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Article

Modified Inertial-Type Subgradient Extragradient Methods for Variational Inequalities and Fixed Points of Finite Bregman Relatively Nonexpansive and Demicontractive Mappings

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Abstract: In this paper, we design two inertial-type subgradient extragradient algorithms with line-search process for solving the pseudomonotone variational inequality problems (VIPs) and common fixed-point problem (CFPP) of finite Bregman relatively nonexpansive mapping and a Bregman relatively demicontractive mapping in p-uniformly convex and uniformly smooth Banach spaces, which are more general than Hilbert spaces. Under mild conditions, we derive weak and strong convergence of the suggested algorithms to a common solution of the VIPs and CFPP, respectively. Additionally, an illustrated example is furnished to back up the feasibility and implementability of our proposed methods.

Keywords: modified inertial-type subgradient extragradient method; variational inequality problem; finite bregman relatively nonexpansive mappings; bregman relatively demicontractive mapping; bregman distance; bregman projection

Mathematics Subject Classification (2010): 47H05; 47H10; 65K15; 65Y05; 68W25

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1. Introduction

Suppose that $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert space with the induced norm $\|\cdot\|$. Let P_C be the metric projection from H onto a nonempty, convex and closed $C \subset H$. Given a nonlinear operator $S : C \to C$. We denote by $\mathrm{Fix}(S)$ the fixed-point set of S. Also, the $\mathbf{R}$, $\to$ and $\rightharpoonup$ are used to represent the set of all real numbers, the strong convergence and the weak convergence, respectively. A mapping $S : C \to C$ is said to be strictly pseudocontractive (see [1]) if $\exists \xi \in [0,1)$ s.t. $\|Sx - Sy\|^2 \le \|x-y\|^2 + \xi\|(I-S)x - (I-S)y\|^2\ \forall x, y \in C$. In particular, in case $\xi = 0$, S reduces to a nonexpansive mapping. Moreover, S is said to be demicontractive if $\mathrm{Fix}(S) \ne \emptyset$ and $\exists \xi \in [0,1)$ s.t. $\|Sx - y\|^2 \le \|x-y\|^2 + \xi\|x - Sx\|^2\ \forall x \in C, y \in \mathrm{Fix}(S)$. In particular, in case $\xi = 0$, S reduces to a quasi-nonexpansive mapping.

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Let $A : H \to H$ be a mapping. Consider the classical variational inequality problem (VIP) of finding $u \in C$ s.t. $\langle Au, v - u\rangle \geq 0\ \forall v \in C$. The solution set of the VIP is denoted by VI(C, A). In 1976, to seek an element of VI(C, A) under weaker conditions, Korpelevich [24] put forward the extragradient approach below, i.e., for any initial $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n) \quad \forall n \geq 0, \end{cases}$$

with $\tau \in (0, \frac{1}{L})$. If VI$(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element in VI(C, A). To the best of our knowledge, the Korpelevich extragradient approach is one of the most effective methods for solving the VIP at present. The literature on the VIP is vast and the Korpelevich extragradient approach has attained wide attention paid by many scholars, who ameliorated it in various forms; see e.g., [1-6, 8-9, 13-16, 19, 21-23, 25-28, 31, 34].

Furthermore, in 2018, Thong and Hieu [21] first put forward the inertial subgradient extragradient method, that is, for any initial $x_0, x_1 \in H$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} u_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \ell A u_n), \\ C_n = \{v \in H : \langle u_n - \ell A u_n - y_n, v - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n}(u_n - \ell A y_n) \quad \forall n \geq 1, \end{cases}$$

with constant $\ell \in (0, \frac{1}{L})$. Under suitable conditions, they proved the weak convergence of $\{x_n\}$ to an element of VI(C, A). Subsequently, Ceng et al. [14] introduced a modified inertial subgradient extragradient method for solving the pseudomonotone VIP and common fixed point problem (CFPP) of finite nonexpansive mappings. Let $S_i : H \to H$ be nonexpansive for $i = 1, ..., N$, $A : H \to H$ be L-Lipschitz continuous pseudomonotone on H, and sequentially weakly continuous on C, s.t. $\Omega = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $f : H \to H$ be a contraction with constant $\delta \in [0, 1)$ and $F : H \to H$ be η-strongly monotone and κ-Lipschitzian s.t. $\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, 2\eta/\kappa^2)$. Presume that $\{\beta_n\}, \{\gamma_n\}, \{\tau_n\}$ are positive sequences s.t. $\beta_n + \gamma_n < 1$, $\sum_{n=1}^\infty \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$ and $\tau_n = o(\beta_n)$. Moreover, one writes $S_n := S_{n \bmod N}$ for integer $n \geq 1$ with the mod function taking values in the set $\{1, ..., N\}$, i.e., if $n = jN + m$ for some integers $j \geq 0$ and $0 \leq m < N$, then $S_n = S_N$ if $m = 0$ and $S_n = S_m$ if $0 < m < N$.

Algorithm 1.1 (see [14, Algorithm 3.1]). **Initialization:** Given $\lambda_1 > 0$, $\alpha > 0$, $\mu \in (0, 1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n s.t. $0 \leq \alpha_n \leq \overline{\alpha_n}$, where

$$\overline{\alpha_n} = \begin{cases} \min\{\alpha, \frac{\tau_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases}$$

Step 2. Compute $w_n = S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1})$ and $y_n = P_C(w_n - \lambda_n A w_n)$.

Step 3. Construct the half-space $C_n := \{z \in H : \langle w_n - \lambda_n A w_n - y_n, z - y_n\rangle \leq 0\}$, and compute $z_n = P_{C_n}(w_n - \lambda_n A y_n)$.

Step 4. Calculate $x_{n+1} = \beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \beta_n \rho F)z_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A w_n - A y_n, z_n - y_n\rangle}, \lambda_n\} & \langle A w_n - A y_n, z_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to Step 1.

Under appropriate conditions, they proved the strong convergence of $\{x_n\}$ to an element of $\Omega = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A)$. In addition, combining the subgradient extragradient method and the Halpern's iteration method, Kraikaew and Saejung [22] proposed the Halpern subgradient extragradient rule for solving the VIP in 2014. They proved the strong convergence of the proposed method to an element in VI(C, A). Recently, Reich et al. [27] introduced two gradient-projection algorithms for solving the VIP for uniformly continuous pseudomonotone mapping. In particular, they used a novel Armijo-type line search to acquire a hyperplane which strictly separates the current iterate from the solutions of the VIP under consideration. They proved the weak and strong convergence

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of two algorithms to a solution of the VIP for uniformly continuous pseudomonotone mapping, respectively.

On the other hand, let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, let E be a p-uniformly convex and uniformly smooth Banach space and C be a convex, closed and nonempty set in E. We denote by E^* the dual space of E. The norm and the duality pairing between E and E^* are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let J_E^p and $J_{E^*}^q$ be the duality mappings of E and E^*, respectively. Let $f_p(u) = \|u\|^p/p \ \forall u \in E$, D_{f_p} be the Bregman distance with respect to (w.r.t) f_p and Π_C be the Bregman projection of E onto C w.r.t. f_p, and presume that $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ s.t. $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Assume that $A : E \to E^*$ is uniformly continuous and pseudo-monotone mapping and $S : C \to C$ is Bregman relatively nonexpansive mapping. Very recently, inspired by the research outcomes in [27], Eskandani et al. [31] invented the hybrid projection method with line-search process for seeking a solution of the VIP with the FPP constraint of S.

Algorithm 1.2 (see [31]). **Initialization:** Let $\nu > 0$, $l \in (0,1)$, $\lambda \in (0, \frac{1}{\nu})$, and put $u, u_1 \in C$ arbitrarily.

Iterative steps: Given the current iterate u_n, calculate u_{n+1} below:

Step 1. Compute $v_n = \Pi_C(J_{E^*}^q(J_E^p u_n - \lambda A u_n))$ and $r_\lambda(u_n) := u_n - v_n$. If $r_\lambda(u_n) = 0$ and $Su_n = u_n$, then stop; $u_n \in \Omega = \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$. Otherwise,

Step 2. Compute $t_n = u_n - \tau_n r_\lambda(u_n)$, where $\tau_n := l^{k_n}$ and k_n is the smallest nonnegative integer k satisfying $\langle Au_n - A(u_n - l^k r_\lambda(u_n)), r_\lambda(u_n)\rangle \le \frac{\nu}{2} D_{f_p}(u_n, v_n)$.

Step 3. Compute $w_n = J_{E^*}^q(\beta_n J_E^p u_n + (1-\beta_n) J_E^p(S\Pi_{C_n} u_n))$ and $u_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p w_n))$, where $C_n := \{v \in C : \hbar_n(v) \le 0\}$ and $\hbar_n(v) = \langle At_n, v - u_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(u_n, v_n)$.

Again set $n := n+1$ and return to Step 1.

Under suitable conditions, they proved the strong convergence of Algorithm 1.2 to $\Pi_\Omega u$. Inspired by the above research outcomes, we design two inertial-type subgradient extragradient algorithms with line-search process for solving the pseudomonotone variational inequality problems (VIPs) and common fixed-point problem (CFPP) of finite Bregman relatively nonexpansive mappings and a Bregman relatively demicontractive mapping in p-uniformly convex and uniformly smooth Banach spaces. Under mild conditions, we prove weak and strong convergence of the suggested algorithms to a common solution of the VIPs and CFPP, respectively. Additionally, an illustrated example is furnished to back up the feasibility and implementability of our proposed approaches.

The structure of this paper is built below: In Sect. 2, we release some concepts and basic results for further use. In Sect. 3, we discuss the convergence analysis of the suggested algorithms. In Sect. 4, our main results are employed to solve the VIPs and CFPP in an illustrated example. Our algorithms are more advantageous and more flexible than the above Algorithms 1.1-1.2 because they involve solving the VIPs for uniformly continuous pseudomonotone mappings and the CFPP of finite Bregman relatively nonexpansive mappings and a Bregman relatively demicontractive mapping. Our results improve and extend the corresponding results announced by some others, e.g., Ceng et al. [14], Eskandani et al. [31] and Reich et al. [27].

2. Preliminaries

Let $(E, \|\cdot\|)$ be a real Banach space, whose dual is denoted by E^*. We use the $u_n \to u$ and $u_n \rightharpoonup u$ to indicate the strong and weak convergence of $\{u_n\}$ to $u \in E$, respectively. Moreover, the set of weak cluster points of $\{u_n\}$ is denoted by $\omega_w(u_n)$, i.e., $\omega_w(u_n) = \{u^\dagger \in E : u_{n_k} \rightharpoonup u^\dagger$ for some $\{u_{n_k}\} \subset \{u_n\}\}$. Let $U = \{u \in E : \|u\| = 1\}$ and $1 < q \le 2 \le p$ with $\frac{1}{p} + \frac{1}{q} = 1$. A Banach space E is referred to as being strictly convex if for each $u, v \in U$ with $u \ne v$, one has $\|u+v\|/2 < 1$. E is referred to as being uniformly convex if $\forall \epsilon \in (0,2]$, $\exists \delta > 0$ s.t. $\forall u, v \in U$ with $\|u - v\| \ge \epsilon$, one has $\|u+v\|/2 \le 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. The modulus of convexity of E is the function $\delta : [0,2] \to [0,1]$ defined by $\delta(\epsilon) = \inf\{1 - \|u+v\|/2 : u, v \in U \text{ with } \|u-v\| \ge \epsilon\}$. It is also known that E is uniformly convex if and only if $\delta(\epsilon) > 0 \ \forall \epsilon \in (0,2]$. Moreover, E is referred to as being p-uniformly convex if $\exists c > 0$ s.t. $\delta(\epsilon) \ge c\epsilon^p \ \forall \epsilon \in [0,2]$.

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The modulus of smoothness $\rho_E : [0,\infty) \to [0,\infty)$ is defined as $\rho_E(\tau) = \sup\{(\|u+\tau v\| + \|u - \tau v\|)/2 - 1 : u, v \in U\}$. E is said to be uniformly smooth if $\lim_{\tau\to 0}\rho_E(\tau)/\tau = 0$, and q-uniformly smooth if $\exists C_q > 0$ s.t. $\rho_E(\tau) \le C_q\tau^q \ \forall \tau > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth. For example, see [32] for more details. Putting $B(0,r) = \{u \in E : \|u\| \le r\}$ for each $r > 0$, we say that $f : E \to \mathbf{R}$ is uniformly convex on bounded sets (see [31]) if $\rho_r(t) > 0 \ \forall r, t > 0$, where $\rho_r(t) : [0,\infty) \to [0,\infty]$ is specified below

$$\begin{aligned}\rho_r(t) \quad &= \inf\{[\alpha f(u) + (1-\alpha)f(v) - f(\alpha u + (1-\alpha)v)]/\alpha(1-\alpha) : \\ &\qquad \alpha \in (0,1) \text{ and } u, v \in B(0,r) \text{ with } \|u - v\| = t\},\end{aligned}$$

for all $t \ge 0$. The function ρ_r is called the gauge of uniform convexity of f. It is known that ρ_r is a nondecreasing function.

Let $f : E \to \mathbf{R}$ be a convex function. If the limit $\lim_{t\to 0^+}\frac{f(u+tv)-f(u)}{t}$ exists for each $v \in E$, then f is referred to as being Gâteaux differentiable at u. In this case, the gradient of f at u is the linear function $\nabla f(u)$, which is defined by $\langle \nabla f(u), v\rangle := \lim_{t\to 0^+}\frac{f(u+tv)-f(u)}{t}$ for each $v \in E$. The function f is referred to as being Gâteaux differentiable if it is Gâteaux differentiable at each $u \in E$. Whenever the limit $\lim_{t\to 0^+}\frac{f(u+tv)-f(u)}{t}$ is attained uniformly for any $v \in U$, we say that f is Fréchet differentiable at u. Besides, f is referred to as being uniformly Fréchet differentiable on a subset $K \subset E$ if the limit $\lim_{t\to 0^+}\frac{f(u+tv)-f(u)}{t}$ is attained uniformly for $(u,v) \in K \times U$. A Banach space E is called smooth if its norm is Gâteaux differentiable.

Let $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. The duality mapping $J_E^p : E \to E^*$ is specified below

$$J_E^p(u) = \{\psi \in E^* : \langle \psi, u\rangle = \|u\|^p \text{ and } \|\psi\| = \|u\|^{p-1}\} \quad \forall u \in E.$$

It is known that E is smooth if and only if J_E^p is single-valued mapping of E into E^*. Also, E is reflexive if and only if J_E^p is surjective, and E is strictly convex if and only if J_E^p is one-to-one. So it follows that, if E is smooth, strictly convex and reflexive Banach space, then J_E^p is a single-valued bijection and in this case, $J_E^p = (J_{E^*}^q)^{-1}$ where $J_{E^*}^q$ is the duality mapping of E^*. Besides, it is known that E is uniformly smooth if and only if the function $f_p(u) = \|u\|^p/p$ is uniformly Fréchet differentiable on bounded sets if and only if J_E^p is single-valued and uniformly continuous on bounded sets. It is also known that E is uniformly convex if and only if the function f_p is uniformly convex (see [32]).

Let $f : E \to \mathbf{R}$ be a Gâteaux differentiable convex function. The Bregman distance w.r.t. f is specified below

$$D_f(u,v) := f(u) - f(v) - \langle \nabla f(v), u - v\rangle \quad \forall u, v \in E.$$

It is worth mentioning that the Bregman distance is not a metric in the usual sense of the term. Clearly, $D_f(u,u) = 0$ but $D_f(u,v) = 0$ can not yield $u = v$. Generally, D_f is not symmetric and does not satisfy the triangle inequality. However, D_f satisfies the three point identity

$$D_f(u,v) + D_f(v,w) = D_f(u,w) - \langle \nabla f(v) - \nabla f(w), u - v\rangle.$$

See [20] for more details on Bregman functions and distances.

It is noteworthy that the duality mapping J_E^p on the smooth Banach space E is the Gâteaux derivative of the function f_p. Then the Bregman distance w.r.t. f_p is specified below

$$\begin{aligned}D_{f_p}(u,v) \quad &= \|u\|^p/p - \|v\|^p/p - \langle J_E^p(v), u - v\rangle \\ &= \|u\|^p/p + \|v\|^p/q - \langle J_E^p(v), u\rangle \\ &= (\|v\|^p - \|u\|^p)/q - \langle J_E^p(v) - J_E^p(u), u\rangle.\end{aligned}$$

In the smooth and p-uniformly convex Banach space E with $2 \le p < \infty$, there holds the following relationship between the metric and Bregman distance:

$$\tau\|u - v\|^p \le D_{f_p}(u,v) \le \langle J_E^p(u) - J_E^p(v), u - v\rangle, \qquad (2.1)$$

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where $\tau > 0$ is some fixed number (see [12]). From (2.1) it is readily known that for any bounded sequence $\{u_n\} \subset E$, the following holds:

$$u_n \to u \quad \Leftrightarrow \quad D_{f_p}(u, u_n) \to 0 \quad (n \to \infty).$$

Let C be a nonempty closed convex subset of reflexive, smooth and strictly convex Banach space E. Bregman projections are defined as minimizers of Bregman distances. The Bregman projection of $u \in E$ onto C w.r.t. f_p is the unique element $\Pi_C u \in C$ s.t. $D_{f_p}(\Pi_C u, u) = \min_{v \in C} D_{f_p}(v, u)$. In Hilbert spaces the Bregman projection w.r.t. f_2 reduces to the metric projection. Using [18, Corollary 4.4] and [30, Theorem 2.1], in uniformly convex Banach spaces Bregman projections can be characterized by the following inequality:

$$\langle J_E^p(u) - J_E^p(\Pi_C u), v - \Pi_C u \rangle \le 0 \quad \forall v \in C. \tag{2.2}$$

Moreover, this inequality is equivalent to the descent property

$$D_{f_p}(v, \Pi_C u) + D_{f_p}(\Pi_C u, u) \le D_{f_p}(v, u) \quad \forall v \in C. \tag{2.3}$$

In case $p = 2$, the duality mapping J_E^p reduces to the normalized duality mapping and is denoted by J. The function $\phi : E^2 \to \mathbf{R}$ is formulated below

$$\phi(u, v) = \|u\|^2 - 2\langle Jv, u \rangle + \|v\|^2 \quad \forall u, v \in E,$$

and $\Pi_C(u) = \operatorname{argmin}_{v \in C} \phi(v, u)$ $\forall u \in E$.

In terms of [31], the function $V_{f_p} : E \times E^* \to [0, \infty)$ associated with f_p is specified below

$$V_{f_p}(u, u^*) = \|u\|^p / p - \langle u^*, u \rangle + \|u^*\|^q / q \quad \forall (u, u^*) \in E \times E^*. \tag{2.4}$$

So, $V_{f_p}(u, u^*) = D_{f_p}(u, J_{E^*}^q(u^*))$ $\forall (u, u^*) \in E \times E^*$. Moreover, by the subdifferential inequality, we obtain

$$V_{f_p}(u, u^*) + \langle v^*, J_{E^*}^q(u^*) - u \rangle \le V_{f_p}(u, u^* + v^*) \quad \forall u \in E,\ u^*, v^* \in E^*. \tag{2.5}$$

In addition, V_{f_p} is convex in the second variable. Thus one has

$$D_{f_p}(z, J_{E^*}^q(\sum_{i=1}^n t_i J_E^p(u_i))) \le \sum_{i=1}^n t_i D_{f_p}(z, u_i) \ \ \forall z \in E, \{u_i\}_{i=1}^n \subset E, \{t_i\}_{i=1}^n \subset [0,1] \text{ with } \sum_{i=1}^n t_i = 1. \tag{2.6}$$

Lemma 2.1 (see [30]). Let E be a uniformly convex Banach space and $\{u_n\}, \{v_n\}$ be two sequences in E such that the first one is bounded. If $\lim_{n\to\infty} D_{f_p}(v_n, u_n) = 0$, then $\lim_{n\to\infty} \|v_n - u_n\| = 0$.

Let $S : C \to C$ be a mapping. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S, that is, $\mathrm{Fix}(S) = \{u \in C : u = Su\}$. A point $u \in C$ is referred to as an asymptotic fixed point of S if $\exists \{u_n\} \subset C$ s.t. $u_n \rightharpoonup u$ and $u_n - Su_n \to 0$. We denote by $\widehat{\mathrm{Fix}}(S)$ the set of asymptotic fixed points of S. The notion of asymptotic fixed points was invented in Reich [11]. A mapping $S : C \to C$ is known as being Bregman relatively ξ-demicontractive w.r.t. f_p if $\mathrm{Fix}(S) = \widehat{\mathrm{Fix}}(S) \neq \emptyset$, and $\exists \xi \in [0, 1)$ s.t. for each bounded $\{v_n\} \subset C$ satisfying $\sup_{n\ge1} \|Sv_n\| < \infty$, the following holds:

$$D_{f_p}(u, Sv_n) \le D_{f_p}(u, v_n) + \xi \rho_b^* \|J_E^p v_n - J_E^p S v_n\| \quad \forall u \in \mathrm{Fix}(S),$$

with $b = \sup_{n\ge1}\{\|v_n\|^{p-1}, \|Sv_n\|^{p-1}\} < \infty$. In particular, putting $b_x = \max\{\|x\|^{p-1}, \|Sx\|^{p-1}\}$ for each $x \in C$, one has

$$D_{f_p}(u, Sx) \le D_{f_p}(u, x) + \xi \rho_{b_x}^* \|J_E^p x - J_E^p S x\| \quad \forall u \in \mathrm{Fix}(S).$$

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In addition, if $\xi = 0$, then S reduces to a Bregman relatively nonexpansive mapping w.r.t. f_p, that is, S is said to be Bregman relatively nonexpansive w.r.t. f_p if $\mathrm{Fix}(S) = \widehat{\mathrm{Fix}}(S) \neq \emptyset$ and $D_{f_p}(u, Sv) \leq D_{f_p}(u, v)\ \forall v \in C, u \in \mathrm{Fix}(S)$.

Definition 2.1. Let C be a nonempty closed convex subset of E. A mapping $A : C \to E^*$ is referred to as being

(i) monotone on C if $\langle Au - Av, u - v\rangle \geq 0\ \forall u, v \in C$;

(ii) pseudo-monotone if $\langle Au, v - u\rangle \geq 0 \Rightarrow \langle Av, v - u\rangle \geq 0\ \forall u, v \in C$;

(iii)L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ s.t. $\|Au - Av\| \leq L\|u - v\| \forall u, v \in C$;

(iv) weakly sequentially continuous if for each $\{x_n\} \subset C$, the relation holds: $x_n \rightharpoonup x \ \Rightarrow\ Ax_n \rightharpoonup Ax$.

Lemma 2.2 (see [31]). Let $r > 0$ be a constant and suppose that $f : E \to \mathbf{R}$ is a uniformly convex function on bounded subsets of a Banach space E. Then

$$f(\sum_{k=1}^{n} \alpha_k x_k) \leq \sum_{k=1}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - x_j\|),$$

$\forall i, j \in \{1, 2, ..., n\}$, $\{x_k\}_{k=1}^n \subset B(0, r)$ and $\{\alpha_k\}_{k=1}^n \subset (0, 1)$ with $\sum_{k=1}^n \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Proof. It is easy to show the conclusion.

Lemma 2.3 (see [28]). Let E_1 and E_2 be two Banach spaces. Suppose that $A : E_1 \to E_2$ is uniformly continuous on bounded subsets of E_1 and D is a bounded subset of E_1. Then $A(D)$ is bounded.

Lemma 2.4 (see [10]). Let $\emptyset \neq C \subset E$ with C being closed and convex in a Banach space E and suppose $A : C \to E^*$ is pseudo-monotone and continuous. Then $x^\dagger \in C$ is a solution to the VIP $\langle Ax^\dagger, y - x^\dagger\rangle \geq 0\ \forall y \in C$, if and only if $\langle Ay, y - x^\dagger\rangle \geq 0\ \forall y \in C$.

Lemma 2.5. Let $2 \leq p < \infty$ and suppose that E is a smooth and p-uniformly convex Banach space with the weakly sequentially continuous duality mapping J_E^p. Let $\{q_n\} \subset E$ and $\emptyset \neq \Omega \subset E$. If $\{D_{f_p}(x, q_n)\}$ converges for each $x \in \Omega$, and $\omega_w(q_n) \subset \Omega$. Then $\{q_n\}$ converges weakly to a point in Ω.

Proof. Using (2.1) we get $\tau\|x - q_n\|^p \leq D_{f_p}(x, q_n)\ \forall x \in \Omega$. This ensures that $\{q_n\}$ is bounded. Hence, from the reflexivity of E we have $\omega_w(q_n) \neq \emptyset$. Also, let us show the weak convergence of $\{q_n\}$ to a point in Ω. Indeed, let $\bar{q}, \hat{q} \in \omega_w(q_n)$ with $\bar{q} \neq \hat{q}$. Then, $\exists\{q_{n_k}\} \subset \{q_n\}$ and $\exists\{q_{m_k}\} \subset \{q_n\}$ s.t. $q_{n_k} \rightharpoonup \bar{q}$ and $q_{m_k} \rightharpoonup \hat{q}$. By the weakly sequential continuity of J_E^p one deduces that $J_E^p(q_{n_k}) \rightharpoonup J_E^p\bar{q}$ and $J_E^p(q_{m_k}) \rightharpoonup J_E^p\hat{q}$. Note that $D_{f_p}(\bar{q}, \hat{q}) + D_{f_p}(\hat{q}, q_n) = D_{f_p}(\bar{q}, q_n) - \langle J_E^p\hat{q} - J_E^p q_n, \bar{q} - \hat{q}\rangle$. So, exploiting the convergence of the sequences $\{D_{f_p}(\bar{q}, q_n)\}$ and $\{D_{f_p}(\hat{q}, q_n)\}$, we deduce that

$$\begin{aligned}
&-\langle J_E^p\hat{q} - J_E^p\bar{q}, \bar{q} - \hat{q}\rangle = \lim_{k\to\infty}[-\langle J_E^p\hat{q} - J_E^p q_{n_k}, \bar{q} - \hat{q}\rangle] \\
&= \lim_{n\to\infty}[D_{f_p}(\bar{q}, \hat{q}) + D_{f_p}(\hat{q}, q_n) - D_{f_p}(\bar{q}, q_n)] \\
&= \lim_{k\to\infty}[-\langle J_E^p\hat{q} - J_E^p q_{m_k}, \bar{q} - \hat{q}\rangle] = -\langle J_E^p\hat{q} - J_E^p\hat{q}, \bar{q} - \hat{q}\rangle = 0,
\end{aligned}$$

which hence yields $\langle J_E^p\bar{q} - J_E^p\hat{q}, \bar{q} - \hat{q}\rangle = 0$. From (2.1) we get $0 < \tau\|\bar{q} - \hat{q}\|^p \leq D_{f_p}(\bar{q}, \hat{q}) \leq \langle J_E^p\bar{q} - J_E^p\hat{q}, \bar{q} - \hat{q}\rangle = 0$. This arrives at a contradiction. Consequently, the sequence $\{q_n\}$ converges weakly to a point in Ω. □

The lemma below was put forth in $\mathbf{R}^m$ by [29]. It is easy to verify that the proof of the lemma in Banach spaces is actually the same as in $\mathbf{R}^m$. Here, we present the lemma but omit the proof in Banach spaces.

Lemma 2.6. Let $\emptyset \neq C \subset E$ with C being closed and convex in a Banach space E. Suppose that $K := \{x \in C : h(x) \leq 0\}$ where h is a real-valued function on E. If $K \neq \emptyset$ and h is Lipschitz continuous on C with modulus $\theta > 0$, then $\theta\mathrm{dist}(x, K) \geq \max\{h(x), 0\}\ \forall x \in C$, where $\mathrm{dist}(x, K)$ stands for the distance of x to K.

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Lemma 2.7 (see [17]). Let $\{\mathbf{\Gamma}_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that, $\exists\{\mathbf{\Gamma}_{n_k}\} \subset \{\mathbf{\Gamma}_n\}$ s.t. $\mathbf{\Gamma}_{n_k} < \mathbf{\Gamma}_{n_k+1}$ $\forall k \ge 1$. Let the sequence $\{\psi(n)\}_{n \ge n_0}$ of integers be defined as $\psi(n) = \max\{k \le n : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\}$, with integer $n_0 \ge 1$ satisfying $\{k \le n_0 : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\} \ne \emptyset$. Then the following hold:

(i) $\psi(n_0) \le \psi(n_0 + 1) \le \cdots$ and $\psi(n) \to \infty$;

(ii) $\mathbf{\Gamma}_{\psi(n)} \le \mathbf{\Gamma}_{\psi(n)+1}$ and $\mathbf{\Gamma}_n \le \mathbf{\Gamma}_{\psi(n)+1}$ $\forall n \ge n_0$.

Lemma 2.8 (see [7]). Let $\{a_n\}$ be a sequence in $[0, \infty)$ satisfying $a_{n+1} \le (1 - \mu_n)a_n + \mu_n \nu_n$ $\forall n \ge 1$, where $\{\mu_n\}$ and $\{\nu_n\}$ both are real sequences such that (i) $\{\mu_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \mu_n = \infty$, and (ii) $\limsup_{n\to\infty} \nu_n \le 0$ or $\sum_{n=1}^{\infty} |\mu_n \nu_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9 (see [33]). Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \le (1 + \delta_n)a_n + b_n$ $\forall n \ge 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3. Main Results

In this section, let E be a p-uniformly convex and uniformly smooth Banach space with $2 \le p < \infty$. Let $\emptyset \ne C \subset E$ with C being closed and convex in E. We are now in a position to state and analyze our iterative algorithms for settling the VIPs for uniformly continuous pseudomonotone mappings and the CFPP of finite Bregman relatively nonexpansive mappings and a Bregman relatively demicontractive mapping in E. Assume always that the conditions hold below:

(C1) For $i = 1, ..., N$, $S_i : C \to C$ is a uniformly continuous and Bregman relatively nonexpansive mapping and $S_0 : C \to C$ is a uniformly continuous and Bregman relatively ξ-demicontractive mapping.

(C2) $\{S_n\}_{n=1}^{\infty}$ is defined as $S_n := S_{n \bmod N}$ for integer $n \ge 1$ with the mod function taking values in the set $\{1, ..., N\}$, i.e., if $n = jN + m$ for some integers $j \ge 0$ and $0 \le m < N$, then $S_n = S_N$ if $m = 0$ and $S_n = S_m$ if $0 < m < N$.

(C3) For $i = 1, 2$, $A_i : E \to E^*$ is pseudomonotone and uniformly continuous on C, s.t. $\|A_i x^\dagger\| \le \liminf_{n\to\infty} \|A_i x_n\|$ $\forall \{x_n\} \subset C$ with $x_n \rightharpoonup x^\dagger$.

(C4) $\Omega = (\bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^{N} \mathrm{Fix}(S_i)) \ne \emptyset$.

Algorithm 3.1. Initialization: Given $x_0, x_1 \in C$ arbitrarily and let $\epsilon > 0$, $\mu_i > 0$, $\lambda_i \in (0, \frac{1}{\mu_i})$, $l_i \in (0, 1)$ for $i = 1, 2$. Choose $\{\ell_n\}, \{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (\xi, 1)$ s.t. $\sum_{n=1}^{\infty} \ell_n < \infty$, $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$ and $\liminf_{n\to\infty}(\alpha_n - \xi)(1 - \alpha_n) > 0$. Moreover, given the iterates x_{n-1} and x_n $(n \ge 1)$, choose ϵ_n s.t. $0 \le \epsilon_n \le \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\|}\} & \text{if } S_n x_n \ne S_n x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterative steps: Calculate x_{n+1} as follows:

Step 1. Calculate $g_n = J_{E^*}^q((1 - \epsilon_n)J_E^p S_n x_n + \epsilon_n J_E^p(2S_n x_n - S_n x_{n-1}))$ and calculate $u_n = J_{E^*}^q(\beta_n J_E^p x_n + (1 - \beta_n)J_E^p g_n)$, $y_n = \Pi_C(J_{E^*}^q(J_E^p u_n - \lambda_1 A_1 u_n))$, $r_{\lambda_1}(u_n) := u_n - y_n$ and $s_n = u_n - \tau_n r_{\lambda_1}(u_n)$, where $\tau_n := l_1^{k_n}$ and k_n is the smallest nonnegative integer k satisfying

$$\langle A_1 u_n - A_1(u_n - l_1^k r_{\lambda_1}(u_n)), u_n - y_n\rangle \le \frac{\mu_1}{2} D_{f_p}(u_n, y_n). \tag{3.1}$$

Step 2. Calculate $w_n = \Pi_{C_n}(u_n)$, with $C_n := \{x \in C : h_n(x) \le 0\}$ and

$$h_n(x) = \langle A_1 s_n, x - u_n\rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(u_n, y_n). \tag{3.2}$$

Step 3. Calculate $\tilde{y}_n = \Pi_C(J_{E^*}^q(J_E^p w_n - \lambda_2 A_2 w_n))$, $r_{\lambda_2}(w_n) := w_n - \tilde{y}_n$ and $t_n = w_n - \tilde{\tau}_n r_{\lambda_2}(w_n)$, where $\tilde{\tau}_n := l_2^{j_n}$ and j_n is the smallest nonnegative integer j satisfying

$$\langle A_2 w_n - A_2(w_n - l_2^j r_{\lambda_2}(w_n)), w_n - \tilde{y}_n\rangle \le \frac{\mu_2}{2} D_{f_p}(w_n, \tilde{y}_n). \tag{3.3}$$

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Step 4. Calculate $v_n = J_{E^*}^q(\alpha_n J_E^p w_n + (1-\alpha_n)J_E^p(S_0 w_n))$ and $x_{n+1} = \Pi_{\widetilde{C}_n \cap Q_n}(w_n)$, where $Q_n := \{x \in C : D_{f_p}(x, v_n) \le D_{f_p}(x, w_n)\}$, $\widetilde{C}_n := \{x \in C : \tilde{h}_n(x) \le 0\}$ and

$$\tilde{h}_n(x) = \langle A_2 t_n, x - w_n\rangle + \frac{\tilde{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \widetilde{y}_n). \tag{3.4}$$

Again set $n := n+1$ and go to Step 1.

The following lemmas are used in the proofs of our main results in the sequel.

Lemma 3.1. Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 3.1. Then the relations hold: $\frac{1}{\lambda_1} D_{f_p}(u_n, y_n) \le \langle A_1 u_n, r_{\lambda_1}(u_n)\rangle$ and $\frac{1}{\lambda_2} D_{f_p}(w_n, \widetilde{y}_n) \le \langle A_2 w_n, r_{\lambda_2}(w_n)\rangle$.

Proof. Observe that the last two relations are similar. Then it suffices to show that the latter relation holds. In fact, using the definition of $\widetilde{y}_n$ and properties of Π_C, one has

$$\langle J_E^p w_n - \lambda_2 A_2 w_n - J_E^p \widetilde{y}_n, y - \widetilde{y}_n\rangle \le 0 \quad \forall y \in C.$$

Setting $y = w_n$ in the last inequality, from (2.1) we get

$$D_{f_p}(w_n, \widetilde{y}_n) \le \langle J_E^p w_n - J_E^p \widetilde{y}_n, w_n - \widetilde{y}_n\rangle \le \lambda_2 \langle A_2 w_n, w_n - \widetilde{y}_n\rangle.$$

This completes the proof. □

Lemma 3.2. The Armijo-type search rules (3.1), (3.3) and the sequence $\{x_n\}$ constructed in Algorithm 3.1 are well defined.

Proof. Observe that the rules (3.1) and (3.3) are similar. Then it suffices to show that the latter rule (3.3) is valid. Using the uniform continuity of A_2 on C, from $l_2 \in (0,1)$ one gets $\lim_{j\to\infty}\langle A_2 w_n - A_2(w_n - l_2^j r_{\lambda_2}(w_n)), r_{\lambda_2}(w_n)\rangle = 0$. In case $r_{\lambda_2}(w_n) = 0$, it is evident that $j_n = 0$. In case $r_{\lambda_2}(w_n) \ne 0$, we know that $\exists j_n \ge 0$ s.t. (3.3) holds.

It is easy to check that for each $n \ge 1$, $\widetilde{C}_n$ and Q_n are convex and closed. We assert that $\Omega \subset \widetilde{C}_n \cap Q_n$. Let $z \in \Omega = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^N \mathrm{Fix}(S_i))$. Using Lemma 2.2 and the Bregman relative ξ-demicontractivity of S_0, from $\{\alpha_n\} \subset (\xi, 1)$ we get

$$\begin{aligned}
&D_{f_p}(z, v_n) \le \alpha_n D_{f_p}(z, w_n) + (1-\alpha_n) D_{f_p}(z, S_0 w_n) \\
&\quad - \alpha_n(1-\alpha_n)\rho^*_{b_{w_n}} \|J_E^P w_n - J_E^P S_0 w_n\| \\
&\le \alpha_n D_{f_p}(z, w_n) + (1-\alpha_n)[D_{f_p}(z, w_n) + \xi \rho^*_{b_{w_n}} \|J_E^P w_n - J_E^P S_0 w_n\|] \\
&\quad - \alpha_n(1-\alpha_n)\rho^*_{b_{w_n}} \|J_E^P w_n - J_E^P S_0 w_n\| \\
&= D_{f_p}(z, w_n) - (\alpha_n - \xi)(1-\alpha_n)\rho^*_{b_{w_n}} \|J_E^P w_n - J_E^P S_0 w_n\| \\
&\le D_{f_p}(z, w_n),
\end{aligned}$$

which hence leads to $z \in Q_n$. Moreover, by Lemma 2.4, we get $\langle A_2 t_n, t_n - z\rangle \ge 0$. Therefore,

$$\begin{aligned}
\tilde{h}_n(z) &= \langle A_2 t_n, z - w_n\rangle + \tfrac{\tilde{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \widetilde{y}_n) \\
&= -\langle A_2 t_n, w_n - t_n\rangle - \langle A_2 t_n, t_n - z\rangle + \tfrac{\tilde{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \widetilde{y}_n) \\
&\le -\tilde{\tau}_n \langle A_2 t_n, r_{\lambda_2}(w_n)\rangle + \tfrac{\tilde{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \widetilde{y}_n).
\end{aligned} \tag{3.5}$$

So it follows from (3.3) that

$$\langle A_2 w_n - A_2 t_n, r_{\lambda_2}(w_n)\rangle \le \frac{\mu_2}{2} D_{f_p}(w_n, \widetilde{y}_n).$$

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Using Lemma 3.1 we have

$$\begin{aligned}\langle A_2 t_n, r_{\lambda_2}(w_n)\rangle &\geq \langle A_2 w_n, r_{\lambda_2}(w_n)\rangle - \tfrac{\mu_2}{2} D_{f_p}(w_n, \widetilde{y}_n)\\ &\geq (\tfrac{1}{\lambda_2} - \tfrac{\mu_2}{2}) D_{f_p}(w_n, \widetilde{y}_n).\end{aligned}$$

This together with (3.5), arrives at

$$\tilde{h}_n(z) \leq -\frac{\tilde{\tau}_n}{2}(\frac{1}{\lambda_2} - \mu_2) D_{f_p}(w_n, \widetilde{y}_n) \leq 0.$$

Consequently, $\Omega \subset \widetilde{C}_n \cap Q_n$. So, the sequence $\{x_n\}$ is well defined. □

Lemma 3.3. Suppose that $\{y_n\}$ and $\{\widetilde{y}_n\}$ are the sequences generated by Algorithm 3.1. If $\lim_{n\to\infty} \|u_n - y_n\| = 0$ and $\lim_{n\to\infty} \|w_n - \widetilde{y}_n\| = 0$, then $\omega_w(u_n) \subset \mathrm{VI}(C, A_1)$ and $\omega_w(w_n) \subset \mathrm{VI}(C, A_2)$.

Proof. Observe that the last two inclusions are similar. Then it suffices to show that the latter inclusion is valid. In fact, let $z \in \omega_w(w_n)$. Then, $\exists\{w_{n_k}\} \subset \{w_n\}$, s.t. $w_{n_k} \rightharpoonup z$ and $\lim_{n\to\infty} \|w_{n_k} - \widetilde{y}_{n_k}\| = 0$. Hence, it is known that $\widetilde{y}_{n_k} \rightharpoonup z$. Since C is of both convexity and closedness, from $\{\widetilde{y}_n\} \subset C$ and $\widetilde{y}_{n_k} \rightharpoonup z$ we get $z \in C$. Next, we consider two cases. If $A_2 z = 0$, then $z \in \mathrm{VI}(C, A_2)$ because $\langle A_2 z, y - z\rangle \geq 0\ \forall y \in C$. If $A_2 z \neq 0$, using the assumption on A_2, instead of the weakly sequential continuity of A_2, we get $0 < \|A_2 z\| \leq \liminf_{k\to\infty} \|A_2 w_{n_k}\|$. So, we could assume that $\|A_2 w_{n_k}\| \neq 0\ \forall k \geq 1$. From (2.2), we get

$$\langle J_E^p w_{n_k} - \lambda_2 A_2 w_{n_k} - J_E^p \widetilde{y}_{n_k}, x - \widetilde{y}_{n_k}\rangle \leq 0 \quad \forall x \in C,$$

and hence

$$\frac{1}{\lambda_2}\langle J_E^p w_{n_k} - J_E^p \widetilde{y}_{n_k}, x - \widetilde{y}_{n_k}\rangle + \langle A_2 w_{n_k}, \widetilde{y}_{n_k} - w_{n_k}\rangle \leq \langle A_2 w_{n_k}, x - w_{n_k}\rangle \quad \forall x \in C. \tag{3.6}$$

According to the uniform continuity of A_2, one knows that $\{A_2 w_{n_k}\}$ is bounded by Lemma 2.3. Note that $\{\widetilde{y}_{n_k}\}$ is bounded as well. So, using the uniform continuity of J_E^p on bounded subsets of E, from (3.6) we have

$$\liminf_{k\to\infty}\langle A_2 w_{n_k}, x - w_{n_k}\rangle \geq 0 \quad \forall x \in C. \tag{3.7}$$

To prove that $z \in \mathrm{VI}(C, A_2)$, we now pick a sequence $\{\tilde{\varepsilon}_k\} \subset (0,1)$ satisfying $\tilde{\varepsilon}_k \downarrow 0$ as $k \to \infty$. For each $k \geq 1$, we denote by l_k the smallest positive integer such that

$$\langle A_2 w_{n_j}, y - w_{n_j}\rangle + \tilde{\varepsilon}_k \geq 0 \quad \forall j \geq l_k. \tag{3.8}$$

Because $\{\tilde{\varepsilon}_k\}$ is decreasing, it is easily known that $\{l_k\}$ is increasing. For convenience, we still denote $\{A_2 w_{n_{l_k}}\}$ by $\{A_2 w_{l_k}\}$. Note that $A_2 w_{l_k} \neq 0\ \forall k \geq 1$ (due to $\{A_2 w_{l_k}\} \subset \{A_2 w_{n_k}\}$). Then, putting $\tilde{g}_{l_k} = \frac{A_2 w_{l_k}}{\|A_2 w_{l_k}\|^{\frac{q}{q-1}}}$, one gets $\langle A_2 w_{l_k}, J_{E^*}^q \tilde{g}_{l_k}\rangle = 1\ \forall k \geq 1$. Indeed, it is evident that $\langle A_2 w_{l_k}, J_{E^*}^q \tilde{g}_{l_k}\rangle = \langle A_2 w_{l_k}, (\frac{1}{\|A_2 w_{l_k}\|^{\frac{q}{q-1}}})^{q-1} J_{E^*}^q A_2 w_{l_k}\rangle = (\frac{1}{\|A_2 w_{l_k}\|^{\frac{q}{q-1}}})^{q-1}\|A_2 w_{l_k}\|^q = 1\ \forall k \geq 1$. So, by (3.8) one has $\langle A_2 w_{l_k}, y + \tilde{\varepsilon}_k J_{E^*}^q \tilde{g}_{l_k} - w_{l_k}\rangle \geq 0\ \forall k \geq 1$. Again from the pseudomonotonicity of A_2 one has

$$\langle A_2(y + \tilde{\varepsilon}_k J_{E^*}^q \tilde{g}_{l_k}), y + \tilde{\varepsilon}_k J_{E^*}^q \tilde{g}_{l_k} - w_{l_k}\rangle \geq 0 \quad \forall y \in C. \tag{3.9}$$

We assert that $\lim_{k\to\infty} \tilde{\varepsilon}_k J_{E^*}^q \tilde{g}_{l_k} = 0$. Indeed, since $\{w_{l_k}\} \subset \{w_{n_k}\}$ and $\tilde{\varepsilon}_k \downarrow 0$ as $k \to \infty$, it follows that

$$0 \leq \limsup_{k\to\infty} \|\tilde{\varepsilon}_k J_{E^*}^q \tilde{g}_{l_k}\| = \limsup_{k\to\infty} \frac{\tilde{\varepsilon}_k}{\|A_2 w_{l_k}\|} \leq \frac{\limsup_{k\to\infty} \tilde{\varepsilon}_k}{\liminf_{k\to\infty} \|A_2 w_{n_k}\|} = 0.$$

Hence one gets $\tilde{\varepsilon}_k J^q_{E^*}\tilde{g}_{l_k} \to 0$ as $k \to \infty$. Thus, taking the limit as $k \to \infty$ in (3.9), by condition (C2) we have $\langle A_2 y, y - z\rangle \geq 0\ \forall y \in C$. By Lemma 2.4 one obtains $z \in \mathrm{VI}(C, A_2)$. □

Lemma 3.4. Suppose that $\{y_n\}$ and $\{\widetilde{y}_n\}$ are the sequences generated by Algorithm 3.1. Then the following hold:

(i) $\lim_{n\to\infty} \tau_n D_{f_p}(u_n, y_n) = 0 \ \Rightarrow\ \lim_{n\to\infty} D_{f_p}(u_n, y_n) = 0$;

(ii) $\lim_{n\to\infty} \tilde{\tau}_n D_{f_p}(w_n, \widetilde{y}_n) = 0 \ \Rightarrow\ \lim_{n\to\infty} D_{f_p}(w_n, \widetilde{y}_n) = 0$.

Proof. Observe that the assertions (i) and (ii) are similar. Then it suffices to show that assertion (ii) is valid. To verify assertion (ii), we consider two cases. In case $\liminf_{n\to\infty} \tilde{\tau}_n > 0$, we might presume that $\exists \tilde{\tau} > 0$ s.t. $\tilde{\tau}_n \geq \tilde{\tau} > 0\ \forall n \geq 1$, which hence arrives at

$$D_{f_p}(w_n, \widetilde{y}_n) = \frac{1}{\tilde{\tau}_n}\tilde{\tau}_n D_{f_p}(w_n, \widetilde{y}_n) \leq \frac{1}{\tilde{\tau}} \cdot \tilde{\tau}_n D_{f_p}(w_n, \widetilde{y}_n). \tag{3.10}$$

This together with $\lim_{n\to\infty} \tilde{\tau}_n D_{f_p}(w_n, \widetilde{y}_n) = 0$, leads to $\lim_{n\to\infty} D_{f_p}(w_n, \widetilde{y}_n) = 0$.

In case $\liminf_{n\to\infty} \tilde{\tau}_n = 0$, we presume that $\limsup_{n\to\infty} D_{f_p}(w_n, \widetilde{y}_n) = a_2 > 0$. Then we know that $\exists\{n_k\} \subset \{n\}$ s.t.

$$\lim_{k\to\infty} \tilde{\tau}_{n_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} D_{f_p}(w_{n_k}, \widetilde{y}_{n_k}) = a_2 > 0.$$

We define $\widehat{t_{n_k}} = \frac{1}{l_2}\tilde{\tau}_{n_k}\widetilde{y}_{n_k} + (1 - \frac{1}{l_2}\tilde{\tau}_{n_k})w_{n_k}$ for each $k \geq 1$. Applying (2.1) and noticing $\lim_{k\to\infty} \tilde{\tau}_{n_k} D_{f_p}(w_{n_k}, \widetilde{y}_{n_k}) = 0$, we have $\lim_{k\to\infty} \tilde{\tau}_{n_k}\|w_{n_k} - \widetilde{y}_{n_k}\|^p = 0$ and hence

$$\lim_{k\to\infty} \|\widehat{t_{n_k}} - w_{n_k}\|^p = \lim_{k\to\infty} \frac{\tilde{\tau}^{p-1}_{n_k}}{l_2^p} \cdot \tilde{\tau}_{n_k}\|w_{n_k} - \widetilde{y}_{n_k}\|^p = 0. \tag{3.11}$$

Because A_2 is uniformly continuous on bounded subsets of C, we obtain

$$\lim_{k\to\infty} \|A_2 w_{n_k} - A_2\widehat{t_{n_k}}\| = 0. \tag{3.12}$$

From the step size rule (3.3) and the definition of $\widehat{t_{n_k}}$, it follows that

$$\langle A_2 w_{n_k} - A_2\widehat{t_{n_k}}, w_{n_k} - \widetilde{y}_{n_k}\rangle > \frac{\mu_2}{2} D_{f_p}(w_{n_k}, \widetilde{y}_{n_k}). \tag{3.13}$$

Now, taking the limit as $k \to \infty$, from (3.12) we have $\lim_{k\to\infty} D_{f_p}(w_{n_k}, \widetilde{y}_{n_k}) = 0$. This, however, reaches a contradiction. So it follows that $\lim_{n\to\infty} D_{f_p}(w_n, \widetilde{y}_n) = 0$. □

Now, we are ready to prove the weak convergence theorem.

Theorem 3.1. Suppose that E is a p-uniformly convex and uniformly smooth Banach space with the weakly sequentially continuous duality mapping J^p_E. If $\{x_n\}$ is the sequence generated by Algorithm 3.1, then $x_n \rightharpoonup z \in \Omega \Leftrightarrow \sup_{n\geq 0}\|x_n\| < \infty$.

Proof. It is clear that the necessity of Theorem 3.1 is valid. Next it suffices to show that the sufficiency is valid. Assume that $\sup_{n\geq 0}\|x_n\| < \infty$. Let $z \in \Omega$. It is clear that $S_n x_n \neq S_n x_{n-1} \Leftrightarrow J^p_E S_n x_n \neq J^p_E(2S_n x_n - S_n x_{n-1})$. Using the definition of ϵ_n, we get $\epsilon_n\|J^p_E S_n x_n - J^p_E(2S_n x_n - S_n x_{n-1})\| \leq \ell_n\ \forall n \geq 1$. From (2.1), (2.6) and the three point identity of D_{f_p} we get

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$$
\begin{aligned}
&D_{f_p}(z,g_n) \le (1-\epsilon_n)D_{f_p}(z,S_nx_n) + \epsilon_n D_{f_p}(z,2S_nx_n - S_nx_{n-1}) \\
&= D_{f_p}(z,S_nx_n) + \epsilon_n[D_{f_p}(z,2S_nx_n - S_nx_{n-1}) - D_{f_p}(z,S_nx_n)] \\
&= D_{f_p}(z,S_nx_n) + \epsilon_n[D_{f_p}(S_nx_n,2S_nx_n - S_nx_{n-1}) \\
&\quad + \langle J_E^p S_nx_n - J_E^p(2S_nx_n - S_nx_{n-1}), z - S_nx_n\rangle] \\
&\le D_{f_p}(z,x_n) + \epsilon_n\langle J_E^p S_nx_n - J_E^p(2S_nx_n - S_nx_{n-1}), z + S_nx_{n-1} - 2S_nx_n\rangle \\
&\le D_{f_p}(z,x_n) + \epsilon_n\|J_E^p S_nx_n - J_E^p(2S_nx_n - S_nx_{n-1})\|\|z + S_nx_{n-1} - 2S_nx_n\| \\
&\le D_{f_p}(z,x_n) + \ell_n M,
\end{aligned}
$$

where $\sup_{n\ge1}\|z + S_nx_{n-1} - 2S_nx_n\| \le M$ for some $M > 0$. Using Lemma 2.2, we get

$$
\begin{aligned}
&D_{f_p}(z,u_n) = V_{f_p}(z,\beta_n J_E^p x_n + (1-\beta_n)J_E^p g_n) \\
&\le \tfrac{1}{p}\|z\|^p - \beta_n\langle J_E^p x_n, z\rangle - (1-\beta_n)\langle J_E^p g_n, z\rangle + \tfrac{\beta_n}{q}\|J_E^p x_n\|^q \\
&\quad + \tfrac{(1-\beta_n)}{q}\|J_E^p g_n\|^q - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\
&= \tfrac{1}{p}\|z\|^p - \beta_n\langle J_E^p x_n, z\rangle - (1-\beta_n)\langle J_E^p g_n, z\rangle + \tfrac{\beta_n}{q}\|x_n\|^p \\
&\quad + \tfrac{(1-\beta_n)}{q}\|g_n\|^p - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\
&= \beta_n D_{f_p}(z,x_n) + (1-\beta_n)D_{f_p}(z,g_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\
&\le \beta_n D_{f_p}(z,x_n) + (1-\beta_n)[D_{f_p}(z,x_n) + \ell_n M] - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\
&\le D_{f_p}(z,x_n) + \ell_n M - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|.
\end{aligned}
$$

Since $w_n = \Pi_{C_n}u_n$, by (2.1) and (2.3) we get

$$
\begin{aligned}
D_{f_p}(z,w_n) &\le D_{f_p}(z,u_n) - D_{f_p}(w_n,u_n) \\
&= D_{f_p}(z,u_n) - D_{f_p}(\Pi_{C_n}u_n,u_n) \\
&\le D_{f_p}(z,u_n) - \tau\|\Pi_{C_n}u_n - u_n\|^p \\
&\le D_{f_p}(z,u_n) - \tau\|P_{C_n}u_n - u_n\|^p \\
&= D_{f_p}(z,u_n) - \tau[\mathrm{dist}(C_n,u_n)]^p.
\end{aligned}
$$

Because $x_{n+1} = \Pi_{\widetilde{C}_n\cap Q_n}w_n$, from (2.1) and (2.3) we get

$$
\begin{aligned}
D_{f_p}(z,x_{n+1}) &\le D_{f_p}(z,w_n) - D_{f_p}(x_{n+1},w_n) \\
&= D_{f_p}(z,w_n) - D_{f_p}(\Pi_{\widetilde{C}_n\cap Q_n}w_n,w_n) \\
&\le D_{f_p}(z,w_n) - D_{f_p}(\Pi_{\widetilde{C}_n}w_n,w_n) \\
&\le D_{f_p}(z,w_n) - \tau\|\Pi_{\widetilde{C}_n}w_n - w_n\|^p \\
&\le D_{f_p}(z,w_n) - \tau\|P_{\widetilde{C}_n}w_n - w_n\|^p \\
&= D_{f_p}(z,w_n) - \tau[\mathrm{dist}(\widetilde{C}_n,w_n)]^p.
\end{aligned}
$$

Combining (3.13) and the last two inequalities, we obtain

$$
\begin{aligned}
&D_{f_p}(z,x_{n+1}) \le D_{f_p}(z,w_n) - D_{f_p}(x_{n+1},w_n) \\
&\le D_{f_p}(z,u_n) - D_{f_p}(w_n,u_n) - D_{f_p}(x_{n+1},w_n) \\
&\le D_{f_p}(z,u_n) - \tau[\mathrm{dist}(C_n,u_n)]^p - \tau[\mathrm{dist}(\widetilde{C}_n,w_n)]^p \\
&\le D_{f_p}(z,x_n) + \ell_n M - \beta_n(1-\beta_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\
&\quad - \tau[\mathrm{dist}(C_n,u_n)]^p - \tau[\mathrm{dist}(\widetilde{C}_n,w_n)]^p,
\end{aligned}
\tag{3.14}
$$

which hence arrives at

$$
D_{f_p}(z,x_{n+1}) \le D_{f_p}(z,x_n) + \ell_n M.
$$

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Since $\sum_{n=1}^{\infty} \ell_n < \infty$, by Lemma 2.9 we deduce that $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists. In addition, by the boundedness of $\{x_n\}$, we conclude that $\{g_n\}, \{u_n\}, \{v_n\}, \{w_n\}, \{y_n\}, \{\tilde{y}_n\}$, $\{s_n\}, \{t_n\}, \{S_n x_n\}$ and $\{S_0 w_n\}$ are also bounded. Using (3.14) we obtain

$$\begin{aligned} &D_{f_p}(w_n, u_n) + D_{f_p}(x_{n+1}, w_n) \le D_{f_p}(z, u_n) - D_{f_p}(z, x_{n+1}) \\ &\le D_{f_p}(z, x_n) + \ell_n M - \beta_n(1-\beta_n)\rho_b^* \| J_E^p x_n - J_E^p g_n \| - D_{f_p}(z, x_{n+1}), \end{aligned}$$

which immediately yields

$$\begin{aligned} &D_{f_p}(w_n, u_n) + D_{f_p}(x_{n+1}, w_n) + \beta_n(1-\beta_n)\rho_b^* \| J_E^p x_n - J_E^p g_n \| \\ &\le D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1}) + \ell_n M. \end{aligned}$$

Since $\lim_{n\to\infty} \ell_n = 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists, it follows that $\lim_{n\to\infty} D_{f_p}(w_n, u_n) = 0$, $\lim_{n\to\infty} D_{f_p}(x_{n+1}, w_n) = 0$, and $\lim_{n\to\infty} \rho_b^* \| J_E^p x_n - J_E^p g_n \| = 0$, which hence yields $\lim_{n\to\infty} \| J_E^p x_n - J_E^p g_n \| = 0$. From $u_n = J_{E^*}^q(\beta_n J_E^p x_n + (1-\beta_n) J_E^p g_n)$, it can be readily seen that $\lim_{n\to\infty} \| J_E^p u_n - J_E^p x_n \| = 0$. Noticing $g_n = J_{E^*}^q((1-\epsilon_n) J_E^p S_n x_n + \epsilon_n J_E^p(2S_n x_n - S_n x_{n-1}))$, we obtain from $\lim_{n\to\infty} \ell_n = 0$ and the definition of ϵ_n that

$$\| J_E^p g_n - J_E^p S_n x_n \| = \epsilon_n \| J_E^p(2S_n x_n - S_n x_{n-1}) - J_E^p S_n x_n \| \le \ell_n \to 0 \quad (n \to \infty).$$

Hence, using (2.1) and uniform continuity of J_E^p on bounded subsets of E, we conclude that $\lim_{n\to\infty} \| g_n - S_n x_n \| = 0$ and

$$\lim_{n\to\infty} \| w_n - u_n \| = \lim_{n\to\infty} \| x_{n+1} - w_n \| = \lim_{n\to\infty} \| x_n - S_n x_n \| = \lim_{n\to\infty} \| u_n - x_n \| = 0. \tag{3.15}$$

Since $\{x_n\}$ is bounded and E is reflexive, then we know that $\omega_w(x_n) \ne \emptyset$. In what follows, we claim that $\omega_w(x_n) \subset \Omega$. Let $z \in \omega_w(x_n)$. Then, $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup z$. From (3.15) one gets $w_{n_k} \rightharpoonup z$. Since $\{A_1 s_n\}$ is bounded, we know that $\exists L_1 > 0$ s.t. $\| A_1 s_n \| \le L_1$. This ensures that for each $x, y \in C_n$,

$$|h_n(x) - h_n(y)| = |\langle A_1 s_n, x - y \rangle| \le \| A_1 s_n \| \| x - y \| \le L_1 \| x - y \|,$$

which implies that $h_n(x)$ is L_1-Lipschitz continuous on C_n. Using Lemma 2.6, we get

$$\operatorname{dist}(C_n, u_n) \ge \frac{1}{L_1} h_n(u_n) = \frac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n). \tag{3.16}$$

Noticing $x_{n+1} \in Q_n$, from the definition of Q_n and (3.14), we have

$$\begin{aligned} D_{f_p}(x_{n+1}, v_n) &\le D_{f_p}(x_{n+1}, w_n) \\ &\le D_{f_p}(z, w_n) - D_{f_p}(z, x_{n+1}) \\ &\le D_{f_p}(z, u_n) - D_{f_p}(z, x_{n+1}) \\ &\le D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1}) + \ell_n M. \end{aligned}$$

Since $\lim_{n\to\infty} \ell_n = 0$ and $\lim_{n\to\infty} D_{f_p}(z, x_n)$ exists, we have $\lim_{n\to\infty} D_{f_p}(x_{n+1}, v_n) = 0$ and hence $\lim_{n\to\infty} \| x_{n+1} - v_n \| = 0$. This together with (3.15), arrives at

$$\lim_{n\to\infty} \| w_n - v_n \| = 0. \tag{3.17}$$

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On the other hand, using Lemma 2.2, we get

$$
\begin{aligned}
& D_{f_p}(z, v_n) = V_{f_p}(z, \alpha_n J_E^p w_n + (1-\alpha_n) J_E^p S_0 w_n) \\
& \leq \tfrac{1}{p}\|z\|^p - \alpha_n \langle J_E^p w_n, z\rangle - (1-\alpha_n)\langle J_E^p S_0 w_n, z\rangle + \tfrac{\alpha_n}{q}\|J_E^p w_n\|^q \\
& \quad + \tfrac{(1-\alpha_n)}{q}\|J_E^p S_0 w_n\|^q - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_0 w_n\| \\
& = \tfrac{1}{p}\|z\|^p - \alpha_n \langle J_E^p w_n, z\rangle - (1-\alpha_n)\langle J_E^p S_0 w_n, z\rangle + \tfrac{\alpha_n}{q}\|w_n\|^p \\
& \quad + \tfrac{(1-\alpha_n)}{q}\|S_0 w_n\|^p - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_0 w_n\| \\
& = \alpha_n D_{f_p}(z, w_n) + (1-\alpha_n) D_{f_p}(z, S_0 w_n) - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_0 w_n\| \\
& \leq \alpha_n D_{f_p}(z, w_n) + (1-\alpha_n)[D_{f_p}(z, w_n) + \xi\rho_b^*\|J_E^p w_n - J_E^p S_0 w_n\|] \\
& \quad - \alpha_n(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_0 w_n\| \\
& = D_{f_p}(z, w_n) - (\alpha_n - \xi)(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_0 w_n\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\alpha_n - \xi)(1-\alpha_n)\rho_b^*\|J_E^p w_n - J_E^p S_0 w_n\| & \leq D_{f_p}(z, w_n) - D_{f_p}(z, v_n) \\
& \leq D_{f_p}(z, w_n) - D_{f_p}(z, v_n) + D_{f_p}(w_n, v_n) \\
& = \langle J_E^p v_n - J_E^p w_n, z - w_n\rangle.
\end{aligned}
$$

Taking the limit in the last inequality as $n \to \infty$, and using uniform continuity of J_E^p on bounded subsets of E, (3.17) and $\liminf_{n\to\infty}(\alpha_n - \xi)(1-\alpha_n) > 0$, we get $\lim_{n\to\infty} \rho_b^*\|J_E^p w_n - J_E^p S_0 w_n\| = 0$ and hence $\lim_{n\to\infty}\|J_E^p w_n - J_E^p S_0 w_n\| = 0$. This together with uniform continuity of $J_{E^*}^q$ on bounded subsets of E^* implies that

$$\lim_{n\to\infty}\|w_n - S_0 w_n\| = 0. \tag{3.18}$$

Now let us show that $z \in \bigcap_{i=1}^2 \mathrm{VI}(C, A_i)$. Since $\{A_2 t_n\}$ is bounded, we know that $\exists L_2 > 0$ s.t. $\|A_2 t_n\| \leq L_2$. This ensures that for each $x, y \in \widetilde{C}_n$,

$$|\tilde{h}_n(x) - \tilde{h}_n(y)| = |\langle A_2 t_n, x - y\rangle| \leq \|A_2 t_n\|\|x - y\| \leq L_2\|x - y\|,$$

which guarantees that $\tilde{h}_n(x)$ is L_2-Lipschitz continuous on $\widetilde{C}_n$. By Lemma 2.6, we get

$$\mathrm{dist}(\widetilde{C}_n, w_n) \geq \frac{1}{L_2}\tilde{h}_n(w_n) = \frac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n). \tag{3.19}$$

Combining (3.14), (3.16) and (3.19), we obtain

$$
\begin{aligned}
& D_{f_p}(z, x_n) - D_{f_p}(z, x_{n+1}) + \ell_n M \\
& \geq D_{f_p}(z, u_n) - D_{f_p}(z, x_{n+1}) \\
& \geq \tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n)]^p + \tau[\tfrac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n)]^p.
\end{aligned} \tag{3.20}
$$

Thus,

$$\lim_{n\to\infty} \tau_n D_{f_p}(u_n, y_n) = \lim_{n\to\infty} \tilde{\tau}_n D_{f_p}(w_n, \tilde{y}_n) = 0.$$

By Lemma 3.4, we get

$$\lim_{n\to\infty}\|u_n - y_n\| = \lim_{n\to\infty}\|w_n - \tilde{y}_n\| = 0.$$

Besides, combining (3.15) and $x_{n_k} \rightharpoonup z$ guarantees that $u_{n_k} \rightharpoonup z$ and $w_{n_k} \rightharpoonup z$. By Lemma 3.3 we deduce that $z \in \omega_w(u_n) \subset \mathrm{VI}(C, A_1)$ and $z \in \omega_w(w_n) \subset \mathrm{VI}(C, A_2)$. Consequently,

$$z \in \bigcap_{i=1}^2 \mathrm{VI}(C, A_i).$$

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Next we claim that $z \in \bigcap_{i=0}^{N} \mathrm{Fix}(S_i)$. Indeed, by (3.15) we immediately get

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - w_n\| + \|w_n - u_n\| + \|u_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.21}$$

We first claim that $\lim_{n\to\infty} \|x_n - S_r x_n\| = 0$ for $r = 1, ..., N$. Actually, by the definition of S_n, we obtain that $S_n \in \{S_1, ..., S_N\}$ $\forall n \geq 1$, which hence leads to $S_{n+i} \in \{S_1, ..., S_N\}$ $\forall n \geq 1, i = 1, ..., N$. Note that for $i = 1, ..., N$,

$$\begin{aligned} \|x_n - S_{n+i}x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| + \|S_{n+i}x_{n+i} - S_{n+i}x_n\| \\ &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i}x_{n+i}\| + \sum_{j=1}^{N} \|S_j x_{n+i} - S_j x_n\|. \end{aligned}$$

Utilizing the uniform continuity of each S_j on C, we deduce from (3.15) and (3.21) that $x_{n+i} - S_{n+i}x_{n+i} \to 0$ and $S_j x_{n+i} - S_j x_n \to 0$ for $i, j = 1, ..., N$. Thus, we get $\lim_{n\to\infty} \|x_n - S_{n+i}x_n\| = 0$ for $i = 1, ..., N$. This immediately implies that

$$\lim_{n\to\infty} \|x_n - S_r x_n\| = 0 \quad \text{for } r = 1, ..., N.$$

So it follows from $x_{n_k} \rightharpoonup z$ that $z \in \widehat{\mathrm{Fix}}(S_r) = \mathrm{Fix}(S_r)$ for $r = 1, ..., N$. Therefore, $z \in \bigcap_{i=1}^{N} \mathrm{Fix}(S_i)$. In addition, from (3.15) and $x_{n_k} \rightharpoonup z$, one has that $w_{n_k} \rightharpoonup z$. Thus, using (3.18) we get $z \in \widehat{\mathrm{Fix}}(S_0) = \mathrm{Fix}(S_0)$. Consequently, $z \in \bigcap_{i=0}^{N} \mathrm{Fix}(S_i)$, and hence $z \in \Omega = (\bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^{N} \mathrm{Fix}(S_i))$. This means that $\omega_w(x_n) \subset \Omega$. As a result, applying Lemma 2.5 we conclude that $x_n \rightharpoonup z$. □

Next, we prove a strong convergence theorem for approximating a common solution of the VIPs for uniformly continuous pseudomonotone mappings and the CFPP of finite Bregman relatively nonexpansive mappings and a Bregman relatively demicontractive mapping in E.

Algorithm 3.2. Initialization: Given $x_0, x_1 \in C$ arbitrarily and let $\epsilon > 0$, $\mu_i > 0$, $l_i \in (0,1)$ and $\lambda_i \in (0, \frac{1}{\mu_i})$ for $i = 1,2$. Choose $\{\ell_n\}, \{\gamma_n\}, \{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (\xi, 1)$ s.t. $\lim_{n\to\infty} \ell_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} (\beta_n - \xi)(1 - \beta_n) > 0$ and $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$. Moreover, given the iterates x_{n-1} and x_n $(n \geq 1)$, choose ϵ_n s.t. $0 \leq \epsilon_n \leq \overline{\epsilon_n}$, where $\sup_{n\geq 1} \frac{\epsilon_n}{\alpha_n} < \infty$ and

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\|}\} & \text{if } S_n x_n \neq S_n x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterative steps: Calculate x_{n+1} as follows:

Step 1. Set $g_n = J_{E^*}^q((1 - \epsilon_n) J_E^p S_n x_n + \epsilon_n J_E^p(2S_n x_n - S_n x_{n-1}))$, and calculate $u_n = J_{E^*}^q(\gamma_n J_E^p x_n + (1 - \gamma_n) J_E^p g_n)$, $y_n = \Pi_C(J_{E^*}^q(J_E^p u_n - \lambda_1 A_1 u_n))$, $r_{\lambda_1}(u_n) := u_n - y_n$ and $s_n = u_n - \tau_n r_{\lambda_1}(u_n)$, where $\tau_n := l_1^{k_n}$ and k_n is the smallest nonnegative integer k satisfying

$$\langle A_1 u_n - A_1(u_n - l_1^k r_{\lambda_1}(u_n)), u_n - y_n\rangle \leq \frac{\mu_1}{2} D_{f_p}(u_n, y_n).$$

Step 2. Calculate $w_n = \Pi_{C_n}(u_n)$, with $C_n := \{x \in C : h_n(x) \leq 0\}$ and

$$h_n(x) = \langle A_1 s_n, x - u_n\rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(u_n, y_n).$$

Step 3. Calculate $\tilde{y}_n = \Pi_C(J_{E^*}^q(J_E^p w_n - \lambda_2 A_2 w_n))$, $r_{\lambda_2}(w_n) := w_n - \tilde{y}_n$ and $t_n = w_n - \tilde{\tau}_n r_{\lambda_2}(w_n)$, where $\tilde{\tau}_n := l_2^{j_n}$ and j_n is the smallest nonnegative integer j satisfying

$$\langle A_2 w_n - A_2(w_n - l_2^j r_{\lambda_2}(w_n)), w_n - \tilde{y}_n\rangle \leq \frac{\mu_2}{2} D_{f_p}(w_n, \tilde{y}_n).$$

Step 4. Set $z_n = \Pi_{\tilde{C}_n}(w_n)$, and calculate $v_n = J_{E^*}^q(\beta_n J_E^p z_n + (1 - \beta_n) J_E^p(S_0 z_n))$ and $x_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1 - \alpha_n) J_E^p v_n))$, where $\tilde{C}_n := \{x \in C : \tilde{h}_n(x) \leq 0\}$ and

$$\tilde{h}_n(x) = \langle A_2 t_n, x - w_n\rangle + \frac{\tilde{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \tilde{y}_n).$$

Again set $n := n + 1$ and go to Step 1.

Theorem 3.2. Suppose that the conditions (C1)-(C3) hold. If $\{x_n\}$ is the sequence generated by Algorithm 3.2, then $x_n \to \Pi_\Omega u \Leftrightarrow \sup_{n\geq 0} \|x_n\| < \infty$.

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Proof. It is clear that the necessity of Theorem 3.2 is valid. Next it suffices to show that the sufficiency is valid. Assume that $\sup_{n\geq 0}\|x_n\| < \infty$. In what follows, we divide our proof into four claims.

Claim 1. We show that

$$(1-\alpha_n)\gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\ \leq \alpha_n D_{f_p}(\hat{u},u) + D_{f_p}(z,x_n) - D_{f_p}(\hat{u},x_{n+1}) + \ell_n M,$$

for some $M > 0$. Indeed, put $\hat{u} = \Pi_\Omega u$. Noticing $w_n = \Pi_{C_n} u_n$ and $z_n = \Pi_{\widetilde{C}_n} w_n$, we deduce from (2.1) and (2.3) that

$$\begin{aligned} D_{f_p}(\hat{u},w_n) &\leq D_{f_p}(\hat{u},u_n) - D_{f_p}(w_n,u_n) \\ &\leq D_{f_p}(\hat{u},u_n) - \tau[\text{dist}(C_n,u_n)]^p, \end{aligned}$$

and

$$\begin{aligned} D_{f_p}(\hat{u},z_n) &\leq D_{f_p}(\hat{u},w_n) - D_{f_p}(z_n,w_n) \\ &\leq D_{f_p}(\hat{u},w_n) - \tau[\text{dist}(\widetilde{C}_n,w_n)]^p. \end{aligned}$$

Using the same inferences as in the proof of Theorem 3.1, we know that

$$\begin{aligned} D_{f_p}(\hat{u},g_n) &\leq D_{f_p}(\hat{u},x_n) + \epsilon_n\|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\| \\ &\quad \times \|\hat{u} + S_n x_{n-1} - 2S_n x_n\| \leq D_{f_p}(\hat{u},x_n) + \ell_n M, \end{aligned}$$

where $\sup_{n\geq 1}\|\hat{u} + S_n x_{n-1} - 2S_n x_n\| \leq M$ for some $M > 0$. This ensures that $\{g_n\}$ is bounded.

Using (2.6) and the last two inequalities, from $\{\gamma_n\} \subset (0,1)$ and $\{\beta_n\} \subset (\xi,1)$ we obtain

$$\begin{aligned} &D_{f_p}(\hat{u},x_{n+1}) \leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)D_{f_p}(\hat{u},v_n) \\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u},z_n) + (1-\beta_n)D_{f_p}(\hat{u},S_0 z_n) \\ &\quad - \beta_n(1-\beta_n)\rho_{b_{z_n}}^*\|z_n - S_0 z_n\|] \\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)\{\beta_n D_{f_p}(\hat{u},z_n) + (1-\beta_n)[D_{f_p}(\hat{u},z_n) + \xi\rho_{b_{z_n}}^*\|z_n - S_0 z_n\|] \\ &\quad - \beta_n(1-\beta_n)\rho_{b_{z_n}}^*\|z_n - S_0 z_n\|\} \\ &= \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)[D_{f_p}(\hat{u},z_n) - (\beta_n - \xi)(1-\beta_n)\rho_{b_{z_n}}^*\|z_n - S_0 z_n\|] \\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)[D_{f_p}(\hat{u},w_n) - D_{f_p}(z_n,w_n)] \\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)[D_{f_p}(\hat{u},u_n) - D_{f_p}(w_n,u_n) - D_{f_p}(z_n,w_n)] \\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)D_{f_p}(\hat{u},u_n) \\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)[\gamma_n D_{f_p}(\hat{u},x_n) + (1-\gamma_n)D_{f_p}(\hat{u},g_n) \\ &\quad - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|] \\ &\leq \alpha_n D_{f_p}(\hat{u},u) + (1-\alpha_n)[D_{f_p}(z,x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|] \\ &\leq \alpha_n D_{f_p}(\hat{u},u) + D_{f_p}(z,x_n) + \ell_n M - (1-\alpha_n)\gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|, \end{aligned}$$

which immediately arrives at the desired claim. In addition, it is easily known that $\{u_n\}, \{v_n\}, \{w_n\}, \{y_n\}, \{\widetilde{y}_n\}, \{z_n\}, \{s_n\}, \{t_n\}$ and $\{S_0 z_n\}$ are also bounded.

Claim 2. We show that

$$D_{f_p}(w_n,u_n) + D_{f_p}(z_n,w_n) \\ \leq D_{f_p}(\hat{u},u_n) - D_{f_p}(\hat{u},x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p\hat{u}, \zeta_n - \hat{u}\rangle.$$

Indeed, set $b = \sup_{n\geq 1}\{\|x_n\|^{p-1}, \|g_n\|^{p-1}, \|z_n\|^{p-1}, \|S_0 z_n\|^{p-1}\}$. By Lemma 2.2 we get

$$\begin{aligned} &D_{f_p}(\hat{u},u_n) = V_{f_p}(\hat{u},\gamma_n J_E^p x_n + (1-\gamma_n)J_E^p g_n) \\ &\leq \tfrac{1}{p}\|\hat{u}\|^p - \gamma_n\langle J_E^p x_n,\hat{u}\rangle - (1-\gamma_n)\langle J_E^p g_n,\hat{u}\rangle + \tfrac{\gamma_n}{q}\|J_E^p x_n\|^q \\ &\quad + \tfrac{(1-\gamma_n)}{q}\|J_E^p g_n\|^q - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\ &= \tfrac{1}{p}\|\hat{u}\|^p - \gamma_n\langle J_E^p x_n,\hat{u}\rangle - (1-\gamma_n)\langle J_E^p g_n,\hat{u}\rangle + \tfrac{\gamma_n}{q}\|x_n\|^p \\ &\quad + \tfrac{(1-\gamma_n)}{q}\|g_n\|^p - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\ &= \gamma_n D_{f_p}(\hat{u},x_n) + (1-\gamma_n)D_{f_p}(\hat{u},g_n) - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\ &\leq \gamma_n D_{f_p}(\hat{u},x_n) + (1-\gamma_n)[D_{f_p}(\hat{u},x_n) + \ell_n M] - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\| \\ &\leq D_{f_p}(\hat{u},x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p g_n\|, \end{aligned} \tag{3.22}$$

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and

$$\begin{aligned} D_{f_p}(\hat{u}, v_n) &= V_{f_p}(\hat{u}, \beta_n J_E^p z_n + (1-\beta_n) J_E^p S_0 z_n) \\ &\leq \tfrac{1}{p}\|\hat{u}\|^p - \beta_n \langle J_E^p z_n, \hat{u}\rangle - (1-\beta_n)\langle J_E^p S_0 z_n, \hat{u}\rangle + \tfrac{\beta_n}{q}\|J_E^p z_n\|^q \\ &\quad + \tfrac{(1-\beta_n)}{q}\|J_E^p S_0 z_n\|^q - \beta_n(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\| \\ &= \tfrac{1}{p}\|\hat{u}\|^p - \beta_n \langle J_E^p z_n, \hat{u}\rangle - (1-\beta_n)\langle J_E^p S_0 z_n, \hat{u}\rangle + \tfrac{\beta_n}{q}\|z_n\|^p \\ &\quad + \tfrac{(1-\beta_n)}{q}\|S_0 z_n\|^p - \beta_n(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\| \\ &= \beta_n D_{f_p}(\hat{u}, z_n) + (1-\beta_n) D_{f_p}(\hat{u}, S_0 z_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\| \\ &\leq \beta_n D_{f_p}(\hat{u}, z_n) + (1-\beta_n)[D_{f_p}(\hat{u}, z_n) + \xi\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\|] \\ &\quad - \beta_n(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\| \\ &= D_{f_p}(\hat{u}, z_n) - (\beta_n - \xi)(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\| \\ &\leq D_{f_p}(\hat{u}, w_n) - (\beta_n - \xi)(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\|. \end{aligned} \tag{3.23}$$

Set $\zeta_n = J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n)$. From (2.5), we have

$$\begin{aligned} &D_{f_p}(\hat{u}, x_{n+1}) \leq D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n)) = V_{f_p}(\hat{u}, \alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n) \\ &\leq V_{f_p}(\hat{u}, \alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n - \alpha_n (J_E^p u - J_E^p \hat{u})) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\ &\leq \alpha_n D_{f_p}(\hat{u}, \hat{u}) + (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\ &= (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\ &\leq (1-\alpha_n)[D_{f_p}(\hat{u}, w_n) - (\beta_n - \xi)(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\|] + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\ &\leq (1-\alpha_n) D_{f_p}(\hat{u}, w_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle. \end{aligned} \tag{3.24}$$

On the other hand, from (3.23) we have

$$\begin{aligned} D_{f_p}(\hat{u}, v_n) &\leq D_{f_p}(\hat{u}, z_n) - (\beta_n - \xi)(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\| \\ &\leq D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n) - (\beta_n - \xi)(1-\beta_n)\rho_b^*\|J_E^p z_n - J_E^p S_0 z_n\| \\ &\leq D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n). \end{aligned}$$

Substituting the above inequality into (3.24), we get

$$\begin{aligned} &D_{f_p}(\hat{u}, x_{n+1}) \leq (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\ &\leq D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\ &\leq D_{f_p}(\hat{u}, u_n) - D_{f_p}(w_n, u_n) - D_{f_p}(z_n, w_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle. \end{aligned}$$

This immediately arrives at

$$\begin{aligned} &D_{f_p}(w_n, u_n) + D_{f_p}(z_n, w_n) \\ &\leq D_{f_p}(\hat{u}, u_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle. \end{aligned} \tag{3.25}$$

Claim 3. We show that

$$\begin{aligned} &(1-\alpha_n)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n)]^p + \tau[\tfrac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n)]^p\} \\ &\leq \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \ell_n M. \end{aligned}$$

Indeed, using the similar inferences to these of (3.20) in the proof of Theorem 3.1, we get

$$\begin{aligned} D_{f_p}(\hat{u}, z_n) &\leq D_{f_p}(\hat{u}, w_n) - \tau[\tfrac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n)]^p \\ &\leq D_{f_p}(\hat{u}, u_n) - \tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n)]^p - \tau[\tfrac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n)]^p. \end{aligned} \tag{3.26}$$

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Applying (3.26), (3.23) and (3.22), we have

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \le D_{f_p}(\hat{u}, J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n)J^p_E v_n))\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n) D_{f_p}(\hat{u}, v_n)\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[D_{f_p}(\hat{u}, z_n) - (\beta_n - \xi)(1-\beta_n)\rho_b^*\|z_n - S_0 z_n\|]\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)\{D_{f_p}(\hat{u}, u_n) - \tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n)]^p - \tau[\tfrac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n)]^p\}\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, u_n) - (1-\alpha_n)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n)]^p + \tau[\tfrac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n)]^p\} \qquad (3.27)\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J^p_E x_n - J^p_E g_n\|\\
&\quad - (1-\alpha_n)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n)]^p + \tau[\tfrac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n)]^p\}\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, x_n) + \ell_n M\\
&\quad - (1-\alpha_n)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n)]^p + \tau[\tfrac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \tilde{y}_n)]^p\}.
\end{aligned}$$

Claim 4. We show that $x_n \to \hat{u}$ as $n \to \infty$. Indeed, since E is reflexive and $\{x_n\}$ is bounded, we know that $\omega_w(x_n) \ne \emptyset$. Let $z \in \omega_w(x_n)$. Then, $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \rightharpoonup z$. For each $n \ge 1$, we write $\mathbf{\Gamma}_n = D_{f_p}(\hat{u}, x_n)$. In what follows, we show the convergence of $\{\mathbf{\Gamma}_n\}$ to zero in the following two possible cases.

Case 1. Suppose that $\exists$ (integer) $n_0 \ge 1$ such that $\{\mathbf{\Gamma}_n\}_{n=n_0}^\infty$ is nonincreasing. Then $\lim_{n\to\infty} \mathbf{\Gamma}_n = d < +\infty$ and $\lim_{n\to\infty}(\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1}) = 0$. From (3.25) and (3.22) we get

$$\begin{aligned}
&D_{f_p}(w_n, u_n) + D_{f_p}(z_n, w_n)\\
&\le D_{f_p}(\hat{u}, u_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle\\
&\le D_{f_p}(\hat{u}, x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J^p_E x_n - J^p_E g_n\| - D_{f_p}(\hat{u}, x_{n+1})\\
&\quad + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle,
\end{aligned}$$

which hence yields

$$\begin{aligned}
&D_{f_p}(w_n, u_n) + D_{f_p}(z_n, w_n) + \gamma_n(1-\gamma_n)\rho_b^*\|J^p_E x_n - J^p_E g_n\|\\
&\le D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \ell_n M + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle\\
&= \mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} + \ell_n M + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty} \ell_n = \lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$, $\lim_{n\to\infty}(\mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1}) = 0$ and the sequence $\{\zeta_n\}$ is bounded, we obtain that $\lim_{n\to\infty} D_{f_p}(w_n, u_n) = 0$, $\lim_{n\to\infty} D_{f_p}(z_n, w_n) = 0$, and $\lim_{n\to\infty} \rho_b^*\|J^p_E x_n - J^p_E g_n\| = 0$, which hence yields $\lim_{n\to\infty}\|J^p_E x_n - J^p_E g_n\| = 0$. From $u_n = J^q_{E^*}(\gamma_n J^p_E x_n + (1-\gamma_n)J^p_E g_n)$, it is easily known that $\lim_{n\to\infty}\|J^p_E u_n - J^p_E x_n\| = 0$. Noticing $g_n = J^q_{E^*}((1-\epsilon_n)J^p_E S_n x_n + \epsilon_n J^p_E(2S_n x_n - S_n x_{n-1}))$, we deduce from $\lim_{n\to\infty} \ell_n = 0$ and the definition of ϵ_n that

$$\|J^p_E g_n - J^p_E S_n x_n\| = \epsilon_n\|J^p_E(2S_n x_n - S_n x_{n-1}) - J^p_E S_n x_n\| \le \ell_n \to 0 \quad (n \to \infty).$$

Hence, using (2.1) and uniform continuity of J^p_E on bounded subsets of E, we conclude that $\lim_{n\to\infty}\|g_n - x_n\| = 0$ and

$$\lim_{n\to\infty}\|w_n - u_n\| = \lim_{n\to\infty}\|z_n - w_n\| = \lim_{n\to\infty}\|x_n - S_n x_n\| = \lim_{n\to\infty}\|u_n - x_n\| = 0. \qquad (3.28)$$

Furthermore, from (3.24) and (3.22) we have

$$\begin{aligned}
&(1-\alpha_n)(\beta_n - \xi)(1-\beta_n)\rho_b^*\|J^p_E z_n - J^p_E S_0 z_n\|\\
&\le (1-\alpha_n)D_{f_p}(\hat{u}, w_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle\\
&\le D_{f_p}(\hat{u}, u_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle\\
&\le D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \ell_n M + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \zeta_n - \hat{u}\rangle.
\end{aligned}$$

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By the similar inferences, we infer that $\lim_{n\to\infty} \|J_E^p z_n - J_E^p S_0 z_n\| = 0$, which hence leads to $\lim_{n\to\infty} \|J_E^p v_n - J_E^p z_n\| = 0$ (due to $v_n = J_{E^*}^q(\beta_n J_E^p z_n + (1-\beta_n) J_E^p S_0 z_n)$). Using uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we get

$$\lim_{n\to\infty} \|z_n - S_0 z_n\| = \lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.29}$$

This together with (3.28) implies that

$$\|v_n - x_n\| \le \|v_n - z_n\| + \|z_n - w_n\| + \|w_n - u_n\| + \|u_n - x_n\| \to 0 \quad (n \to \infty).$$

It is clear that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.30}$$

Let us show that $z \in \bigcap_{i=0}^N \mathrm{Fix}(S_i)$. Indeed, since $\zeta_n = J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n)$, it can be readily seen that

$$\lim_{n\to\infty} \|\zeta_n - x_n\| = 0. \tag{3.31}$$

In addition, using (2.3), (3.22) and (3.23), we have

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \le D_{f_p}(\hat{u}, \zeta_n) - D_{f_p}(x_{n+1}, \zeta_n) \\
&= D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n) - D_{f_p}(x_{n+1}, \zeta_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n) D_{f_p}(\hat{u}, v_n) - D_{f_p}(x_{n+1}, \zeta_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, w_n) - D_{f_p}(x_{n+1}, \zeta_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, u_n) - D_{f_p}(x_{n+1}, \zeta_n) \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, x_n) + \ell_n M - D_{f_p}(x_{n+1}, \zeta_n),
\end{aligned}$$

which hence arrives at

$$\begin{aligned}
D_{f_p}(x_{n+1}, \zeta_n) &\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \ell_n M \\
&= \alpha_n D_{f_p}(\hat{u}, u) + \mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} + \ell_n M.
\end{aligned}$$

So it follows that $\lim_{n\to\infty} D_{f_p}(x_{n+1}, \zeta_n) = 0$ and hence $\lim_{n\to\infty} \|x_{n+1} - \zeta_n\| = 0$. This together with (3.31), leads to

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - \zeta_n\| + \|\zeta_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.32}$$

Observe that for $i = 1, ..., N$,

$$\begin{aligned}
\|x_n - S_{n+i} x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i} x_{n+i}\| + \|S_{n+i} x_{n+i} - S_{n+i} x_n\| \\
&\le \|x_n - x_{n+i}\| + \|x_{n+i} - S_{n+i} x_{n+i}\| + \sum_{j=1}^N \|S_j x_{n+i} - S_j x_n\|.
\end{aligned}$$

Exploiting the uniform continuity of each S_j on C, we deduce from (3.28) and (3.32) that $x_{n+i} - S_{n+i} x_{n+i} \to 0$ and $S_j x_{n+i} - S_j x_n \to 0$ for $i, j = 1, ..., N$. Thus, we get $\lim_{n\to\infty} \|x_n - S_{n+i} x_n\| = 0$ for $i = 1, ..., N$. This immediately implies that $\lim_{n\to\infty} \|x_n - S_r x_n\| = 0$ for $r = 1, ..., N$. So it follows from $x_{n_k} \rightharpoonup z$ that $z \in \widehat{\mathrm{Fix}}(S_r) = \mathrm{Fix}(S_r)$ for $r = 1, ..., N$. Therefore, $z \in \bigcap_{i=1}^N \mathrm{Fix}(S_i)$. In addition, from (3.30) and $x_{n_k} \rightharpoonup z$, one has that $z_{n_k} \rightharpoonup z$. Thus, using (3.29) we get $z \in \widehat{\mathrm{Fix}}(S_0) = \mathrm{Fix}(S_0)$. Consequently, $z \in \bigcap_{i=0}^N \mathrm{Fix}(S_i)$,

In what follows, we show that $z \in \bigcap_{i=1}^2 \mathrm{VI}(C, A_i)$. From (3.27), we have

$$\begin{aligned}
&(1-\alpha_n)\{\tau[\tfrac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \widetilde{y}_n)]^p\} \\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \ell_n M \\
&= \alpha_n D_{f_p}(\hat{u}, u) + \mathbf{\Gamma}_n - \mathbf{\Gamma}_{n+1} + \ell_n M.
\end{aligned}$$

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So it follows that $\lim_{n\to\infty} \frac{\tau_n}{2\lambda_1 L_1} D_{f_p}(u_n, y_n) = \lim_{n\to\infty} \frac{\tilde{\tau}_n}{2\lambda_2 L_2} D_{f_p}(w_n, \widetilde{y}_n) = 0$, and hence

$$\lim_{n\to\infty} \tau_n D_{f_p}(u_n, y_n) = \lim_{n\to\infty} \tilde{\tau}_n D_{f_p}(w_n, \widetilde{y}_n) = 0. \tag{3.33}$$

Using Lemma 3.4, we infer that

$$\lim_{n\to\infty} \|u_n - y_n\| = \lim_{n\to\infty} \|w_n - \widetilde{y}_n\| = 0. \tag{3.34}$$

Applying Lemma 3.3 and (3.34), we obtain that $z \in \bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)$. Hence we get $\omega_w(x_n) \subset \bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)$. Consequently, $\omega_w(x_n) \subset \Omega = (\bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^{N} \mathrm{Fix}(S_i))$. Lastly, we show that $\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \le 0$. We can pick a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u}\rangle = \lim_{j\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u}\rangle.$$

Because E is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_j} \rightharpoonup \tilde{z}$. So it follows from (2.2) and $\tilde{z} \in \Omega$ that

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u}\rangle = \lim_{j\to\infty} \langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u}\rangle = \langle J_E^p u - J_E^p \hat{u}, \tilde{z} - \hat{u}\rangle \le 0. \tag{3.35}$$

This together with (3.31) ensures that

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \le 0.$$

From (3.24) and (3.22), we get

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \le (1-\alpha_n) D_{f_p}(\hat{u}, w_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&\le (1-\alpha_n) D_{f_p}(\hat{u}, u_n) + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&\le (1-\alpha_n)[D_{f_p}(\hat{u}, x_n) + \epsilon_n \|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\| \\
&\quad \times \|\hat{u} + S_n x_{n-1} - 2S_n x_n\|] + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&\le (1-\alpha_n) D_{f_p}(\hat{u}, x_n) + \epsilon_n \|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\| \\
&\quad \times \|\hat{u} + S_n x_{n-1} - 2S_n x_n\| + \alpha_n \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \\
&= (1-\alpha_n) D_{f_p}(\hat{u}, x_n) + \alpha_n \{\tfrac{\epsilon_n}{\alpha_n} \|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\| \\
&\quad \times \|\hat{u} + S_n x_{n-1} - 2S_n x_n\| + \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle\}.
\end{aligned} \tag{3.36}$$

Using uniform continuity of each S_i $(1 \le i \le N)$ on C, and uniform continuity of J_E^p on bounded subsets of E, from (3.32) and the boundedness of $\{x_n\}$ we get

$$\lim_{n\to\infty} \|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\| \|\hat{u} + S_n x_{n-1} - 2S_n x_n\| = 0.$$

Noticing $\sup_{n\ge 1} \frac{\epsilon_n}{\alpha_n} < \infty$ and $\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle \le 0$, we infer that

$$\begin{aligned}
&\limsup_{n\to\infty} \{\tfrac{\epsilon_n}{\alpha_n} \|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\| \\
&\qquad \times \|\hat{u} + S_n x_{n-1} - 2S_n x_n\| + \langle J_E^p u - J_E^p \hat{u}, \zeta_n - \hat{u}\rangle\} \le 0.
\end{aligned}$$

Since $\{\alpha_n\} \subset (0,1)$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, applying Lemma 2.8 to (3.36) we obtain that $\lim_{n\to\infty} D_{f_p}(\hat{u}, x_n) = 0$ and hence $\lim_{n\to\infty} \|\hat{u} - x_n\| = 0$.

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Case 2. Suppose that $\exists\{\mathbf{\Gamma}_{n_k}\} \subset \{\mathbf{\Gamma}_n\}$ s.t. $\mathbf{\Gamma}_{n_k} < \mathbf{\Gamma}_{n_k+1}$ $\forall k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\psi : \mathcal{N} \to \mathcal{N}$ by

$$\psi(n) := \max\{k \le n : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\}.$$

By Lemma 2.7, we get

$$\mathbf{\Gamma}_{\psi(n)} \le \mathbf{\Gamma}_{\psi(n)+1} \quad \text{and} \quad \mathbf{\Gamma}_n \le \mathbf{\Gamma}_{\psi(n)+1}. \tag{3.37}$$

From (3.25) and (3.22) it follows that

$$\begin{aligned} &D_{f_p}(w_{\psi(n)}, u_{\psi(n)}) + D_{f_p}(z_{\psi(n)}, w_{\psi(n)}) + \gamma_{\psi(n)}(1-\gamma_{\psi(n)})\rho_b^* \| J_E^p x_{\psi(n)} - J_E^p g_{\psi(n)} \| \\ &\le \mathbf{\Gamma}_{\psi(n)} - \mathbf{\Gamma}_{\psi(n)+1} + \ell_{\psi(n)} M + \alpha_{\psi(n)} \langle J_E^p u - J_E^p \hat{u}, \zeta_{\psi(n)} - \hat{u} \rangle. \end{aligned}$$

Noticing $g_{\psi(n)} = J_{E^*}^q((1-\epsilon_{\psi(n)})J_E^p S_{\psi(n)} x_{\psi(n)} + \epsilon_{\psi(n)} J_E^p(2S_{\psi(n)}x_{\psi(n)} - S_{\psi(n)}x_{\psi(n)-1}))$ and $u_{\psi(n)} = J_{E^*}^q(\gamma_{\psi(n)} J_E^p x_{\psi(n)} + (1-\gamma_{\psi(n)}) J_E^p g_{\psi(n)}))$, we obtain that $\lim_{n\to\infty} \|g_{\psi(n)} - x_{\psi(n)}\| = 0$ and

$$\lim_{n\to\infty} \|w_{\psi(n)} - u_{\psi(n)}\| = \lim_{n\to\infty} \|z_{\psi(n)} - w_{\psi(n)}\| = \lim_{n\to\infty} \|x_{\psi(n)} - S_{\psi(n)} x_{\psi(n)}\| = \lim_{n\to\infty} \|u_{\psi(n)} - x_{\psi(n)}\| = 0. \tag{3.38}$$

Also, from (3.24) and (3.22) we have

$$\begin{aligned} &(1-\alpha_{\psi(n)})(\beta_{\psi(n)} - \xi)(1-\beta_{\psi(n)})\rho_b^* \| J_E^p z_{\psi(n)} - J_E^p S_0 z_{\psi(n)} \| \\ &\le \mathbf{\Gamma}_{\psi(n)} - \mathbf{\Gamma}_{\psi(n)+1} + \ell_{\psi(n)} M + \alpha_{\psi(n)} \langle J_E^p u - J_E^p \hat{u}, \zeta_{\psi(n)} - \hat{u} \rangle. \end{aligned}$$

Noticing $v_{\psi(n)} = J_{E^*}^q(\beta_{\psi(n)} J_E^p z_{\psi(n)} + (1-\beta_{\psi(n)}) J_E^p S_0 z_{\psi(n)})$ and using the similar inferences to those in Case 1, we get

$$\lim_{n\to\infty} \|z_{\psi(n)} - S_0 z_{\psi(n)}\| = \lim_{n\to\infty} \|v_{\psi(n)} - z_{\psi(n)}\| = 0.$$

This together with (3.38) implies that

$$\lim_{n\to\infty} \|v_{\psi(n)} - x_{\psi(n)}\| = \lim_{n\to\infty} \|z_{\psi(n)} - x_{\psi(n)}\| = 0. \tag{3.39}$$

Noticing $\zeta_{\psi(n)} = J_{E^*}^q(\alpha_{\psi(n)} J_E^p u + (1-\alpha_{\psi(n)}) J_E^p v_{\psi(n)})$, from (3.39) we get

$$\lim_{n\to\infty} \|\zeta_{\psi(n)} - x_{\psi(n)}\| = 0. \tag{3.40}$$

Using the similar inferences to those in Case 1, we conclude that $\lim_{n\to\infty} \|x_{\psi(n)+1} - x_{\psi(n)}\| = 0$,

$$\lim_{n\to\infty} \|u_{\psi(n)} - y_{\psi(n)}\| = \lim_{n\to\infty} \|w_{\psi(n)} - \tilde{y}_{\psi(n)}\| = 0, \tag{3.41}$$

and

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \hat{u}, \zeta_{\psi(n)} - \hat{u} \rangle \le 0. \tag{3.42}$$

Using (3.36), we get

$$\begin{aligned} D_{f_p}(\hat{u}, x_{\psi(n)+1}) &\le (1-\alpha_{\psi(n)}) D_{f_p}(\hat{u}, x_{\psi(n)}) \\ &+ \alpha_{\psi(n)} \{ \tfrac{\epsilon_{\psi(n)}}{\alpha_{\psi(n)}} \| J_E^p S_{\psi(n)} x_{\psi(n)} - J_E^p(2S_{\psi(n)} x_{\psi(n)} - S_{\psi(n)} x_{\psi(n)-1}) \| \\ &\times \| \hat{u} + S_{\psi(n)} x_{\psi(n)-1} - 2S_{\psi(n)} x_{\psi(n)} \| + \langle J_E^p u - J_E^p \hat{u}, \zeta_{\psi(n)} - \hat{u} \rangle \}, \end{aligned} \tag{3.43}$$

which together with (3.37), hence yields

$$\begin{aligned} \mathbf{\Gamma}_{\psi(n)} \le &\tfrac{\epsilon_{\psi(n)}}{\alpha_{\psi(n)}} \| J_E^p S_{\psi(n)} x_{\psi(n)} - J_E^p(2S_{\psi(n)} x_{\psi(n)} - S_{\psi(n)} x_{\psi(n)-1}) \| \\ &\times \| \hat{u} + S_{\psi(n)} x_{\psi(n)-1} - 2S_{\psi(n)} x_{\psi(n)} \| + \langle J_E^p u - J_E^p \hat{u}, \zeta_{\psi(n)} - \hat{u} \rangle. \end{aligned}$$

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As a result, from (3.42) we deduce that

$$\lim_{n\to\infty} \mathbf{\Gamma}_{\psi(n)} = 0. \tag{3.44}$$

From (3.42), (3.43) and (3.44), one has that

$$\lim_{n\to\infty} \mathbf{\Gamma}_{\psi(n)+1} = 0. \tag{3.45}$$

Again from (3.37), we have $\lim_{n\to\infty} D_{f_p}(\hat{u}, x_n) = \lim_{n\to\infty} \mathbf{\Gamma}_n = 0$. Hence $\lim_{n\to\infty} \|x_n - \hat{u}\| = 0$. This completes the proof. □

Remark 3.1. It can be easily seen from the proof of Theorem 3.2 that if the assumption that $\lim_{n\to\infty} \frac{\ell_n}{\alpha_n} = 0$, is used in place of the one that $\lim_{n\to\infty} \ell_n = 0$ and $\sup_{n\geq 1} \frac{\epsilon_n}{\alpha_n} < \infty$, then Theorem 3.2 is still valid.

Setting $A_2 = 0$ in Algorithm 3.1, we immediately obtain the following algorithm for finding an element of $\Omega = \mathrm{VI}(C, A_1) \cap (\bigcap_{i=0}^N \mathrm{Fix}(S_i))$.

Algorithm 3.3. Initialization: Given $x_0, x_1 \in C$ arbitrarily and let $\epsilon > 0$, $\mu_1 > 0$, $\lambda_1 \in (0, \frac{1}{\mu_1})$, $l_1 \in (0,1)$. Choose $\{\ell_n\}, \{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (\xi, 1)$ s.t. $\sum_{n=1}^\infty \ell_n < \infty$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\liminf_{n\to\infty}(\alpha_n - \xi)(1-\alpha_n) > 0$. Moreover, given the iterates x_{n-1} and x_n $(n \geq 1)$, choose ϵ_n s.t. $0 \leq \epsilon_n \leq \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|J_E^p S_n x_n - J_E^p(2S_n x_n - S_n x_{n-1})\|}\} & \text{if } S_n x_n \neq S_n x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterative steps: Calculate x_{n+1} as follows:

Step 1. Set $g_n = J_{E^*}^q((1-\epsilon_n)J_E^p S_n x_n + \epsilon_n J_E^p(2S_n x_n - S_n x_{n-1}))$, and calculate $u_n = J_{E^*}^q(\beta_n J_E^p x_n + (1-\beta_n)J_E^p g_n)$, $y_n = \Pi_C(J_{E^*}^q(J_E^p u_n - \lambda_1 A_1 u_n))$, $r_{\lambda_1}(u_n) := u_n - y_n$ and $s_n = u_n - \tau_n r_{\lambda_1}(u_n)$, where $\tau_n := l_1^{k_n}$ and k_n is the smallest nonnegative integer k satisfying

$$\langle A_1 u_n - A_1(u_n - l_1^k r_{\lambda_1}(u_n)), u_n - y_n\rangle \leq \frac{\mu_1}{2} D_{f_p}(u_n, y_n).$$

Step 2. Calculate $w_n = \Pi_{C_n}(u_n)$, with $C_n := \{x \in C : h_n(x) \leq 0\}$ and

$$h_n(x) = \langle A_1 s_n, x - u_n\rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(u_n, y_n).$$

Step 3. Calculate $v_n = J_{E^*}^q(\alpha_n J_E^p w_n + (1-\alpha_n)J_E^p(S_0 w_n))$ and $x_{n+1} = \Pi_{Q_n}(w_n)$, where $Q_n := \{x \in C : D_{f_p}(x, v_n) \leq D_{f_p}(x, w_n)\}$.

Again set $n := n+1$ and go to Step 1.

Corollary 3.1. Suppose that the conditions (C1)-(C3) with $A_2 = 0$, hold, and $\Omega = \mathrm{VI}(C, A_1) \cap (\bigcap_{i=0}^N \mathrm{Fix}(S_i)) \neq \emptyset$. Let $\{x_n\}$ be the sequence constructed in Algorithm 3.3. Then $x_n \rightharpoonup z \in \Omega \Leftrightarrow \sup_{n\geq 0} \|x_n\| < \infty$.

Next, let $S_1 : E \to C$ be a Bregman relatively nonexpansive mapping and $S_i = S = I$ the identity mapping of E for $i = 2, ..., N$. Then we get $\Omega = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^N \mathrm{Fix}(S_i)) = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap \mathrm{Fix}(S_1)$. In this case, Algorithm 3.2 reduces to the following iterative scheme for solving a pair of VIPs and the FPP of S_1. By Theorem 3.2 we obtain the following strong convergence result.

Corollary 3.2. Suppose that the condition (C3) holds, and let $\Omega = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap \mathrm{Fix}(S_1) \neq \emptyset$. For initial $x_0, x_1 \in C$, choose ϵ_n s.t. $0 \leq \epsilon_n \leq \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|J_E^p S_1 x_n - J_E^p(2S_1 x_n - S_1 x_{n-1})\|}\} & \text{if } S_1 x_n \neq S_1 x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

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Suppose that $\{x_n\}$ is the sequence constructed by

$$\begin{cases} g_n = J^q_{E^*}((1-\epsilon_n)J^p_E S_1 x_n + \epsilon_n J^p_E(2S_1x_n - S_1x_{n-1})), \\ u_n = J^q_{E^*}(\gamma_n J^p_E x_n + (1-\gamma_n)J^p_E g_n), \\ y_n = \Pi_C(J^q_{E^*}(J^p_E u_n - \lambda_1 A_1 u_n)), \\ s_n = (1-\tau_n)u_n + \tau_n y_n, \\ w_n = \Pi_{C_n} u_n, \\ \widetilde{y}_n = \Pi_C(J^q_{E^*}(J^p_E w_n - \lambda_2 A_2 w_n)), \\ t_n = (1-\widetilde{\tau}_n)w_n + \widetilde{\tau}_n \widetilde{y}_n, \\ z_n = \Pi_{\widetilde{C}_n} w_n, \\ x_{n+1} = \Pi_C(J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n)J^p_E z_n) \quad \forall n \ge 1, \end{cases}$$

where $\tau_n := l_1^{k_n}$, $\widetilde{\tau}_n := l_2^{j_n}$ and k_n, j_n are the smallest nonnegative integers k and j satisfying, respectively,

$$\langle A_1 u_n - A_1(u_n - l_1^k(u_n - y_n)), u_n - y_n\rangle \le \frac{\mu_1}{2} D_{f_p}(u_n, y_n),$$

$$\langle A_2 w_n - A_2(w_n - l_2^j(w_n - \widetilde{y}_n)), w_n - \widetilde{y}_n\rangle \le \frac{\mu_2}{2} D_{f_p}(w_n, \widetilde{y}_n),$$

and the sets $C_n, \widetilde{C}_n$, are constructed below

(i) $C_n := \{x \in C : h_n(x) \le 0\}$ and $h_n(x) = \langle A_1 s_n, x - u_n\rangle + \frac{\tau_n}{2\lambda_1} D_{f_p}(u_n, y_n)$;

(ii) $\widetilde{C}_n := \{x \in C : \widetilde{h}_n(x) \le 0\}$ and $\widetilde{h}_n(x) = \langle A_2 t_n, x - w_n\rangle + \frac{\widetilde{\tau}_n}{2\lambda_2} D_{f_p}(w_n, \widetilde{y}_n)$.

Then, $x_n \to \Pi_\Omega u \Leftrightarrow \sup_{n\ge 0}\|x_n\| < \infty$.

4. Examples

In this section, we provide an illustrated example to demonstrate the feasibility and implementability of our proposed approaches. Put $\epsilon = \frac{1}{3}$, $\mu_i = 1$ and $l_i = \lambda_i = \frac{1}{3}$ for $i = 1,2$. We first provide an example of uniformly continuous and pseudomonotone mappings $A_i : E \to E^*, i = 1,2$, Bregman relatively nonexpansive mapping $S_1 : C \to C$ and Bregman relatively demicontractive mapping $S_0 : C \to C$ with $\Omega = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^1 \mathrm{Fix}(S_i)) \ne \emptyset$. Let $C = [-2,2]$ and $E = H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. The initial points x_0, x_1 are randomly chosen in C. For $i = 1,2$, let $A_i : H \to H$ be defined as $A_1 x := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$ and $A_2 x := x + \sin x$ for all $x \in H$. Now, we first show that A_1 is Lipschitz continuous and pseudomonotone. Indeed, for all $x, y \in H$ we have

$$\begin{aligned} \|A_1 x - A_1 y\| &= |\tfrac{1}{1+\|\sin x\|} - \tfrac{1}{1+\|x\|} - \tfrac{1}{1+\|\sin y\|} + \tfrac{1}{1+\|y\|}| \\ &\le |\tfrac{\|y\|-\|x\|}{(1+\|x\|)(1+\|y\|)}| + |\tfrac{\|\sin y\| - \|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}| \\ &\le \|x-y\| + \|\sin x - \sin y\| \le 2\|x-y\|. \end{aligned}$$

This implies that A_1 is Lipschitz continuous. Also, we show that A_1 is pseudomonotone. For each $x, y \in H$, it is easy to see that

$$\begin{aligned} &\langle A_1 x, y - x\rangle = (\tfrac{1}{1+|\sin x|} - \tfrac{1}{1+|x|})(y-x) \ge 0 \\ &\Rightarrow \langle A_1 y, y - x\rangle = (\tfrac{1}{1+|\sin y|} - \tfrac{1}{1+|y|})(y-x) \ge 0. \end{aligned}$$

It is readily known that A_2 is Lipschitz continuous and monotone. Indeed, we deduce that $\|A_2 x - A_2 y\| \le \|x-y\| + \|\sin x - \sin y\| \le 2\|x-y\|$ and

$$\langle A_2 x - A_2 y, x - y\rangle = \|x-y\|^2 + \langle \sin x - \sin y, x - y\rangle \ge \|x-y\|^2 - \|x-y\|^2 = 0.$$

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Now, let $S_1 : C \to C$ and $S_0 : C \to C$ be defined as $S_1x = \sin x$ and $S_0x = \frac{1}{5}x + \frac{3}{5}\sin x$. It is easy to verify that $\mathrm{Fix}(S_1) = \mathrm{Fix}(S_0) = \{0\}$ and $S_1 : C \to C$ is Bregman relatively nonexpansive. Also, $S_0 : C \to C$ is Bregman relatively ξ-demicontractive with $\xi = \frac{1}{5}$. Indeed, note that

$$\|S_0x - S_0y\|^2 = \|\frac{1}{5}(x-y) + \frac{3}{5}(\sin x - \sin y)\|^2 \leq \|x-y\|^2 + \frac{2}{5}\|(I-S_0)x - (I-S_0)y\|^2.$$

Consequently,

$$\Omega = (\bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^{1} \mathrm{Fix}(S_i)) = \{0\} \neq \emptyset.$$

In addition, putting $\beta_n = \frac{n+2}{2(n+1)}$ $\forall n \geq 1$, we obtain

$$\lim_{n\to\infty}(\beta_n - \xi)(1-\beta_n) = \lim_{n\to\infty}(\frac{n+2}{2(n+1)} - \frac{1}{5})(1 - \frac{n+2}{2(n+1)}) = (\frac{1}{2} - \frac{1}{5})(1 - \frac{1}{2}) = \frac{3}{20} > 0$$

In this case, the conditions (C1)-(C3) are satisfied.

Example 4.1. Let $\ell_n = \frac{1}{2(n+1)^2}$ and $\alpha_n = \beta_n = \frac{n+2}{2(n+1)}$ $\forall n \geq 1$. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose ϵ_n s.t. $0 \leq \epsilon_n \leq \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|S_1x_n - S_1x_{n-1}\|}\} & \text{if } S_1x_n \neq S_1x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Algorithm 3.1 is rewritten as follows:

$$\begin{cases} g_n = S_1x_n + \epsilon_n(S_1x_n - S_1x_{n-1}), \\ u_n = \frac{n+2}{2(n+1)}x_n + \frac{n}{2(n+1)}g_n, \\ y_n = P_C(u_n - \frac{1}{3}A_1u_n), \\ s_n = (1-\tau_n)u_n + \tau_n y_n, \\ w_n = P_{C_n}u_n, \\ \widetilde{y}_n = P_C(w_n - \frac{1}{3}A_2w_n), \\ t_n = (1-\tilde{\tau}_n)w_n + \tilde{\tau}_n\widetilde{y}_n, \\ v_n = \frac{n+2}{2(n+1)}w_n + \frac{n}{2(n+1)}S_0w_n, \\ Q_n = \{x \in C : \|x - v_n\| \leq \|x - w_n\|\}, \\ x_{n+1} = P_{\widetilde{C}_n \cap Q_n}w_n \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where for each $n \geq 1$, the sets $C_n, \widetilde{C}_n$ and the step-sizes $\tau_n, \tilde{\tau}_n$ are chosen as in Algorithm 3.1. Then, by Theorem 3.1, we deduce that $\{x_n\}$ converges to $0 \in \Omega = (\bigcap_{i=1}^{2} \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^{1} \mathrm{Fix}(S_i))$.

Example 4.2. Let $\ell_n = \frac{1}{2(n+1)^2}$, $\alpha_n = \frac{1}{2(n+1)}$ and $\beta_n = \gamma_n = \frac{n+2}{2(n+1)}$ $\forall n \geq 1$. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose ϵ_n s.t. $0 \leq \epsilon_n \leq \overline{\epsilon_n}$, where

$$\overline{\epsilon_n} = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|S_1x_n - S_1x_{n-1}\|}\} & \text{if } S_1x_n \neq S_1x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Algorithm 3.2 is rewritten as follows:

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$$\begin{cases} g_n = S_1x_n + \epsilon_n(S_1x_n - S_1x_{n-1}), \\ u_n = \frac{n+2}{2(n+1)}x_n + \frac{n}{2(n+1)}g_n, \\ y_n = P_C(u_n - \frac{1}{3}A_1u_n), \\ s_n = (1-\tau_n)u_n + \tau_n y_n, \\ w_n = P_{C_n}u_n, \\ \widetilde{y}_n = P_C(w_n - \frac{1}{3}A_2w_n), \\ t_n = (1-\tilde{\tau}_n)w_n + \tilde{\tau}_n\widetilde{y}_n, \\ z_n = P_{\widetilde{C}_n}w_n, \\ v_n = \frac{n+2}{2(n+1)}z_n + \frac{n}{2(n+1)}S_0z_n, \\ x_{n+1} = P_C(\frac{1}{2(n+1)}u + \frac{2n+1}{2(n+1)}v_n) \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where for each $n \geq 1$, the sets $C_n, \widetilde{C}_n$ and the step-sizes $\tau_n, \tilde{\tau}_n$ are chosen as in Algorithm 3.2. Then, by Theorem 3.2, we deduce that $\{x_n\}$ converges to $0 \in \Omega = (\bigcap_{i=1}^2 \mathrm{VI}(C, A_i)) \cap (\bigcap_{i=0}^1 \mathrm{Fix}(S_i))$.

5. Conclusions

Let $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let E be a p-uniformly convex and uniformly smooth Banach space. Then its dual space E^* is q-uniformly smooth Banach space with $1 < q \leq 2$. Utilizing the geometric properties of E and E^*, we design two inertial-type subgradient extragradient algorithms with line-search process for solving the pseudomonotone variational inequality problems (VIPs) and common fixed-point problem (CFPP), where the geometric properties involve the properties of the generalized duality mappings $J_E^p, J_{E^*}^q$ and Bregman projection operator Π_C. Here the CFPP indicates the common fixed-point problem of finite Bregman relatively nonexpansive mapping and a Bregman relatively demicontractive mapping in E. Under the properties of the generalized duality mappings $J_E^p, J_{E^*}^q$ and Bregman projection operator Π_C, we prove weak and strong convergence of the suggested algorithms to a common solution of the VIPs and CFPP, respectively. Additionally, an illustrated example is furnished to demonstrate the feasibility and implementability of our proposed approaches. In the end, it is noteworthy that part of our future research is aimed at attaining the weak and strong convergence results for the modifications of our proposed approaches with Nesterov double inertial-type extrapolation steps (see [34]) and adaptive stepsizes.

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