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Article

Continuous Mapping of Covering Approximate Space and Topology Induced by Arbitrary Covering Relation

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Abstract: In the study of rough sets, there are many covering approximation spaces, how to classify covering approximation spaces has become a hot issue. In this paper, we propose concepts covering approximation T_1-space, $F-$symmtry, covering rough continuous mapping, covering rough homeomorphism mapping to solve this question. We also propose a new method for constructing topology in Theorem 5.1, and get the following properties: (1) For each $x \in U$, $\{X_i : i \in I\} \subseteq \mathcal{P}(U)$ is all the subsets of U which contains x and $*$ is a reflexive relation on U. If $V \in \tau$ is a subset of U and $x \in V$, then $\underline{*}(\bigcap_{i\in I} X_i)$ is the smallest subset of U and $x \in \underline{*}(\bigcap_{i\in I} X_i) \subseteq V$. Denoted by $C(x) = \bigcap\{\underline{*}(X_i) : x \in \underline{*}(X_i), i \in I\}$. (2)If $V \in \tau$ is a subset of U, then $V = \bigcup_{x\in V} C(x)$. (3) Let $\{\underline{*}(X_i) : x \notin \underline{*}(X_i), i \in I\}$, then $\overline{\{x\}} = U \setminus \bigcup_{x\notin\underline{*}(X_i), i\in I} \underline{*}(X_i)$; (4) Let $*$ be a reflexive relation on U. For every $X \subseteq U$, we have $int(\underline{*}(X)) = \underline{*}(X)$. Where $int(\underline{*}(X))$ represents the interior of $\underline{*}(X)$. (5) Let $\{\underline{*}(X_i) : x \in \underline{*}(X_i), i \in I\}$ be a family subsets of U, then $\{\underline{*}(X_i) : x \in \underline{*}(X_i), i \in I\}$ is a base for (U, τ) at the point x.

Keywords: covering approximation space; covering rough continuous mapping; relation; topology

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1. Introduction

With the development of science and technology, the task of data classification becomes more and more important in many fields, such as medical diagnosis, decision making, data mining among others. Data exists in various spaces, information processing must rely on the relevant properties of space. How to get the properties of space has become an urgent problem. To address this issue, fuzzy set theory and granular computing are proposed separately by L. A. Zadeh in 1965 and 1996 [1,2]. They have been used in various fields, which include machine learning, approximation reasoning and knowledge discovering. Although they can be used to deal with uncertainties and incompleteness, but using it requires adding human factors which lead inaccurate results. Classical rough set theory is a suitable tool for dealing with incomplete and imprecise information systems, and also an effective tool for dealing with uncertain knowledge. Therefore, it has been successfully applied to many fields [3–7]. lower and upper approximation operators are defined by classical rough sets based on equivalence relation [8–12], which exhibit many limitations because equivalence relation is too strong. In order to remove this restriction, Zakowski used the covering relation replace the equivalence relation and defined the concept of covering rough set [13]. Many scholars have done a lot of work in this fields, for example, Zhang Y.l. et al. studied the invariance of covering rough sets by using compatible mappings [14], and Li J.J. studied covering generalized approximation spaces by using topological methods [15,16]. Wang P. et al. studied the necessary and sufficient conditions for the covering

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upper approximation operator to become a topological closure operator and investigated membership functions [17,18]. Zhang W.X. et al. study the rough set of general relations, and so on [19–22]. There are many ways to generate approximation operators from the above, due to the large coverage space, it is unrealistic to study its properties one by one. whether we can classify the covering approximation spaces and study the nature of the class is a question worth thinking about.

Topological structure is one of the most important structures of Mathematics, which provide mathematic tool for dealing with information systems and rough sets [22]. It is a very effective method to use topological method to study covering approximate space. For example, Yang L. Y, et al. defined topology by lower approximation operator. Z. Zhao proposed topology induced by the covering[24]. Yu H, et al. obtain topology by lower approximation operator and upper approximation operator[25] and so on.

Continuous mapping plays a key role in general topology and other fields. So we can investigate the relationship between two topological spaces can by continuous mapping, such as homeomorphism, Separation, connectedness and so on. Covering approximation space is an important part of generalized approximation space. In order to study the properties of covering approximation space. Many scholars have studied the properties of covering approximation spaces by means of various relations, The important topological concept of continuity is not used. We define covering rough continuous mapping, covering rough homeomorphism mapping, obtain covering approximation T_1-space and provide a classification method. Beside this, we also propose a method to construct topology, with the help of this method, we can give a unified situation about using covering to construct topology and relation to construct topology. Compared with the topology we constructed, their topology becomes a special case. In other words, we generalize the method of constructing topology.

The organization of the paper is as follows. In Section I, the background is introduced. In Section II, a review of several preliminary concepts in rough approximation space are briefly recalled. In Section III, we propose some new concepts covering rough continuous mapping $F-$symmetry, covering rough homeomorphism mapping and obtain many interesting results. We also establish classification for covering approximation spaces by these definitions. In Section IV, through analyzing the properties of covering spaces by covering rough continuous mapping and covering rough homeomorphism mapping, we define separation property of covering approximation spaces and obtain T_1 covering approximation space and also distinguish covering approximation spaces by T_1 separation, and then establish another classification method for covering approximation spaces. In Section V, we successfully apply rough set to construct new topological space, obtain a new topology induced by $*$. We also compared it with other topologies, get this topology is finer than others' and research the properties of with it. Section VI concludes the paper.

2. Preliminaries

Let U be a non-empty set and $*$ a arbitrary relation on U. Then $(U, *)$ is called an approximation space. For any $X \subseteq U$, the lower and upper approximation of X are defined as follows:

$$\underline{R}(X) = \{x \in U : *(x) \subseteq X\}$$
$$\overline{R}(X) = \{x \in U : *(x) \cap X \neq \emptyset\}$$

If $*$ is an equivalence relation on U, then the $*(x) = [x]_*$; if $*$ is a binary relation on U, then the $*(x) = *_s(x)$; if $*$ is a covering relation on U, then the $*(x) = \{C \in \mathcal{C} : x \in C\}$. we have:

$$\underline{D}(X) = \cup\{R_s(x) : R_s(x) \subseteq X\}$$
$$\overline{D}(X) = \cup\{R_s(x) : R_s(x) \cap X \neq \emptyset\}$$

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A binary relation R on U is said to be reflexive if for any $x \in U$, then $(x, x) \in R$; R is said to be symmetric, if for any $x, y \in U$, then $(x, y) \in R$ implies $(y, x) \in R$; if $(x, y) \in R$ and $(x, z) \in R$ implies$(y, z) \in R$ for any $x, y, z \in U$, then R is called a Euclidean relation.

Definition 2.1 (**Covering**[13]). Let U be an universe of discourse, $\mathcal{C}$ is a family of subsets of$U$, and none subsets in $\mathcal{C}$ is empty. If $\cup\mathcal{C} = U$, then $\mathcal{C}$ is called a covering of U.

Definition 2.2 (**Covering approximation space**[13]). Let U be an universe of discourse and $\mathcal{C}$ a covering of U. then we call U together with covering $\mathcal{C}$ a covering approximation space. denoted by $(U, \mathcal{C})$.

Definition 2.3. Let $(U, \mathcal{C})$ be a covering approximation space. For any $X \subseteq U$, the operators are defined as follows:

$$\underline{apr}(X) = \{y \in U : N(y) \subseteq X, N(y) = \cap C_i, \forall C_i \in \mathcal{C}, y \in C_i\}$$
$$\overline{apr}(X) = \{y \in U : N(y) \cap X \neq \emptyset, N(y) = \cap C_i, \forall C_i \in \mathcal{C}, y \in C_i\}$$

Definition 2.4. [23] A topological space is a pair $(X, \mathcal{O})$ consisting of a set X and a family $\mathcal{O}$ of subsets of X satisfying the following conditions:
(O_1) $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$;
(O_2) If $U_1 \in \mathcal{O}$ and $U_2 \in \mathcal{O}$, then $U_1 \cap U_2 \in \mathcal{O}$;
(O_3) $\mathcal{A} \subseteq \mathcal{O}$, then $\cup\mathcal{A} \in \mathcal{O}$.

The set X is called a space, the elements of X are called points of the space, and the subsets of X belonging to $\mathcal{O}$ are called open in the space. The family $\mathcal{O}$ of open subsets of X is also called a topology on X

Definition 2.5. [23] Let $(X, \mathcal{O})$ be a topological space; a set $F \subseteq X$ is called closed in the space if its complement $X \backslash F$ is an open set. If for some $x \in X$ and open set $U \subseteq X$ we have $x \in U$, we say that U is a neighborhood of x.

Definition 2.6. [23] A family $\mathcal{B} \subseteq \mathcal{O}$ is called a base for a topological space $(X, \mathcal{O})$ if $\mathcal{B}$ has the following properties:
(B_1) For any $U_1, U_2 \in \mathcal{B}$ and every point $x \in U_1 \cap U_2$, there exists a $U \in \mathcal{B}$ such that $x \in U \subseteq x \in U_1 \cap U_2$;
(B_2) For every $x \in X$, there exists a $U \in \mathcal{B}$ such that $x \in U$.

Definition 2.7. [23] A family $\mathcal{B}(x)$ of neighbourhoods of x is called a base for a topological space $(X, \mathcal{O})$ at the point x, if for any neighbourhood V of x, there exists a $U \in \mathcal{B}(x)$ such that $x \in U \subseteq V$. The collection $\{\mathcal{B}(x) : x \in X\}$ is called a neighbourhood system for the topological space $(X, \mathcal{O})$.

Definition 2.8. [23] An operator H: $P(U) \to P(U)$ is called a topological closure operator on U if it satisfied the following conditions: for any $X, Y \subseteq U$,
$(C_1) H(X \cup Y) = H(X) \cup H(Y)$;

(C$_2$)$X \subseteq H(X)$;
(C$_3$)$H(\varnothing) = \varnothing$;
(C$_4$)$H(H(X)) = H(X)$.

Lemma 2.1. Let X and Y be two sets, and $f : X \to Y$ a mapping, then:
(1)For any $A \subseteq X$, we have $A \subseteq f^{-1}(f(A))$;
(2)For any $B \subseteq Y$, we have $f(f^{-1}(B)) \subseteq B$, if f is surjection, then there is $f(f^{-1}(B)) = B$;
(3)For any $A \subseteq X$ and $B \subseteq Y$, there is $f(A) \subseteq B$ if and only if $A \subseteq f^{-1}(B)$.

3. Covering rough continuous mapping and covering rough homeomorphism mapping

It is well known that covering approximation space is the most important generalized approximation space. Many scholars have defined many covering approximation operators. It's very important to distinguish these operators. In this section, we propose the covering rough continuous mapping and the covering rough homeomorphism mapping by using the upper operator $\overline{apr}$ of covering approximation space, and obtain some properties of the covering rough continuous mapping and the covering rough homeomorphism mapping.

Lemma 3.1. Let $(U, \mathcal{C})$ be a covering approximation space. For any $X \subseteq U$ and $X_i \subseteq U(i \in I)$, we have:
(1) $\overline{apr}(X^c) = (\underline{apr}(X))^c$;
(2) $\underline{apr}(\bigcap_{i\in I} X_i) = \bigcap_{i\in I} \underline{apr}(X_i)$;
(3) $\overline{apr}(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} \overline{apr}(X_i)$;
(4) $\overline{apr}(X) = \bigcup_{x\in X} \overline{apr}(\{x\})$, where X^c represents the complement of X in U.

proof (1)For any $y \in \overline{apr}(X^c)$, by Definition 2.3, we have $N(y) \bigcap (U \backslash X) \neq \varnothing$, then $N(y) \nsubseteq X$. Hence $y \notin \underline{apr}(X)$ by Definition 2.3, it follows that $y \in (\underline{apr}(X))^c$, which implies $\overline{apr}(X^c) \subseteq (\overline{apr}(X))^c$.

We only need to prove the converse. For any $x \in (\underline{apr}(X))^c$, we have $x \notin \underline{apr}(X)$. By Definition 2.3, we can obtain $N(x) \nsubseteq X$, and claim that $N(x) \bigcap (U \backslash X) \neq \varnothing$. If $N(x) \bigcap (U \backslash X) = \varnothing$, then $N(x) \subseteq X$. It is contradiction to $N(x) \nsubseteq X$. In other words, $N(x) \bigcap X^c \neq \varnothing$. From Definition 2.3, we have $x \in \overline{apr}(X^c)$. It follows that $\overline{apr}(X^c) = (\underline{apr}(X))^c$.

(2) For any $i \in I$, since $\bigcap_{i\in I} X_i \subseteq X_i$, then $\underline{apr}(\bigcap_{i\in I} X_i) \subseteq \underline{apr}(X_i)$, thus $\underline{apr}(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} \underline{apr}(X_i)$, we shall to prove the converse.

For any $x \in \bigcap_{i\in I} \underline{apr}(X_i)$, we have $x \in \underline{apr}(X_i)$ for any $i \in I$. According to Definition 2.3, we can obtain $N(x) \subseteq X_i$. By the arbitrariness of i, we have $N(x) \subseteq \bigcap_{i\in I} X_i$. Thus we have $x \in \underline{apr}(\bigcap_{i\in I} X_i)$. From the above, we know that $\underline{apr}(\bigcap_{i\in I} X_i) = \bigcap_{i\in I} \underline{apr}(X_i)$.

(3)The proof is similar to (2).

(4)It is easy to prove $\bigcup_{x\in X} \overline{apr}(\{x\}) \subseteq \overline{apr}(X)$ by (2), we only need to prove $\overline{apr}(X) \subseteq \bigcup_{x\in X} \overline{apr}(\{x\})$. For any $y \in \overline{apr}(X)$, according to Definition 2.3, we obtain $N(y) \bigcap X \neq \varnothing$. Thus there exists $x_0 \in N(y) \bigcap X$ such that $N(y) \bigcap \{x_0\} \neq \varnothing$. Therefore, $y \in \overline{apr}(\{x_0\})$. By the arbitrariness of y, we can obtain $\overline{apr}(X) \subseteq_{x\in X} \overline{apr}(\{x\})$. From the above, it follows that $\overline{apr}(X) = \bigcup_{x\in X} \overline{apr}(\{x\})$.

Definition 3.1. Let $(U_1, \mathcal{C}_1)$ and $(U_2, \mathcal{C}_2)$ be two covering approximation spaces and $f : U_1 \to U_2$ a mapping. If $f(\overline{apr}(X)) \subseteq \overline{apr}(f(X))$ for any $X \subseteq U_1$, then we call f is covering rough continuous mapping from $(U_1, \mathcal{C}_1)$ to $(U_2, \mathcal{C}_2)$; If f is a bijective, f and f^{-1} are covering rough continuous mappings, then f is called covering rough homeomorphism mapping from $(U_1, \mathcal{C}_1)$ to $(U_2, \mathcal{C}_2)$. Where $f(X) = \{f(x) : x \in X\}$.

From Definition 3.1, we can obtain properties of covering rough continuous mapping as following:

Proposition 3.1. Let $(U_1, \mathcal{C}_1)$ and $(U_2, \mathcal{C}_2)$ be two covering approximation spaces and $f : U_1 \to U_2$ is a mapping. Then the followings are equivalent:

(1)f is a covering rough continuous mapping from $(U_1, \mathcal{C}_1)$ to $(U_2, \mathcal{C}_2)$;
(2)For any $Y \subseteq U_2$, $\overline{apr}(f^{-1}(Y)) \subseteq f^{-1}(\overline{apr}(Y))$;
(3)For any $Y \subseteq U_2$, $f^{-1}(\underline{apr}(Y)) \subseteq \underline{apr}(f^{-1}(Y))$;
(4)For any $x \in U_1$, $f(\overline{apr}\{x\}) \subseteq \overline{apr}(f(x))$.

Proof $(1) \Rightarrow (2)$. For any $Y \subseteq U_2$, by Definition 3.1 and Lemma 3.1 (2), it is not difficult to prove $f(\overline{apr}(f^{-1}(Y))) \subseteq \overline{apr}(f(f^{-1}(Y))) \subseteq \overline{apr}(X)$. It follows that $\overline{apr}(f^{-1}(Y)) \subseteq f^{-1}(\overline{apr}(Y))$ according to Lemma 3.1 (3).

$(2) \Rightarrow (1)$. For any $X \subseteq U_1$, we have $f^{-1}(\overline{apr}(f(X))) \supseteq \overline{apr}(f^{-1}(f(X))) \supseteq \overline{apr}(X)$ by (2). So $f(\overline{apr}(X)) \subseteq \overline{apr}(f(X))$ by Lemma 3.1 (3). Hence f is a covering rough continuous mapping from (U_1, C_1) to (U_2, C_2).

$(2) \Rightarrow (3)$. For any $Y \subseteq U_2$, $f^{-1}(\overline{apr}(Y^c)) \supseteq \overline{apr}(f^{-1})(Y^c))$ by (2). From Lemma 3.1 (1), we have $f^{-1}(\overline{apr}(Y^c)) = f^{-1}((\underline{apr}(Y))^c) = (f^{-1}(\underline{apr}(Y)))^c$, which implies that $f^{-1}(\underline{apr}(Y)) \subseteq \underline{apr}(f^{-1}(Y))$ for any $Y \subseteq U_2$.

$(3) \Rightarrow (2)$. It is easy to prove, so we omit the proof.

$(1) \Rightarrow (4)$. It is obvious by Definition 3.1.

$(4) \Rightarrow (1)$. For any $X \subseteq U_1$, $X = \cup_{x \in X}\{x\}$. We have $f(\overline{apr}\{x\}) \subseteq \overline{apr}(f(x))$ by (4). From Lemma 3.1 (4) and f preserves the union operations, thus $f(\overline{apr}(X)) = f(\overline{apr}(U_{x \in X}\{x\})) = U_{x \in X} f(\overline{apr}(\{x\})) \subseteq U_{x \in X}\overline{apr}(f(x)) = \overline{apr}(f(x))$. therefore, $(1) \Leftrightarrow (4)$.

Theorem 3.1. *Let $(U_1, \mathcal{C}_1)$, $(U_2, \mathcal{C}_2)$ and $(U_3, \mathcal{C}_3)$ be covering approximation spaces. If $f : (U_1, \mathcal{C}_1) \to (U_2, \mathcal{C}_2)$ and $g : (U_2, \mathcal{C}_2) \to (U_3, \mathcal{C}_3)$ are both covering rough continuous mappings, then $g \circ f : (U_1, C_1) \to (U_3, C_3)$ is also a covering rough continuous mapping.*

Proof For any $X \subseteq U_1$, f is a covering rough continuous mapping, according to Definition 3.1, we have $f(\overline{apr}(X)) \subseteq \overline{apr}(f(X))$. Since $f(X) \subseteq U_2$ and g is a covering rough continuous mapping. It is easy to obtain $g(\overline{apr}(f(X))) \subseteq \overline{apr}(g(f(X)))$. Therefore, $g(f(\overline{apr}(X))) \subseteq \overline{apr}(g(f(X)))$, thus $g \circ f(\overline{apr}(X)) \subseteq \overline{apr}(g \circ f(X))$. From Definition 3.1, we have $g \circ f$ is a covering rough continuous mapping.

By Definition 3.1, Proposition 3.1 and Theorem 3.1, it is easy to obtain theorems of covering rough homeomorphism mapping:

Theorem 3.2. *Let $(U_1, \mathcal{C}_1)$ and $(U_2, \mathcal{C}_2)$ be two covering approximation spaces and f a bijective mapping. Then the following cases are equivalent:*

(1)If f is a covering rough homeomorphism mapping from $(U_1, \mathcal{C}_1)$ to $(U_2, \mathcal{C}_2)$, then f^{-1} is a covering rough homeomorphism mapping from $(U_2, \mathcal{C}_2)$ to $(U_1, \mathcal{C}_1)$;
(2) For any $X \subseteq U_1$, then $f(\overline{apr}(X)) = \overline{apr}(f(X))$;
(3) For any $Y \subseteq U_2$, then $f^{-1}(\underline{apr}(Y)) = \underline{apr}(f^{-1}(Y))$;
(4) For any $Y \subseteq U_2$, then $f^{-1}(\overline{apr}(X)) = \overline{apr}(f^{-1}(Y))$.

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Theorem 3.3. *Let $(U_1,\mathcal{C}_1)$,$(U_2,\mathcal{C}_2)$ and $(U_3,\mathcal{C}_3)$ be covering approximation spaces, we have:*

(1) Identity mapping $i:(U_1,\mathcal{C}_1)\to(U_1,\mathcal{C}_1)$ is a covering rough homeomorphism mapping.

(2)If $f:(U_1,\mathcal{C}_1)\to(U_2,\mathcal{C}_2)$ is a covering rough homeomorphic mapping, then $f^{-1}:(U_2,\mathcal{C}_2)\to(U_1,\mathcal{C}_1)$ is also a covering rough homeomorphism mapping.

(3)If $f:(U_1,\mathcal{C}_1)\to(U_2,\mathcal{C}_2)$ and $g:(U_2,\mathcal{C}_2)\to(U_3,\mathcal{C}_3)$ are covering rough homeomorphism mappings, then $g\circ f:(U_1,\mathcal{C}_1)\to(U_3,\mathcal{C}_3)$ is also a covering rough homeomorphism mapping.

4. Separation properties of covering approximation Spaces

Separation properties play an important role in many spaces. In this section, we will define T_1-space and discuss its properties. In the segment, we provide a new method for classification of covering approximation spaces.

Definition 4.1. Let $(U,\mathcal{C})$ be a covering approximation space and for any $x\in U$, if $\overline{apr}(\{x\})=\{x\}$, then covering approximation space $(U,\mathcal{C})$ is called a T_1-space.

Example 4.1. Let U be an arbitrary non-empty universe, and $\mathcal{C}=\{\{x\}:x\in U\}$. By Definition 2.3 and Definition 4.1, it is easy to check that $(U,\mathcal{C})$ is a covering approximation space and also a T_1-space.

Example 4.2. Let U be an arbitrary non-empty universe, taking $\mathcal{C}=\{U\}$. By Definition 4.1, We can easily obtain that $(U,\mathcal{C})$ is a covering approximation space but it's not a T_1-space.

Let U be an arbitrary non-empty universe and $\mathcal{C}$ a covering of U. Then $\mathcal{C}$ can induce a topology τ. Since the τ is related with $\mathcal{C}$, we use $\tau_{\mathcal{C}}$ instead of τ. We call the topological space $(U,\tau_{\mathcal{C}})$ is induced by covering approximation space $(U,\mathcal{C})$.

Lemma 4.1. Topological space $(U,\tau_{\mathcal{C}})$ is a topological T_1-space if and only if every single point set in $(U,\tau_{\mathcal{C}})$ is closed set.

Theorem 4.1. *Let $(U,\mathcal{C})$ be a covering approximation space and $(U,\tau_{\mathcal{C}})$ a topological space induced by $(U,\mathcal{C})$. Then covering approximation space $(U,\mathcal{C})$ is a T_1-space if and only if $(U,\tau_{\mathcal{C}})$ is a topological T_1-space.*

Proof ($\Rightarrow$) Let covering approximation space $(U,\mathcal{C})$ be a T_1-space, by Definition 4.1, we claim that $\{x\}=N(x)$ for any $x\in U$. Otherwise, there exists $a\in N(x)$ such that $x\neq a$, from Definition 2.3, we have $x\in\overline{apr}(\{x\})$. Since $\overline{apr}(\{x\})$ has more than two elements, it contradicts that $(U,\mathcal{C})$ is a T_1-space.

We shall prove that $\overline{\{x\}}$ is a closed set. $\overline{\{x\}}=\bigcap\{C:C\in\mathcal{C}\}=N(x)$=$\{x\}$. By Lemma 4.1, we obtain that $(U,\tau_{\mathcal{C}})$ is a topological T_1-space.

($\Leftarrow$) Let $(U,\tau_{\mathcal{C}})$ be a topological T_1-space. Then for any $x\in U$, by Definition 2.3, $\overline{apr}(\{x\})=\{y:N(y)\bigcap\{x\}\neq\emptyset\}=\{y:x\in N(y)\}=\{y:x\in\{y\}\}=\{x\}$, Therefore, covering approximation space $(U,\mathcal{C})$ is a T_1-space.

Theorem 4.2. *Let $(U_1,\mathcal{C}_1)$,$(U_2,\mathcal{C}_2)$ be two covering approximation spaces and f a covering rough homeomorphism mapping from covering approximation space $(U_1,\mathcal{C}_1)$ to covering approximation space $(U_2,\mathcal{C}_2)$, then the covering approximation space $(U_1,\mathcal{C}_1)$ is a T_1-space if and only if the covering approximation space $(U_2,\mathcal{C}_2)$ is a T_1-space.*

Proof ($\Rightarrow$) Let covering approximation space $(U_1, \mathcal{C}_1)$ be T_1-space and f a mapping from the covering approximation space $(U_1, \mathcal{C}_1)$ to the covering approximate space $(U_2, \mathcal{C}_2)$, then for any $y \in U_2$, there exists a unique $x \in U_1$ such that $x = f^{-1}(y)$. Since the covering approximation space $(U_1, \mathcal{C}_1)$ is a T_1-space, then $\overline{apr}(\{x\}) = \{x\} = f^{-1}(y)$, therefore, $f(\overline{apr}(\{x\})) = \{y\}$, by Theorem 3.2, we obtain $\overline{apr}(f(\{x\})) = \overline{apr}(\{y\}) = \{y\}$, thus $(U_2, \mathcal{C}_2)$ is a T_1-space.

($\Leftarrow$) We can use the similar method to prove the converse, therefore we omit it here.

Topological operator plays an important role in general topology and rough sets, which provides a method for exchange information systems[22]. We discuss the necessary and sufficient conditions for $\overline{D}$ to be a topological closure operator.

Theorem 4.3. *Let U be a universe, R is a preorder and Euclidean relation on U, then $\overline{D}$ is a topological closure operator on U.*

Proof Let R be a binary on U, R is a preorder and Euclidean relation. We shall prove $\overline{D}$ is a topological closure operator on U. Since R is a preorder, then For any $X \subseteq U$, we can obtain $\underline{D}(X) \subseteq X$, $X \subseteq \overline{D}(X)$ and $\overline{D}(\overline{D}(X)) \subseteq \overline{D}(X)$, by Definition 13, $\overline{D}(X)$ satisfies (C_2) and (C_4). We only need to prove $\overline{D}$ satisfies (C_1) and (C_3). By the definition $\overline{D}$, we can obtain $\overline{D}(\emptyset) = \emptyset$. We shall prove (C_1). For any $X, Y \subseteq U$, we know that $\overline{D}(X) \subseteq \overline{D}(X \cup Y)$ and $\overline{D}(Y) \subseteq \overline{D}(X \cup Y)$ by $X \subseteq X \cup Y$, thus $\overline{D}(X) \cup \overline{D}(Y) \subseteq \overline{D}(X \cup Y)$. We prove the converse.

For any $y \in \overline{D}(X \cup Y)$, by the definition $\overline{D}$, there exists $x_0 \in U$ such that $y \in R_s(x_0)$ and $R_s(x_0) \cap (X \cup Y) \neq \emptyset$. Therefore $(R_s(x_0) \cap X) \cup (R_s(x_0) \cap Y) \neq \emptyset$, thus $R_s(x_0) \cap X \neq \emptyset$ or $R_s(x_0) \cap Y \neq \emptyset$. By the definition $\overline{D}$, we have $y \in \overline{D}(X)$ or $y \in \overline{D}(Y)$.

In Theorem 4.3, the converse may not true.

Example 4.3.

Let $U = \{a, b\}$, $R = \{(a, a), (a, b), (b, b)\}$. By the definition $\overline{D}$, We have $R_s(\{a\}) = \{a, b\}, R_s(\{b\}) = \{b\}$:

(1)$\overline{D}(\emptyset) = \emptyset$;

(2)$\overline{D}(\overline{D}(\emptyset)) = \emptyset = \overline{D}(\emptyset)$;
$\overline{D}(\overline{D}(\{a\})) = \{a, b\} = \overline{D}(\{a\})$;
$\overline{D}(\overline{D}(\{b\})) = \{a, b\} = \overline{D}(\{b\})$;
$\overline{D}(\overline{D}(\{a, b\})) = \{a, b\} = \overline{D}(\{a, b\})$;

(3)$\emptyset \subseteq \overline{D}(\emptyset)$;
$\{a\} \subseteq \overline{D}(\{a\})$,
$\{b\} \subseteq \overline{D}(\{b\})$;
$\{a, b\} \subseteq \overline{D}(\{a, b\})$;

(4) $\overline{D}(\emptyset) \cup \overline{D}(\emptyset) = \emptyset = \overline{D}(\emptyset \cup \emptyset)$;
$\overline{D}(\emptyset) \cup \overline{D}(\{a\}) = \{a, b\} = \overline{D}(\emptyset \cup \{a\})$;
$\overline{D}(\emptyset) \cup \overline{D}(\{b\}) = \{a, b\} = \overline{D}(\emptyset \cup \{b\})$;
$\overline{D}(\emptyset) \cup \overline{D}(\{a, b\}) = \{a, b\} = \overline{D}(\emptyset \cup \{a, b\})$;
$\overline{D}(\{a\}) \cup \overline{D}(\{b\}) = \{a, b\} = \overline{D}(\{a\} \cup \{b\})$,
$\overline{D}(\{a\}) \cup \overline{D}(\{a, b\}) = \{a, b\} = \overline{D}(\{a\} \cup \{a, b\})$,
$\overline{D}(\{b\}) \cup \overline{D}(\{a, b\}) = \{a, b\} = \overline{D}(\{b\} \cup \{a, b\})$,
$\overline{D}(\{a, b\}) \cup \overline{D}(\{a, b\}) = \{a, b\} = \overline{D}(\{a, b\} \cup \{a, b\})$,
$b \in R_s(\{a\})$ and $a \in R_s(\{a\})$, but $a \notin R_s(\{b\})$.

from the above, we know that $\overline{D}$ is a topological closure operator on U, but R is not a Euclidean relation.

It is natural to enguive about the necessary and sufficient conditions for it to be a topological closure operator?

Definition 4.2. [26] Let $* = R$ be an arbitrary binary relation on U. The smallest transitive relation on U containing the relation R is called the transitive closure of R. They denote the transitive closure of R by $t(R)$.

Yu H. et al. obtained if R is a reflexive relation, then $t(R)$ is also a reflexive relation in [26]. D, Pei et al. show that R is a binary relation on U, then $t(R) = R \cup R^2 \cup ...R^n...$ in [27]. It is natural to ask what other properties does it have? We will discuss this question.

Definition 4.3. Let U be a non-empty universe and R a binary relation on U. R is called $F-$smmety if for any $x, y \in U$, we have $(x, y) \in R$ and $(y, x) \in R$.

Theorem 4.4. *Let U be a universe, R a $F-$symmtry relation on U and $n \geq 2$, then $t(R)$ is a reflexive and $F-$smmety relation on U.*

Proof Since $R \subseteq t(R)$, we have $t(R)$ is a $F-$symmetry relation. We only need to prove $t(R)$ is a reflexive relation. For any $x, y \in U$, we have $(x, y) \in R$ and $(y, x) \in R$. Thus $(x, x) \in R^2$ and $(y, y) \in R^2$ by $n \geq 2$ and composition of R^2.

5. Topologies induced by relation $*$

Zhao defined topology by coverings in [24] and only used in covering approximation space. L. Yang et al. defined open set by $\underline{R}(X) = A^0$ and construct topology. We shall use relation $*$ and a subset of $\mathcal{P}(U)$ to construct topology. If $*$ is a covering relation, the topology induced by $*$ is finer than zhao's; if $*$ is a relation R, the topology induced by R is finer than Yang's. In other words, we propose a new method of constructing topology by $*$ and make the topology they constructed become our special case. That is to say, we generalize their methods of constructing topology.

Lemma 5.1. [23] Let X be a topological space. For every $A \subseteq X$ the following conditions are equivalent:

(1) The point x belong to $\overline{A}$;

(2)For every neighborhood U of x we have $U \cap A \neq \emptyset$.

Lemma 5.2. [24] Let $(U, \mathcal{C})$ be a covering approximation space. The topology $\mathcal{T}$ induced by the covering $\mathcal{C}$ is defined as follows: A subset G of U is said to be open in U if for each $x \in G$, there are finite elements $C_1, C_2, ..., C_n$ of $\mathcal{C}$ such that $x \in \bigcap_{i=1}^{n} \subseteq G$.

Theorem 5.1. *Let U be an arbitrary non-empty universe and $*$ a reflexive relation on U. We construct topology τ by the $*$ as follows: $V \subseteq U$ is called open set in U if for any $x \in V$, there exists a finite family $\{X_i : i \in I\} \subseteq \mathcal{P}(U)$ such that $x \in \underline{*}(\bigcap\limits_{i \in I} X_i) \subseteq V$.*

Proof We prove that τ satisfies Definition 2.4.

(1) It is obvious $\emptyset$ and U in τ by the definition condition of open set. So they satisfy (O_1) of Definition 2.4.

(2) For any $V_1, V_2 \in \tau$, we show that $V_1 \bigcap V_2 \in \tau$. For any $x \in V_1 \bigcap V_2$, there exist finite family $\{X_{i_k} : k \in I\}$ and $\{X_{j_m} : m \in L\}$ such that $x \in \underline{*}(\bigcap\limits_{k \in I} X_{i_k}) \subseteq V_1$ and $x \in \underline{*}(\bigcap\limits_{m \in L} X_{i_m}) \subseteq V_2$.

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Thus $x \in (\underline{*}(\bigcap\limits_{k\in I} X_{i_k}))\bigcap(\underline{*}(\bigcap\limits_{m\in L} X_{i_m})) = \underline{*}((\bigcap\limits_{m\in L} X_{i_m})\bigcap(\bigcap\limits_{k\in I} X_{i_k})) \subseteq V_1 \bigcap V_2$. This prove that any finite intersections of elements of τ are still in τ. It satisfies (O_2) of Definition 2.4.

(3) Let $\{V_\alpha : \alpha \in J\}$ be a family of elements of τ. We need to prove that $V = \bigcup\limits_{\alpha\in G} V_\alpha \in \tau$. For any $x \in V$, it must exist indexed $\alpha_0 \in J$ such that $x \in V_{\alpha_0}$. Since V_{α_0} is open and $*$ a reflexive relation on U, we can find a finite family $\{X_i : i \in I\} \subseteq \mathcal{P}(U)$ such that $x \in \underline{*}(\bigcap\limits_{i\in I} X_i) \subseteq V_{\alpha_0} \subseteq V$, so V is open, which satisfies (O_3) of Definition 2.4. Therefore the desired result is proved.

Example 5.1. Let $U = \{a, b, c\}$ be an arbitrary non-empty universe, taking $\mathcal{C} = \{\{a, c\}, \{b, c\}\}$, $\mathcal{A} = \{\{b\}, \{a, b\}\} \subseteq \mathcal{P}(U)$ and $X = \{b\} \subseteq U$. By Definition 2.2, we have $(U, \mathcal{C})$ is a covering approximation space, but X is not an open set by Lemma 5.2. We also know that $\mathcal{A} = \{\{b\}, \{a, b\}\}$ is not a covering of U, but we can obtain $b \in \underline{*}(\bigcap \mathcal{A}) \subseteq X$, thus X is an open set in U by Theorem 5.1.

From the example 5.1 and Lemma 5.2, we obtain a topology which is $\{\emptyset, U, \{b, c\}\}$. This topology no longer contains more elements. From this angle, we can say that the method of defining topology in Lemma 5.2 is limited by covering. We can construct the finer topology which is $\mathcal{P}(U)$ by the example 5.1 and Theorem 5.1. Our method solves the limitation of covering.

Remark 5.1. (1) For any $X \subseteq U$, if $x \in X$ and $*$ is a reflexive relation on U, We have $\underline{*}(X) \in \tau$ by Theorem 5.1.

(2) If $*$ is a reflexive relation and $\underline{*}(X) = X^0$, we have $\underline{*}(\bigcap\limits_{i\in I} X_i) \subseteq \underline{*}(X) = X^0$ by Theorem 5.1 and $X^0 \in \tau$. therefore, our topology τ induced by $*$ is finer than Yang's. We generalize Yang's results.

(3) If $*$ is a reflexive relation and $\{\underline{*}(X) : X \subseteq U\} = \mathcal{C}$ a covering of U, then $\mathcal{T} = \tau$ by Lemma 5.2 and Theorem 5.1, we can conclude that our topology is much finer than Zhao's because $\underline{*}(\bigcap\limits_{i\in I} X_i) \subseteq \underline{*}(X) \subseteq X$. We also generalize his definition, their definitions become a special case of our definition.

Theorem 5.2. *Let U be an arbitrary non-empty universe, $*$ a reflexive relation on U and τ a topology induced by $*$. Then we have the following properties:*

(1) For each $x \in U$, $\{X_i : i \in I\} \subseteq \mathcal{P}(U)$ is all the subsets of U which contains x and $*$ is a reflexive relation on U. If $V \in \tau$ is a subset of U and $x \in V$, then $\underline{*}(\bigcap\limits_{i\in I} X_i)$ is the smallest subset of U and $x \in \underline{*}(\bigcap\limits_{i\in I} X_i) \subseteq V$. Denoted by $C(x) = \bigcap\{\underline{*}(X_i) : x \in \underline{*}(X_i), i \in I\}$.

(2)If $V \in \tau$ is a subset of U, then $V = \bigcup\limits_{x\in V} C(x)$.

(3) Let $\{\underline{*}(X_i) : x \notin \underline{*}(X_i), i \in I\}$, then $\overline{\{x\}} = U \setminus \bigcup\limits_{x\notin \underline{*}(X_i), i\in I} \underline{*}(X_i)$;

(4) Let $*$ be a reflexive relation on U. For every $X \subseteq U$, we have $int(\underline{*}(X)) = \underline{*}(X)$. Where $int(\underline{*}(X))$ represents the interior of $\underline{*}(X)$.

(5) Let $\{\underline{*}(X_i) : x \in \underline{*}(X_i), i \in I\}$ be a family subsets of U, then $\{\underline{*}(X_i) : x \in \underline{*}(X_i), i \in I\}$ is a base for (U, τ) at the point x.

Proof (1) Let $V \in \tau$ be a subset of U and $x \in V$. There is a finite subset family $\{X_k : k \in K\} \subseteq \mathcal{P}(U)$ such that $x \in \underline{*}(\bigcap\limits_{i\in I} X_i) \subseteq V$ by Theorem 5.1. $\{X_i : i \in I\} \subseteq \mathcal{P}(U)$ is all the subsets of U which contains x, then $C(x)$ is the smallest subset of U containing x and $C(x) \subseteq \underline{*}(\bigcap\limits_{k\in K} X_k)$, thus $x \in C(x) \subseteq \underline{*}(\bigcap\limits_{k\in K} X_k) \subseteq V$.

(2)Let $V \in \tau$ be a subset of U and $x \in V$, we can obtain $C(x) \subseteq V$ by (1). Since $\{C(x) : x \in V\}$ is a cover of V, then $V \subseteq \bigcup\limits_{x\in G} C(x)$, thus we have $V = \bigcup\limits_{x\in V} C(x)$.

(3)Pick $M(x) = \bigcup\limits_{x \notin \underline{*}(X_i), i \in I} \underline{*}(X_i)$. $\underline{*}(X_i)$ is an open subset of U for very $i \in I$ by Remark 5.1 (1), the union $\bigcup\limits_{x \notin \underline{*}(X_i), i \in I} \underline{*}(X_i)$ is open by (O_3). It follows that the set $U \setminus \bigcup\limits_{x \notin \underline{*}(X_i), i \in I} \underline{*}(X_i)$ is closed and $x \in (U \setminus \bigcup\limits_{x \notin \underline{*}(X_i), i \in I} \underline{*}(X_i))$. It is easily seen that $\overline{\{x\}}$ is the smallest closed set containing x. Obviously, $\overline{\{x\}} \subseteq U \setminus \bigcup\limits_{x \notin \underline{*}(X_i), i \in I} \underline{*}(X_i)$. We shall prove the converse.

For any $y \in U \setminus \overline{\{x\}}$. Since $U \setminus \overline{\{x\}}$ is an open set containing y, then $C(y) \subseteq U \setminus \overline{\{x\}}$ by (1), therefore $C(y) \cap \overline{\{x\}} = \emptyset$ and $x \notin C(y)$. There exists a $\underline{*}(X_{i_0})$ of $\{\underline{*}(X_i) : x \notin \underline{*}(X_i), i \in I\}$ such that $y \in \underline{*}(X_{i_0}) \wedge x \notin \underline{*}(X_{i_0})$. Thus $y \in M(x)$ and $U \setminus \overline{\{x\}} \subseteq M(x)$, therefore $U \setminus M(x) \subseteq \overline{\{x\}}$.

(4) Let $*$ be a reflexive relation on U, for every $X \subseteq U$, from Remark 5.1(1) it follows that $\underline{*}(X)$ is an open set. then we obtain $\underline{*}(X) = int(\underline{*}(X))$ directly by definition of open set.

(5)Let $\mathcal{B}(x)$=$\{\underline{*}(X_i) : x \in \underline{*}(X_i), i \in I\}$. For any $V_1, V_2 \in \mathcal{B}(x)$ and every $x \in V_1 \cap V_2$, there exist finite collections $\{X_{i_k} : k \in I\}$ and $\{X_{j_m} : m \in L\}$ such that $x \in \underline{*}(\bigcap\limits_{k \in I} X_{i_k}) \subseteq V_1$ and $x \in \underline{*}(\bigcap\limits_{m \in L} X_{i_m}) \subseteq V_2$. Thus $x \in (\underline{*}(\bigcap\limits_{k \in I} X_{i_k})) \bigcap (\underline{*}(\bigcap\limits_{m \in L} X_{i_m})) = \underline{*}((\bigcap\limits_{m \in L} X_{i_m}) \bigcap (\bigcap\limits_{k \in I} X_{i_k})) \subseteq V_1 \bigcap V_2$. Pick $X_0 = (\bigcap\limits_{m \in L} X_{i_m}) \bigcap (\bigcap\limits_{k \in I} X_{i_k})$, then $\underline{*}(X_0) \in \mathcal{B}(x)$ which satisfies (B_1) of Definition 2.6. $\mathcal{B}(x)$ is obvious that it satisfies (B_2) of Definition 2.6. We have $\{\underline{*}(X_i) : x \in \underline{*}(X_i), i \in I\}$ is a base for (U, τ) at the point x.

Theorem 5.3. *Let U be an arbitrary non-empty universe, $*$ a reflexive relation on U and τ a topology induced by $*$, then we have the following properties:*

(1) For any $A \subseteq U$ and $x \in U$, we have $x \in \overline{A}$ if and only if $C(x) \cap A \neq \emptyset$ if and only if there exists a base $\mathcal{B}(x)$ at x such that for every $V \in \mathcal{B}(x)$, $V \cap A \neq \emptyset$.

(2) For any $x, y \in U$ and $x \neq y$, then $C(x) = C(y)$ if and only if $\overline{\{x\}} = \overline{\{y\}}$.

(3) Let $\mathcal{P}_x^{\{\underline{*}(X_i):i \in I\}}$=$\{y \in U : \forall i \in I, \underline{*}(X_i) \in \{\underline{*}(X_i) : i \in I\}(x \in \underline{*}(X_i) \longleftrightarrow y \in \underline{*}(X_i))\}$. For any $x \in U$, then $\mathcal{P}_x^{\{\underline{*}(X_i):i \in I\}}$=$N(x) \cap \overline{\{x\}}$.

(4) Let F be a closed subset of U, then $F = \bigcup\limits_{x \in F} \overline{\{x\}}$= $\bigcup\limits_{x \in F} \mathcal{P}_x^{\{\underline{*}(X_i):i \in I\}}$.

Proof (1) For any $x \in \overline{A}$, from Lemma 5.1, $x \in \overline{A}$ if and only if $C(x) \cap A \neq \emptyset$ can be proved in a similar way.

If $C(x) \cap A \neq \emptyset$, then $V \cap A \neq \emptyset$ is obvious. It remains to show that if there exists a base $\mathcal{B}(x)$ at x such that for every $V \in \mathcal{B}(x)$, $V \cap A \neq \emptyset$, then $x \in \overline{A}$. Suppose that $x \in \overline{A}$ dose not hold, i.e., $x \notin \overline{A}$. There exists a closed set F such that $x \notin F$. For the open set $V = U \backslash F$ we have $x \in V$ and $V \cap A = \emptyset$. For every base $\mathcal{B}(x)$ at x, there exists a $V' \in \mathcal{B}(x)$ such that $x \in V' \subseteq V$ and from $V \cap A = \emptyset$, it follows that $V' \cap A = \emptyset$,i,e, for every $V \in \mathcal{B}(x)$, $V \cap A \neq \emptyset$ dose not hold.

(2)If $C(x) = C(y)$, we have $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$ from (1). It is well known that $\overline{\{x\}}$ is the smallest closed set containing x, thus $\overline{\{x\}} \subseteq \overline{\{y\}}$. Similarly, $\overline{\{y\}} \subseteq \overline{\{x\}}$, therefore, we have $\overline{\{x\}} = \overline{\{y\}}$. If $\overline{\{x\}} = \overline{\{y\}}$, we shall prove $C(x) = C(y)$. Since $x \in \overline{\{y\}}$, we have $\{y\} \cap C(x) \neq \emptyset$, thus $y \in C(x)$, therefore $C(y) \subseteq C(x)$. Similarly, we have $C(x) \subseteq C(y)$, thus $C(x) = C(y)$.

(3) It is not difficult to prove that $\mathcal{P}_x^{\{\underline{*}(X_i):i \in I\}}$=$\{y \in U : \forall i \in I, \underline{*}(X_i) \in \{\underline{*}(X_i) : i \in I\}(x \in \underline{*}(X_i) \longleftrightarrow y \in \underline{*}(X_i))\} = \{y \in U : C(x) = C(y)\}$. Suppose y is an element of $\mathcal{P}_x^{\{\underline{*}(X_i):i \in I\}}$, then $x \in C(y)$, thus $y \in \overline{\{x\}}$, and then $y \in C(x) \cap \overline{\{x\}}$, therefore $\mathcal{P}_x^{\{\underline{*}(X_i):i \in I\}} \subseteq C(x) \cap \overline{\{x\}}$. We shall prove the converse. For any $y \in C(x) \cap \overline{\{x\}}$. We have $y \in C(x)$ and $x \in C(y)$ by (1); therefore, $C(x) = C(y)$, thus we have $y \in \mathcal{P}_x^{\{\underline{*}(X_i):i \in I\}}$.

(4) For any $x \in F$, we have $\overline{\{x\}} \subseteq F$, and then $\bigcup\limits_{x\in F} \overline{\{x\}} \subseteq F$. $\{\overline{\{x\}} : x \in F\}$ is an closed covering of F, thus we have $F \subseteq \bigcup\limits_{x\in F} \overline{\{x\}}$, therefore, $F = \bigcup\limits_{x\in F} \overline{\{x\}}$. $F \subseteq \bigcup\limits_{x\in F} \mathcal{P}_x^{\{\underline{*}(X_i):i\in I\}} \subseteq \bigcup\limits_{x\in F} \overline{\{x\}} = F$ by (3).

Zhao defined the upper approximation operator $\mathrm{COM}^+(X)$ in [24] by $U/\sim$ as follows: $\mathrm{COM}^+(X) = \bigcup\{Y_i \in U/\sim: Y_i \cap X \neq \emptyset\}$. The following property gives its characterizations.

Proposition 5.1. $\mathrm{COM}^+(X) = \bigcup\{Y_i \in U/\sim: Y_i \cap X \neq \emptyset\}$ is a topological closure operator on U.

Proof $\mathrm{COM}^+(\emptyset) = \bigcup\{Y_i \in U/\sim: Y_i \cap \emptyset \neq \emptyset\} = \emptyset$ satisfies (C_3) by Definition 1.6; for any $X \subseteq U$, we claim that $X \subseteq \mathrm{COM}^+(X) = \bigcup\{Y_i \in U/\sim: Y_i \cap X \neq \emptyset\}$. $\forall x \in X$, since $U/\sim$ is a partition of U, there exists $Y_{i_0} \in U/\sim$ such that $x \in Y_{i_0}$, thus $Y_{i_0} \cap \{x\} \neq \emptyset$, so we have $x \in COM^+(\{x\})$, therefore, $X \subseteq COM^+(X)$ satisfies (C_2) by Definition 1.6. We shall prove COM^+ satisfies (C_1). For any $X, Y \subseteq U$, $\mathrm{COM}^+(X) \cup \mathrm{COM}^+(Y) \subseteq COM^+(X \cup Y)$ is obvious. We only to prove the converse.

For any $x \in COM^+(X \cup Y)$, there exists a $Y_{i_0} \in U/\sim$ such that $x \in Y_{i_0}$ and $Y_{i_0} \cap (X \cup Y) \neq \emptyset$. Thus $(Y_{i_0} \cap X) \cup (Y_{i_0} \cap Y) \neq \emptyset$, and then $(Y_{i_0} \cap X) \neq \emptyset$ and $(Y_{i_0} \cap Y) \neq \emptyset$ is satisfied at least one. Therefore, $x \in COM^+(X)$ or $x \in COM^+(Y)$. We have $COM^+(X \cup Y) \subseteq \mathrm{COM}^+(X) \cup \mathrm{COM}^+(Y)$ which satisfies (C_1).

Finally, we prove COM^+ satisfies (C_4). For any $X \subseteq U$, $COM^+(X) \subseteq COM^+(COM^+(X))$ is obvious. We need to prove the converse. For any $x \in COM^+(COM^+(X))$, there exists $Y_{i_0} \in U/\sim$ such that $x \in Y_{i_0}$ and $Y_{i_0} \cap COM^+(X) \neq \emptyset$. Then there exists a x_0 such that $x_0 \in Y_{i_0} \cap COM^+(X)$, therefore, there exists $Y_{i_1} \in U/\sim$ such that $x_0 \in Y_{i_1}$ and $Y_{i_1} \cap X \neq \emptyset$. Since $U/\sim$ is a partition of U and $x_0 \in Y_{i_0}$, $x_0 \in Y_{i_1}$, then $Y_{i_0} = Y_{i_1}$. Thus $x \in Y_{i_1}$ and $Y_{i_1} \cap X \neq \emptyset$, and then $x \in COM^+(X)$. We can obtain $COM^+(COM^+(X)) \subseteq COM^+(X)$.

6. Conclusions

In this paper, we have proposed new concepts and investigated relationship among covering approximation spaces. We have obtained some interesting properties of covering spaces by covering rough continuous mapping and covering rough homeomorphism mapping. We can provide a method for classification of covering approximation spaces. In future work, it is worth investigating the properties of the other approximation spaces and obtain the method for general approximation spaces.

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