 1, 2)$$

in form (12). For the control action $f(q_1, q_2, 0) = 1$ the relation

$$\frac{1}{\theta_1}(q_1c_1 + q_2c_2) + V(q_1 - 1, q_2, 1)$$

is equal to the right hand side of the inequality (13). The difference of the total average cost given by the relation in (14) for equivalent actions must be equal to zero. Therefore, if $c_2\tilde{\mu}_1 = c_1\tilde{\mu}_2$, the control action 1 and 2 are equivalent. □

It is assumed now that new customers can arrive to the system, i.e. $\lambda_1 > 0, \lambda_2 > 0$. We expect that the same $c\mu$-rule defined in (7) will be optimal for the system with arrivals but from technical reasons it is quite difficult to derive expressions for the mean overall service times of the ith-class customers. Therefore, to analyse the properties of a an optimal control policy we have to introduce a queueing model with a constraint on possible arrivals. This model differs from original one, since we assume that a new arrival can occur only in state $x = (q_1, q_2, 0), q_1, q_2 \geq 0$, where the server is empty. The dynamic-programming approach, see e.g. [9], [14,15], is used to establish the properties of the optimal control policy in the following Proposition. Note that the state space of the corresponding Markov decision process (MDP) is infinite and the costs are unbounded. The existence of an average cost optimal stationary policy and convergence of the value-iteration algorithm can be verified in the same way as it was done in [13], where the authors generalized the existence of the optimal policy for the discounted expected total cost minimization to the average cost criterion.

Proposition 2. *In state $x = (q_1, q_2, 0)$ the optimal allocation policy for the system with a constraint on arrivals can be defined in form*

$$f(x) = \begin{cases} 1, & \text{if } c_2m_1 \leq c_1m_2,\ \frac{\alpha_1+\beta}{m_1\mu_1} \leq \frac{\alpha_2+\beta}{m_2\mu_2}, \\ 2 & \text{if } c_2m_1 \geq c_1m_2,\ \frac{\alpha_1+\beta}{m_1\mu_1} \geq \frac{\alpha_2+\beta}{m_2\mu_2} \end{cases}. \tag{15}$$

In case of equalities $c_2m_1 = c_1m_2$ and $\frac{\alpha_1+\beta}{m_1\mu_1} = \frac{\alpha_2+\beta}{m_2\mu_2}$, the control actions 1 and 2 are equivalent.

Proof. For the introduced cost structure the average cost stationary optimal policy exists. This policy can be found as a solution of the system of optimality equations for the dynamic-programming relative value function $v : E \to \mathbb{R}$ and gain g,

$$\begin{aligned} v(q_1, q_2, 0) = q_1c_1 + q_2c_2 - g + \lambda_1 v(q_1 + 1, q_2, 0) + \lambda_2 v(q_1, q_2 + 1, 0) \\ + \min\Big\{\theta_1 v(q_1 - 1, q_2, 1) + (1 - \theta_1 - \lambda_1 - \lambda_2)v(q_1, q_2, 0), \\ \theta_2 v(q_1, q_2 - 1, 2) + (1 - \theta_2 - \lambda_1 - \lambda_2)v(q_1, q_2, 0)\Big\}, \end{aligned} \tag{16}$$

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where $q_1,\ q_2 \geq 1$ and $c(q_1, q_2, 0) = q_1 c_1 + q_2 c_2$ is an immediate cost of the corresponding MDP. In states with the only one nonempty queue the optimal policy consists in service of the corresponding customer, the relative value functions are of the form,

$$\begin{aligned} v(q_1,0,0) &= q_1 c_1 - g + \lambda_1 v(q_1+1,0,0) + \lambda_2 v(q_1,1,0) \\ &\quad + \theta_1 v(q_1-1,0,1) + (1-\theta_1-\lambda_1-\lambda_2) v(q_1,0,0), \end{aligned} \tag{17}$$

$$\begin{aligned} v(0,q_2,0) &= q_2 c_2 - g + \lambda_1 v(1,q_2,0) + \lambda_2 v(0,q_2+1,0) \\ &\quad + \theta_2 v(0,q_2-1,2) + (1-\theta_2-\lambda_1-\lambda_2) v(0,q_2,0), \end{aligned} \tag{18}$$

where $q_1,\ q_2 \geq 1$. The value $v(0,0,0) = 0$, then $g = \lambda_1 v(1,0,0) + \lambda_2 v(0,1,0)$. The equations for states where the server is busy or failed are

$$\begin{aligned} v(q_1,q_2,1) &= (q_1+1)c_1 + q_2 c_2 - g + \mu_1 v(q_1,q_2,0) + \alpha_1 v(q_1+1,q_2,3) \\ &\quad + (1-\mu_1-\alpha_1) v(q_1,q_2,1), \end{aligned} \tag{19}$$

$$\begin{aligned} v(q_1,q_2,2) &= q_1 c_1 + (q_2+1)c_2 - g + \mu_2 v(q_1,q_2,0) + \alpha_2 v(q_1,q_2+1,3) \\ &\quad + (1-\mu_2-\alpha_2) v(q_1,q_2,2), \end{aligned} \tag{20}$$

$$v(q_1,q_2,3) = q_1 c_1 + q_2 c_2 + \beta v(q_1,q_2,0) + (1-\beta) v(q_1,q_2,3). \tag{21}$$

The equation (16) can be expressed using (19)–(21) in the following way,

$$\begin{aligned} v(q_1,q_2,0) &= q_1 c_1 + q_2 c_2 - g + \lambda_1 v(q_1+1,q_2,0) + \lambda_2 v(q_1,q_2+1,0) \\ &\quad + \min\Big\{ \frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}(q_1 c_1 + q_2 c_2 - g) + \frac{\mu_1\theta_1}{\alpha_1+\mu_1} v(q_1-1,q_2,0) \\ &\quad + \Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1} + (1-\lambda_1-\lambda_2-\theta_1)\Big) v(q_1,q_2,0), \\ &\qquad \frac{\theta_2}{\beta}\frac{\alpha_2+\beta}{\alpha_2+\mu_2}(q_1 c_1 + q_2 c_2 - g) + \frac{\mu_2\theta_2}{\alpha_2+\mu_2} v(q_1,q_2-1,0) \\ &\quad + \Big(\frac{\alpha_2\theta_2}{\alpha_2+\mu_2} + (1-\lambda_1-\lambda_2-\theta_2)\Big) v(q_1,q_2,0) \Big\}. \end{aligned} \tag{22}$$

We rewrite equations (17) and (18) in the same way,

$$\begin{aligned} v(q_1,0,0) &= q_1 c_1 - g + \lambda_1 v(q_1+1,0,0) + \lambda_2 v(q_1,1,0) + \frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}(c_1 q_1 - g) \\ &\quad + \frac{\mu_1\theta_1}{\alpha_1+\mu_1} v(q_1-1,0,0) + \Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1} + (1-\lambda_1-\lambda_2-\theta_1)\Big) v(q_1,0,0), \end{aligned} \tag{23}$$

$$\begin{aligned} v(0,q_2,0) &= q_2 c_2 - g + \lambda_1 v(1,q_2,0) + \lambda_2 v(0,q_2+1,0) + \frac{\theta_2}{\beta}\frac{\alpha_2+\beta}{\alpha_2+\mu_2}(c_2 q_2 - g) \\ &\quad + \frac{\mu_2\theta_2}{\alpha_2+\mu_2} v(0,q_2-1,0) + \Big(\frac{\alpha_2\theta_2}{\alpha_2+\mu_2} + (1-\lambda_1-\lambda_2-\theta_2)\Big) v(0,q_2,0). \end{aligned} \tag{24}$$

The solution for the system of optimality equations can be calculated recursively using equivalent discrete-time model on a finite horizon obtained by an uniformization procedure. The corresponding recursive relations have almost the same structure. Namely, the equations (22)–(24) can be rewritten in such a way that on the left-hand side the function $v(x)$ is replaced by the $n+1$-stage cost function $v_{n+1}(x)$ and on the right-hand side we put $g = 0$ and function $v(x)$ is replaced by the n-stage cost

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function $v_n(x)$. It is assumed that the initial condition is $v_0(x) = 0, x \in E$. Let the inequalities in the first raw of (15) hold. Consider the term for action selection in obtained recursive relations. In case

$$\begin{aligned}
&\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}(q_1c_1+q_2c_2)+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}v_n(q_1-1,q_2,0)\\
&+\Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1}+(1-\lambda_1-\lambda_2-\theta_1)\Big)v_n(q_1,q_2,0)\\
&\leq\frac{\theta_2}{\beta}\frac{\alpha_2+\beta}{\alpha_2+\mu_2}(q_1c_1+q_2c_2)+\frac{\mu_2\theta_2}{\alpha_2+\mu_2}v_n(q_1,q_2-1,0)\\
&+\Big(\frac{\alpha_2\theta_2}{\alpha_2+\mu_2}+(1-\lambda_1-\lambda_2-\theta_2)\Big)v_n(q_1,q_2,0)
\end{aligned} \tag{25}$$

for any $n \in \mathbb{N}_0$, the optimal control action is $f(q_1,q_2,0) = 1$ for an arbitrary n. The statement is proved by induction on n. If $n = 0$, the inequality (25) obviously holds. Assume the validity of this inequality for some n. Then it must be proved that (25) holds for $n \to n+1$. Expressions for $v_{n+1}(q_1-1,q_2,0)$, $v_{n+1}(q_1,q_2-1,0)$ and $v_{n+1}(q_1,q_2,0)$ can be obtained from (22)–(24). The first terms by $q_1c_1+q_2c_2$ in inequality (25) are multiplied by the factor $\nu = 1$ defined as

$$\nu=\lambda_1+\lambda_2+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}+\frac{\alpha_1\theta_1}{\alpha_1+\mu_1}+\Big(1-\lambda_1-\lambda_2-\theta_1\Big).$$

In case $q_1 \geq 2, q_2 \geq 1$, we obtain from (25) the following inequality,

$$\begin{aligned}
&\nu\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}(q_1c_1+q_2c_2)+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}\Big[\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)((q_2-1)c_1+q_2c_2)\\
&+\lambda_1v_n(q_1,q_2,0)+\lambda_2v_n(q_1-1,q_2+1,0)+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}v_n(q_1-2,q_2,0)\\
&+\Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1}+(1-\lambda_1-\lambda_2-\theta_1)\Big)v_n(q_1-1,q_2,0)\Big]\\
&+\Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1}+(1-\lambda_1-\lambda_2-\theta_1)\Big)\Big[\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)\times\\
&\times(q_1c_1+q_2c_2)+\lambda_1v_n(q_1+1,q_2,0)+\lambda_2v_n(q_1,q_2+1,0)\\
&+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}v_n(q_1-1,q_2,0)+\Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1}+(1-\lambda_1-\lambda_2-\theta_1)\Big)v_n(q_1,q_2,0)\Big]\\
&\leq\nu\frac{\theta_2}{\beta}\frac{\alpha_2+\beta}{\alpha_2+\mu_2}(q_1c_1+q_2c_2)+\frac{\mu_2\theta_2}{\alpha_2+\mu_2}\Big[\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)(q_1c_1+(q_2-1)c_2)\\
&+\lambda_1v_n(q_1+1,q_2-1,0)+\lambda_2v_n(q_1,q_2,0)+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}v_n(q_1-1,q_2-1,0)\\
&+\Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1}+(1-\lambda_1-\lambda_2-\theta_1)\Big)v_n(q_1,q_2-1,0)\Big]\\
&+\Big(\frac{\alpha_2\theta_2}{\alpha_2+\mu_2}+(1-\lambda_1-\lambda_2-\theta_2)\Big)\Big[\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)\times\\
&\times(q_1c_1+q_2c_2)+\lambda_1v_n(q_1+1,q_2,0)+\lambda_2v_n(q_1,q_2+1,0)\\
&+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}v_n(q_1-1,q_2,0)+\Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1}+(1-\lambda_1-\lambda_2-\theta_1)\Big)v_n(q_1,q_2,0)\Big],
\end{aligned} \tag{26}$$

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where we multiply the first term in the left- and right-hand side by the factor $\nu = 1$. The inequality (26) which is valid due to the following reasons. For the terms with the factor λ_1 by adding to the left-hand side the item $c_1 \frac{\theta_1}{\beta} \frac{\alpha_1+\beta}{\alpha_1+\mu_1}$ and to the right-hand side the item $c_1 \frac{\theta_2}{\beta} \frac{\alpha_2+\beta}{\alpha_2+\mu_2}$ we get,

$$\begin{aligned} &\frac{\theta_1}{\beta} \frac{\alpha_1+\beta}{\alpha_1+\mu_1}((q_1+1)c_1+q_2c_2) + \frac{\mu_1\theta_1}{\alpha_1+\mu_1} v_n(q_1,q_2,0) \\ &+ \Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1} + (1-\lambda_1-\lambda_2-\theta_1)\Big) v_n(q_1+1,q_2,0) \leq \\ &\frac{\theta_2}{\beta} \frac{\alpha_2+\beta}{\alpha_2+\mu_2}((q_1+1)c_1+q_2c_2) + \frac{\mu_2\theta_2}{\alpha_2+\mu_2} v_n(q_1+1,q_2-1,0) \\ &+ \Big(\frac{\alpha_2\theta_2}{\alpha_2+\mu_2} + (1-\lambda_1-\lambda_2-\theta_2)\Big) v_n(q_1+1,q_2,0). \end{aligned} \tag{27}$$

The inequality (27) holds due to the induction assumption (25) in state $x = (q_1+1, q_2, 0)$. In the same way prove the inequality for the terms with the factor λ_2. In this case we add to the left-hand side and right-hand side of the corresponding inequality respectively the elements $c_2 \frac{\theta_1}{\beta} \frac{\alpha_1+\beta}{\alpha_1+\mu_1}$ and to the right-hand side $c_2 \frac{\theta_2}{\beta} \frac{\alpha_2+\beta}{\alpha_2+\mu_2}$. The the induction assumption (25) in state $x = (q_1, q_2+1, 0)$ can be applied. The inequality obtained for the terms with a factor $\frac{\mu_1\theta_1}{\alpha_1+\mu_1}$ by subtracting $c_1 \frac{\theta_1}{\beta} \frac{\alpha_1+\beta}{\alpha_1+\mu_1}$ and $c_1 \frac{\theta_2}{\beta} \frac{\alpha_2+\beta}{\alpha_2+\mu_2}$ respectively from the left-hand side and right-hand side holds as well due to the induction assumption in state $x = (q_1-1, q_2, 0)$. The inequalities for the terms with factors $\frac{\alpha_1\theta_1}{\alpha_1+\mu_1}$ and $(1-\lambda_1-\lambda_2-\theta)$ satisfy directly the induction assumption (25). The rest of terms builds the inequality of the form

$$\begin{aligned} &\frac{\theta_1}{\beta} \frac{\alpha_1+\beta}{\alpha_1+\mu_1}\Big(-\lambda_1c_1-\lambda_2c_2+c_1\frac{\mu_1\theta_1}{\alpha_1+\mu_1}\Big) + \frac{\mu_1\theta_1}{\alpha_1+\mu_1}\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)((q_1-1)c_1+q_2c_2) \\ &+ \frac{\alpha_1\theta_1}{\alpha_1+\mu_1}\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)(q_1c_1+q_2c_2) + (1-\lambda_1-\lambda_2-\theta_1)\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)(q_1c_1+q_2c_2) \\ &\leq \frac{\theta_2}{\beta} \frac{\alpha_2+\beta}{\alpha_2+\mu_2}\Big(-\lambda_1c_1-\lambda_2c_2+c_1\frac{\mu_1\theta_1}{\alpha_1+\mu_1}\Big) + \frac{\mu_2\theta_2}{\alpha_2+\mu_2}\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)(q_1c_1+(q_2-1)c_2) \\ &+ \frac{\alpha_2\theta_2}{\alpha_2+\mu_2}\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)(q_1c_1+q_2c_2) + (1-\lambda_1-\lambda_2-\theta_2)\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)(q_1c_1+q_2c_2). \end{aligned} \tag{28}$$

After some simple algebra we get from (28) the inequality

$$\begin{aligned} &\frac{\theta_1}{\beta} \frac{\alpha_1+\beta}{\alpha_1+\mu_1}\Big(-\lambda_1c_1-\lambda_2c_2+c_2\frac{\mu_2\theta_2}{\alpha_2+\mu_2}\Big) - c_1\frac{\mu_1\theta_1}{\alpha_1+\mu_1} \\ &\leq \frac{\theta_2}{\beta} \frac{\alpha_2+\beta}{\alpha_2+\mu_2}\Big(-\lambda_1c_1-\lambda_2c_2+c_1\frac{\mu_1\theta_1}{\alpha_1+\mu_1}\Big) - c_2\frac{\mu_2\theta_2}{\alpha_2+\mu_2} \end{aligned}$$

or by means of the variable m_i it is of the form

$$\begin{aligned} &\frac{\alpha_1+\beta}{m_1\mu_1\beta}\Big(-\lambda_1c_1-\lambda_2c_2+\frac{c_2}{m_2}\Big) - \frac{c_1}{m_1} \\ &\leq \frac{\alpha_2+\beta}{m_2\mu_2\beta}\Big(-\lambda_1c_1-\lambda_2c_2+\frac{c_1}{m_1}\Big) - \frac{c_2}{m_2}. \end{aligned} \tag{29}$$

The inequality (29) holds due to the assumptions for the control action 1, i.e. $\frac{c_2}{m_2} \leq \frac{c_1}{m_1}$ and $\frac{\alpha_1+\beta}{m_1\mu_1} \leq \frac{\alpha_2+\beta}{m_2\mu_2}$ and the inequality

$$\lambda_1 m_1 + \lambda_2 m_2 = \lambda_1 \frac{\alpha_1+\mu_1}{\mu_1\theta_1} + \lambda_2 \frac{\alpha_2+\mu_2}{\mu_2\theta_2} < 1$$

obtained directly from the stability condition (6). In fact, the expressions in brackets of the inequality (29) can be rewritten respectively as

$$\frac{c_2}{m_2}(1-\lambda_2 m_2) - \frac{c_1}{m_1}\lambda_1 m_1 \text{ and } \frac{c_1}{m_1}(1-\lambda_1 m_1) - \frac{c_2}{m_2}\lambda_2 m_2 \geq 0. \tag{30}$$

The second expression in (30) is obviously non-negative, due to conditions $\frac{c_1}{m_1} \geq \frac{c_2}{m_2}$ and $1-\lambda_1 m_1 > \lambda_2 m_2$. If

$$\frac{c_2}{m_2}(1-\lambda_2 m_2) - \frac{c_1}{m_1}\lambda_1 m_1 \leq 0,$$

then (29) is true. If

$$\frac{c_2}{m_2}(1-\lambda_2 m_2) - \frac{c_1}{m_1}\lambda_1 m_1 \geq 0,$$

then

$$\frac{c_2}{m_2}(1-\lambda_2 m_2) - \frac{c_1}{m_1}\lambda_1 m_1 - \frac{c_1}{m_1}(1-\lambda_1 m_1) + \frac{c_2}{m_2}\lambda_2 m_2 = \frac{c_2}{m_2} - \frac{c_1}{m_1} \leq 0$$

that confirms the validity of the inequality (29).

The inequality (25) for $q_1 = 1$ can be proved using the same technique as before. Indeed, the inequality (26) is converted to

$$\begin{aligned}
&\nu\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}(c_1+q_2c_2) + \frac{\mu_1\theta_1}{\alpha_1+\mu_1}\Big[\Big(\frac{\theta_2}{\beta}\frac{\alpha_2+\beta}{\alpha_2+\mu_2}+1\Big)q_2c_2 + \lambda_1 v_n(1,q_2,0) + \lambda_2 v_n(0,q_2+1,0)\\
&+\frac{\mu_2\theta_2}{\alpha_2+\mu_2}v_n(0,q_2-1,0) + \Big(\frac{\alpha_2\theta_2}{\alpha_2+\mu_2} + (1-\lambda_1-\lambda_2-\theta_2)\Big)v_n(0,q_2,0)\Big]\\
&+\Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1} + (1-\lambda_1-\lambda_2-\theta_1)\Big)\Big[\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)\times\\
&\times(c_1+q_2c_2) + \lambda_1 v_n(2,q_2,0) + \lambda_2 v_n(1,q_2+1,0)\\
&+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}v_n(0,q_2,0) + \Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1} + (1-\lambda_1-\lambda_2-\theta_1)\Big)v_n(1,q_2,0)\Big] \leq\\
&\nu\frac{\theta_2}{\beta}\frac{\alpha_2+\beta}{\alpha_2+\mu_2}(c_1+q_2c_2) + \frac{\mu_2\theta_2}{\alpha_2+\mu_2}\Big[\Big(\frac{\theta_1}{\beta}\frac{\alpha_1+\beta}{\alpha_1+\mu_1}+1\Big)\times\\
&\times(c_1+(q_2-1)c_2) + \lambda_1 v_n(2,q_2-1,0) + \lambda_2 v_n(1,q_2,0) + \frac{\mu_1\theta_1}{\alpha_1+\mu_1}v_n(0,q_2-1,0)\\
&+\Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1} + (1-\lambda_1-\lambda_2-\theta_1)\Big)v_n(1,q_2-1,0)\Big]\\
&+\Big(\frac{\alpha_2\theta_2}{\alpha_2+\mu_2} + (1-\lambda_1-\lambda_2-\theta_2)\Big)\Big[\Big(\frac{\theta_2}{\beta}\frac{\alpha_2+\beta}{\alpha_2+\mu_2}+1\Big)\times\\
&\times(c_1+q_2c_2) + \lambda_1 v_n(2,q_2,0) + \lambda_2 v_n(1,q_2+1,0)\\
&+\frac{\mu_1\theta_1}{\alpha_1+\mu_1}v_n(0,q_2,0) + \Big(\frac{\alpha_1\theta_1}{\alpha_1+\mu_1} + (1-\lambda_1-\lambda_2-\theta_1)\Big)v_n(1,q_2,0)\Big].
\end{aligned}$$

By further comparing the terms of the corresponding inequality for the parameters of the system, we obtain inequality (29) using the proof by induction as for the case $q_2 \geq 2$. And finally we note that the inequalities in (15) automatically lead to the $c\mu$-rule defined in (7), but this doesn't always hold true vice - versa. □

Conjecture 1. *We expect that the policy defined by (15) is also optimal for the system where a constraint on arrival is omitted. This can be explained by the fact that the proportion of the class-i customers arrived to the system during the time when the server was idle will also be maintained when customers arrive in states where the server is busy or in a failed state. Therefore, the incentive to service the customer of a certain class remains the same and hence the policy (15) seems to be valid for the original queueing system with arrivals.*

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Example 1. *Consider the system with fixed parameters $(\lambda_1, \alpha_1, \beta, c_1) = (0.13, 0.20, 5, 1.00)$ and six cases of varied parameters given in Table 1. The last two columns of the table represents the values of the average cost g evaluated using a simulation technique for the policy with $f(q_1, q_2, 0) = 1$ and $f(q_1, q_2, 0) = 2, q_1, q_2 \geq 1$.*

Table 1. Simulation results

λ_2	μ_1	μ_2	θ_1	θ_2	α_2	c_2	$g^{f=1}$	$g^{f=2}$
0.30	5	3	0.55	0.5	0.10	1.00	25.657	29.403
0.26	5	3	0.70	0.40	0.10	2.00	36.909	34.238
0.26	5	3	0.70	0.43	0.12	1.64	18.223	18.291
0.30	3	5	0.51	0.5	0.10	1.00	25.430	22.955
0.30	3	5	0.51	0.5	0.10	0.90	23.421	24.709
0.30	5	3	0.55	0.5	0.10	1.20	33.348	31.610

The inequalities from (15) are:

$$
\begin{aligned}
\textit{Case 1: } c_2\tilde{\mu}_1 &= 2.098 < 2.407 = c_1\tilde{\mu}_2 \Rightarrow f = 1,\\
c_2 m_1 &= 1.890 < 2.067 = c_1 m_2,\\
\frac{\alpha_1 + \beta}{m_1\mu_1} &= 0.550 < 0.823 = \frac{\alpha_2 + \beta}{m_2\mu_2};\\
\textit{Case 2: } c_2\tilde{\mu}_1 &= 3.387 > 2.923 = c_1\tilde{\mu}_2 \Rightarrow f = 2,\\
c_2 m_1 &= 2.971 > 2.583 = c_1 m_2,\\
\frac{\alpha_1 + \beta}{m_1\mu_1} &= 0.700 > 0.658 = \frac{\alpha_2 + \beta}{m_2\mu_2};\\
\textit{Case 3: } c_2\tilde{\mu}_1 &= 2.779 = 2.779 = c_1\tilde{\mu}_2 \Rightarrow f = 1 \equiv 2,\\
c_2 m_1 &= 2.438 = 2.438 = c_1 m_2,\\
\frac{\alpha_1 + \beta}{m_1\mu_1} &= 0.700 = 0.700 = \frac{\alpha_2 + \beta}{m_2\mu_2};\\
\textit{Case 4: } c_2\tilde{\mu}_1 &= 2.438 > 2.244 = c_1\tilde{\mu}_2 \Rightarrow f = 2,\\
c_2 m_1 &= 2.091 > 2.040 = c_1 m_2,\\
\frac{\alpha_1 + \beta}{m_1\mu_1} &= 0.828 > 0.500 = \frac{\alpha_2 + \beta}{m_2\mu_2};\\
\textit{Case 5: } c_2\tilde{\mu}_1 &= 2.194 < 2.244 = c_1\tilde{\mu}_2 \Rightarrow f = 1,\\
c_2 m_1 &= 1.882 < 2.040 = c_1 m_2,\\
\frac{\alpha_1 + \beta}{m_1\mu_1} &= 0.829 > 0.500 = \frac{\alpha_2 + \beta}{m_2\mu_2};\\
\textit{Case 6: } c_2\tilde{\mu}_1 &= 2.519 > 2.407 = c_1\tilde{\mu}_2 \Rightarrow f = 2,\\
c_2 m_1 &= 2.269 > 2.067 = c_1 m_2,\\
\frac{\alpha_1 + \beta}{m_1\mu_1} &= 0.550 < 0.823 = \frac{\alpha_2 + \beta}{m_2\mu_2}.
\end{aligned}
$$

In Case 3 we get equivalent policies, the simulation results for the average cost are very similar. In Cases 5 and 6 the relations are converse, but the optimal policy, as expected, still follows the rule (7).

For the system with finite buffer capacities the optimal allocation policy can have in general another structure then the $c\mu$-rule due to the influence of boundary states. In the next example we illustrate such a result.

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Example 2. *Consider the system with parameters of Case 4 from Example 1 and finite buffer capacity for both of queues* $N_1 = N_2 = 20$*. The state-dependent optimal control actions* $f(q_1, q_2, 0)$ *evaluated by a dynamic-programming approach are summarized as a matrix represented in Table 2. The columns describe the number of customers in a queue 1 and the rows – in a queue 2. It can be seen that the optimal policy is not a static any more.*

Table 2. Matrix of optimal control actions $f(q_1, q_2, 0)$

$q_2 \backslash q_1$	0	1	2	3	4	5	6	7	8	9	10	...
0	0	2	2	2	2	2	2	2	2	2	2	...
1	1	2	2	2	2	2	2	2	2	2	2	...
2	1	2	2	2	2	2	2	2	2	2	2	...
3	1	2	2	2	2	2	2	2	2	2	1	...
4	1	2	2	2	2	2	2	2	2	1	1	...
5	1	2	2	2	2	2	2	1	1	1	1	...
6	1	2	2	2	2	1	1	1	1	1	1	...
7	1	2	2	2	1	1	1	1	1	1	1	...
8	1	2	1	1	1	1	1	1	1	1	1	...
9	1	1	1	1	1	1	1	1	1	1	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

The optimal average cost here is $g = 9.980$*. The average cost for the policy* $f(q_1, q_2, 0) = 2, q_1, q_2 \geq 1$*, is equal to* $g = 11.591$ *and for the policy* $f(q_1, q_2, 0) = 1, q_1, q_2 \geq 1$*, the average cost takes lower value* $g = 10.077$*. This results for the optimal policy differ from those obtained for higher buffer capacities. As* N_1 *and* N_2 *increase, the boundary between areas 1 and 2 in a control matrix shifts right. In infinite buffer case, the optimal policy is defined exclusively by actions* $f(q_1, q_2, 0) = 2, q_1, q_2 \geq 1$*, with the average cost* $g = 22.955$*, while the alternative policy* $f(q_1, q_2, 0) = 1, q_1, q_2 \geq 1$*, leads now to the higher average cost* $g = 25.430$*.*

4. Conclusion

In this paper we have analyzed the optimal routing problem for the unreliable single-server two-class queueing system with constant retrial rates. We derived conditions for the optimality of a static policy to serve the customers from a certain queue. The system without new arrivals can be treated as a ordinary multi-class system with a generally distributed service time. For the system with new arrivals a $c\mu$-rule cannot be used directly. We have provided a dynamic programming approach to find explicit conditions when static control policies are guaranteed to be optimal.

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