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Finding an Extension of the Expected Value That Is Unique, Finite, and Natural for All Functions in Prevalent Subset of the Set of All Functions

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Article

Finding an Extension of the Expected Value That Is Unique, Finite, and "Natural" for All Functions in Prevalent Subset of the Set of All Functions

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Abstract: Suppose for $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$. If set A, using the Hausdorff outer measure, is measurable in the sense of Carathèodory; we want to find an extension of the expected value, w.r.t the Hausdorff measure, that's unique, finite and "natural" (defined on §2.3 & §2.4) for all f in a prevalent subset of $\mathbb{R}^A$. The issue is current extensions of the expected value are finite for all functions in *only* a shy subset of $\mathbb{R}^A$. The reason this issue wasn't resolved is mathematicians have not thought of the problem, focusing on application rather than generalization. Despite the lack of potential use, we'll attempt to solve the problem by defining a choice function—this shall choose a unique set of *equivalent* sequences of sets $(F_k^{\star\star\star})_{k\in\mathbb{N}}$, where the set-theoretic limit of $F_k^{\star\star\star}$ is the graph of f; the measure H^h is the h-Hausdorff measure, such for each $k \in \mathbb{N}$, $0 < H^h(F_k^{\star\star\star}) < +\infty$; and $(f_k^{\star})_{k\in\mathbb{N}}$ is a sequence of functions where $\{(x, f_k^{\star}(x)) : x \in \text{dom}(F_k^{\star\star\star})\} = F_k^{\star\star\star}$. Thus, the extended expected value of f *or* $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ is: $\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(k \in \mathbb{N})\left(k \geq N \Rightarrow \left|\frac{1}{H^h\left(\text{dom}(F_k^{\star\star\star})\right)}\int_{\text{dom}(F_k^{\star\star\star})} f_k^{\star}\, dH^h - \mathbb{E}^{**}[f, F_k^{\star\star\star}]\right| < \epsilon\right)$ which should be unique, finite, and "natural" (defined on §2.3 & §2.4) for all f in a prevalent subset of $\mathbb{R}^A$. Note we guessed the choice function using computer programming but we don't use mathematical proofs due to the lack of expertise in the subject matter. Despite this, the biggest use of this research is the extension of the expected value is finite for "almost all" functions: this is easier to use in application when finding the "average" of functions covering an infinite expanse of space.

Keywords: Expected Value; Hausdorff measure; (Exact) dimension function; Function Space; Prevalent and Shy sets; Entropy; Choice Function

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0. Introduction

According to an article in Quanta Magazine [1] Wood writes, "No known mathematical procedure can meaningfully average an infinite number of objects covering an infinite expanse of space in general. The path integral is more of a physics philosophy than an exact mathematical recipe." The cited paper [2] presents a constructive approach to Wood's statement using filters over families of finite set; however, the average in the approach is not unique: the method determines the average value of functions with a range that lies in any algebraic structure for which the finite averages make sense. In this paper, we will explore a more constructive approach where the average unique, finite, and "natural" (defined in §2.3 & §2.4) for a prevalent subset [3] of the set of all functions.

We begin with describing "the infinite objects" which cover "an infinite expanse of space" as unbounded functions, since the definition is more approachable from a mathematical standpoint. Moreover, for $n \in \mathbb{N}$, set $A \subseteq \mathbb{R}^n$ and function $f : A \to \mathbb{R}$; suppose we get a *prevalent* subset of a function-space means "almost all" functions are in that space, and a *shy* subset of a function-space means "almost no" functions are in that space. Using the Hausdorff outer measure (for A measurable in the Carathèodory sense); we then get the set of unbounded f where the expected value is infinite or undefined, forms a prevalent subset of $\mathbb{R}^A$. Furthermore, the set of all f with finite expected values forms *only* a shy subset of $\mathbb{R}^A$, meaning "almost no" functions have finite expected values.

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Therefore, after we define prevalent and shy sets with mathematics in §1.1; we define two attempts to answer the thesis [1] of the first paragraph in §1.2. Note neither attempts give complete answers: they extend the Hausdorff measure of A to be positive and finite but do not guarantee unbounded functions will have finite expected values. Infact, the expected value from both attempts are positive and finite for *only* a shy subset of $\mathbb{R}^A$.

Hence, we define a sequence of sets called $\star$-*sequence of sets* (def. 4) whose properties allow for finite expected values for all f in a prevalent subset $\mathbb{R}^A$. Note these $\star$-sequences of sets converge to the graph of f i.e. $\{(x, f(x)) : x \in A\}$ rather than A; otherwise, the *generalized expected value* of f w.r.t to their own $\star$-sequence (def. 5) cannot be finite for all f in a prevalent subset of $\mathbb{R}^A$. Moreover, since there are functions where there are multiple $\star$-sequences of sets which we may choose, with the generalized expected values of f w.r.t each $\star$-sequence different and non-unique—we must have a choice function choosing a unique set of equivalent $\star$-sequences with the same, unique expected value.

For defining the choice function, we ask a question in §2.4 where with previous sections; we define equivalent & non-equivalent $\star$-sequences of sets for §2.1, and "natural" expected values for §2.3. We attempt to answer the question in §2.4 by redefining linear/super-linear convergence (def. 8) in terms of Entropy and Samples, where the samples are derived by taking points of each partitions of the domain of a $\star$-sequence of sets, such that the partitions have equal Hausdorff measure. Since all samples have finite points; we order the x-values of the points from least to greatest, take the difference between consecutive pairs of x-values in the sample, multiply the differences by a constant so they add up to one (i.e. a discrete probability distribution), and use the Entropy of the distribution [4] to redefine def. 8 as def. 12. We then use the redefined definition to create a choice function.

In the case that a choice function does not give a unique expected value in equation 4.1.11; we'll use iterations of choice function C (eq. 4.1.9) in §4.2, to increase the chance of choosing non-equivalent $\star$-sequences of sets, such that the generalized expected values of f w.r.t each $\star$-sequence is the same.

1. Preliminary Definitons/Motivation

Other than integration with filters [2], there is no constructive approach to finding a meaningful average of functions covering an infinite expanse of space; however, there are two constructive approaches to making the average unique, finite, and "natural". Before beginning, consider the following mathematical definitions:

1.1. Preliminary Definitions

Let X be a completely metrizable topological vector space.

Definition 1 (**Prevalent Subset of** X)**.** *A Borel set $E \subset X$ is said to be **prevalent** if there exists a Borel measure μ on X such that:*

1. $0 < \mu(C) < \infty$ *for some compact subset C of X, and*
2. *the set $\{E + x : x \in X\}$ has full μ-measure (that is, the complement of $\{E + x : x \in X\}$ has measure zero).*

More generally, a subset F of X is prevalent if F contains a prevalent Borel Set. Also note:

Definition 2 (**Shy Subset of** X)**.** *The complement of a prevalent set is called a shy set.*

Therefore, we can use definitions 1 and 2 to prove or disprove:

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[1] We want to find an extension of the expected value, w.r.t the Hausdorff measure, that's unique, finite and "natural" for all f in a prevalent subset of $\mathbb{R}^A$

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Theorem 1. *The set of unbounded functions forms a prevalent subset of the set of all functions.*

Moreover, let (V, d) be a metric space. If set $A \subseteq V$, where we restrict A using Hausdorff outer measure to sets measurable in the Carathèodory sense; let H^{α} be the α-dimensional Hausdorff measure on A, where $\alpha \in [0, +\infty)$ and $\dim_{\mathrm{H}}(A)$ is the Hausdorff dimension of set A. In addition, when $\dim_{\mathrm{H}}(A) \in \mathbb{N}$, suppose $H^{\dim_{\mathrm{H}}(A)}(A)$ equals the $\dim_{\mathrm{H}}(A)$-dimensional Lebesgue measure with the expected value w.r.t the Hausdorff measure defined to be the following:

Definition 3 (Expected Value of f)**.** *If $n \in \mathbb{N}$, where set $A \subseteq \mathbb{R}^n$, the expected value of function $f : A \to \mathbb{R}$ is*

$$\mathbb{E}[f] = \frac{1}{H^{dim_{H}(A)}(A)} \int_A f \, dH^{dim_{H}(A)}$$

where we can see there are cases where $\mathbb{E}[f]$ is undefined or infinite (e.g. $H^{\dim_{\mathrm{H}}(A)}(A)$ is zero, $+\infty$ or f is unbounded). In this case, if topological vector space X is $\mathbb{R}^A$ (see §1.1), we also must prove:

Theorem 2. *The expected value $\mathbb{E}[f]$ is finite for all f in only a shy subset of $\mathbb{R}^A$*

1.2. Extended Expected Values

Two solutions to getting a finite expected value for "larger" subset of $\mathbb{R}^A$ is:

1. Defining a **dimension function**; i.e., $h : [0, +\infty) \to [0, +\infty]$, that's monotonically increasing, strictly positive and right continuous, such that when R denotes the radius of a ball in a covering for the definition of the Hausdorff Measure, we replace $R^{\dim_{\mathrm{H}}(A)}$ with $h(R)$ so $H^h(A)$: the h**-Hausdorff measure**, is positive and finite. This leads to the extended expected value $\mathbb{E}^{\star}[f]$, where:

$$\mathbb{E}^{\star}[f] = \frac{1}{H^h(A)} \int_A f \, dH^h$$

   Note, however, not all A has dimension function h which leads to:

2. If A is fractal but has no gauge function, we could use this paper [5] which is an extension of the Lebesgue density theorem and this paper [6] which is an extension of the Hausdorff measure using Hyperbolic Cantor sets. Note, however, when A is non-fractal (e.g. countably infinite) or f is unbounded, there is a possibility that the expected value is infinite or undefined. Infact, we need to prove:

Theorem 3. *The extended expected value in (1) and (2) is finite for all f in only a shy subset of $\mathbb{R}^A$*

The suspicion is either extensions extend the Hausdorff measure to be positive and finite for the *most* subsets A of $\mathbb{R}^n$. However, if every subset of $\mathbb{R}^n$ had positive and finite measure; when f is unbounded, the expected value w.r.t these measures/densities are still infinite or undefined for all f in a prevalent subset of $\mathbb{R}^A$. This means all unbounded f with finite expected values and bounded f form *only* a shy subset of the set of all f.

2. Attempt to Answer Thesis

Suppose h is the dimension function and H^h is the h-Hausdorff measure (§1.2, crit. 1).

Definition 4 ($\star$**-Sequence of Sets).** *If we define a sequence of sets $(F_r^{\star})_{r \in \mathbb{N}}$, where h is the dimension function, then when:*

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1. *the set theoretic limit of a sequence of sets* $(F_r^\star)_{r\in\mathbb{N}}$ *is* $\{(x,f(x)) : x \in A\}$ *(i.e.* $(F_r^\star)_{r\in\mathbb{N}}$ *converges to* $\{(x,f(x)) : x \in A\}$*) where:*

$$\limsup_{r\to\infty} F_r^\star = \liminf_{r\to\infty} F_r^\star = \{(x,f(x)) : x \in A\}$$

2. *For all* $r \in \mathbb{N}$, $0 < H^h(F_r^\star) < +\infty$
3. *we define sequence of functions* $(f_r^\star)_{r\in\mathbb{N}}$ *where* $f_r^\star : dom(F_r^\star) \to range(F_r^\star)$ *such that* $\{(x, f_r^\star(x)) : x \in dom(F_r^\star)\} = F_r^\star$

we have $(F_r^\star)$ *is a* ***$\star$-sequence of sets*** *or* ***starred-sequence of sets****.*

Note this will lead to a new extension of the expected value where when there's at least one starred-sequence of sets where the extension is finite, the extension *could* be finite for all f in a prevalent subset of $\mathbb{R}^A$.

Definition 5 (Generalized Expected Value). *If* $(F_r^\star)_{r\in\mathbb{N}}$ *is a* $\star$*-sequence of sets (def. 4), the generalized expected value of* f *w.r.t* $(F_r^\star)_{r\in\mathbb{N}}$ *is* $E^{**}[f, F_r^\star]$ *where:*

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N \Rightarrow \left|\frac{1}{H^h\left(dom\left(F_r^\star\right)\right)}\int_{dom(F_r^\star)} f_r^\star \, dH^h - \mathbb{E}^{**}[f, F_r^\star]\right| < \epsilon\right) \quad (2.0.1)$$

2.1. Equivalent and Non-Equivalent $\star$-sequences of Sets

Next, we define set V', where we want the generalized expected value to exist for all $f \in V'$ w.r.t at least one sequence (in a set of $\star$-sequences of sets) where

Definition 6 (Equivalent Starred-Sequences of Sets). *All starred-sequences of sets are equivalent (in the set of* $\star$*-sequences of sets), if we get for all* $f \in V'$*; the generalized expected value of* f *w.r.t each starred-sequence of sets has the same value.*

Definition 7 (Non-Equivalent Starred-Sequences of Sets). *All starred-sequences of sets are non-equivalent (in a set of* $\star$*-sequences of sets), if there exists an* $f \in V'$*, where the generalized expected values of* f *w.r.t each starred-sequence of sets has two or more different values; e.g., defined vs undefined values.*

However, proving that two or more starred-sequences of sets are equivalent or non-equivalent (using def. 6 or 7) is tedious. Therefore, we ask the following:

2.1.1. Question 1

Is there are a simpler definition of equivalent and non-equivalent $\star$-sequences of sets.

2.2. Motivation for Question

For all f in a prevalent subset of $\mathbb{R}^A$ (def. 1), we may choose a $\star$-sequence of sets $(F_r^\star)_{r\in\mathbb{N}}$ where the generalized expected value of f w.r.t least one starred-sequence is finite. However, consider the following problem:

Theorem 4. *The set of all* f*, where the generalized expected values of* f *w.r.t two or more non-equivalent* $\star$*-sequences of sets has different values, form a prevalent subset of* $\mathbb{R}^A$*.*

This means "almost all" functions have *several* generalized expected values depending on the starred-sequence chosen. Therefore, we need to choose a unique $\star$-sequence of sets where the new extended expected value is also "natural" and unique:

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2.3. Essential Definitions for a "Natural" Expected Value

Suppose $(F_r^\star)_{r\in\mathbb{N}}$ and $(F_j^{\star\star})_{j\in\mathbb{N}}$ are non-equivelant starred-sequences of sets (def. 4 & 7): we have the following is essential for a "natural" expected value.

Definition 8 (Linear & Super-linear Convergence of a ⋆-Sequence of Sets To That Of Another ⋆-Sequence of Sets). *If we define function $S:\mathbb{R}\to\mathbb{R}$, where $r,j\in\mathbb{N}$ such that:*

$$H^h(F_r^\star)=\mathcal{O}(S(H^h(F_j^{\star\star})))$$

*where we have $\mathcal{O}$ as the Big-O notation and $0<\lim_{x\to\infty}S(x)/x$, then $(F_r^\star)_{r\in\mathbb{N}}$ converges to the graph of f: $\{(x,f(x)):x\in A\}$ at a **linear** or **super-linear** rate compared to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$.*

Now we may combine the previous definitions into a main question with an answer that solves the thesis [2].

2.4. Main Question

Does there exist a choice function that chooses a unique set (of equivalent ⋆-sequences of sets) such that:

1. The chosen starred-sequences of sets converge to $\{(x,f(x)):x\in A\}$ at a rate *linear* or *super-linear* (def. 8) to the rate non-equivalent ⋆-sequences of sets converge to $\{(x,f(x)):x\in A\}$
2. The *generalized expected value* (def. 5) of f w.r.t the chosen (and equivalent) starred-sequences of sets is finite.
3. The choice function chooses a unique set of equivalent ⋆-sequences of sets which satisfy (1) and (2), for all $f\in Q$ such that Q is a prevalent subset of $\mathbb{R}^A$.
4. Out of all the choice functions which satisfy (1), (2) and (3), we choose the one with the *simplest form*, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers (excluding those with quantifiers)?

Note 5 (Notes On Question). *Note, the unique set of equivalent and chosen starred-sequences of sets is defined using notation $\sim(F_k^{\star\star\star})_{k\in\mathbb{N}}$, where $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ is a starred-sequence in $\sim(F_k^{\star\star\star})_{k\in\mathbb{N}}$. Therefore, after we define the choice function, the answer should be:*

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(k\in\mathbb{N})\left(k\geq N\Rightarrow\left|\frac{1}{H^h\left(dom\left(F_k^{\star\star\star}\right)\right)}\int_{dom(F_k^{\star\star\star})}f_k^\star\,dH^h-\mathbb{E}^{**}[f,F_k^{\star\star\star}]\right|<\epsilon\right) \tag{2.4.1}$$

Also, consider the following: if the solution to the main question is extraneous, what other criteria can be included to get a unique choice function? (Note if the solution is always extraneous, we want to replace "equivelant starred-sequences of sets" with the following: "the set of all ⋆-sequences of sets, where the generalized expected values of f w.r.t each starred-sequence is the same".)

3. Solution To The Main Question Of Section 2.4

Suppose h is the dimension function, H^h is the h-Hausdorff measure (§1.2, crit. 1), and $(F_r^\star)_{r\in\mathbb{N}}$ is the starred-sequence of sets (def. 4). We will use an alternative approach to definition 8 or def. 12 so we can define a choice function which solves the main question. Read from the second sentence of second-to-last paragraph of the intro of §0 for a summary.

[2] We want to find an extension of the expected value, w.r.t the Hausdorff measure, that's unique, finite and "natural" for all f in a prevalent subset of $\mathbb{R}^A$

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3.1. Preliminary Definitions

Definition 9 (Uniform ε coverings of each term of a $\star$-sequence of sets). *We define the uniform ε coverings of each term of $(dom(F_r^\star))_{r\in\mathbb{N}}$ ($dom(F_r^\star)$ for some r) as a group of pair-wise disjoint sets covering $dom(F_r^\star)$, such that when taking dimension function h of $dom(F_r^\star)$, we want H^h of each of the sets covering $dom(F_r^\star)$ to have the same value $\varepsilon \in range(H^h)$, where $\varepsilon > 0$ and the total sum of H^h of the coverings is minimized. In shorter notation, if*

- *The element $t \in \mathbb{N}$*
- *The set $T \supset \mathbb{N}$ is arbitrary and uncountable.*

and set Ω is defined as:

$$\Omega = \begin{cases} \{1,\cdots,t\} & \text{if there are } t \text{ ways of writing uniform } \varepsilon \text{ coverings of } dom(F_r^\star) \\ \mathbb{N} & \text{if there are countably infinite ways of writing uniform } \varepsilon \text{ coverings of } dom(F_r^\star) \\ T & \text{if there are uncountable ways of writing uniform } \varepsilon \text{ coverings of } dom(F_r^\star) \end{cases} \tag{3.1.1}$$

then for every $\omega \in \Omega$, the set of uniform ε coverings is defined using $\mathscr{U}(\epsilon, dom(F_r^\star), \omega)$ where ω "enumerates" all possible uniform ε coverings of $dom(F_r^\star)$ for every $r \in \mathbb{N}$.

Definition 10 (Sample of the uniform ε coverings of each term of a $\star$-sequence of sets). *The sample of uniform ε coverings of each term of $(dom(F_r^\star))_{r\in\mathbb{N}}$ (or $dom(F_r^\star)$ for some r) is the set of points where for every $\varepsilon \in range(H^h)$ and $r \in \mathbb{N}$, we take a point from each pair-wise disjoint set in the uniform ε coverings of $dom(F_r^\star)$ (def. 9). In shorter notation, if*

- *The element $k \in \mathbb{N}$*
- *The set $\mathcal{K} \supset \mathbb{N}$ is arbitrary and uncountable.*

and set Ψ_ω is defined as:

$$\Psi_\omega = \begin{cases} \{1,\cdots,k\} & \text{if there are } k \text{ ways of writing the sample of uniform } \varepsilon \text{ coverings of } dom(F_r^\star) \\ \mathbb{N} & \text{if there are countably infinite ways of writing the sample of uniform } \varepsilon \text{ coverings of } dom(F_r^\star) \\ \mathcal{K} & \text{if there are uncountable ways of writing the sample of uniform } \varepsilon \text{ coverings of } dom(F_r^\star) \end{cases} \tag{3.1.2}$$

then for every $\psi \in \Psi_\omega$, the set of all samples of the set of uniform ε coverings is defined using $\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi)$, such that ψ "enumerates" all possible samples of $\mathscr{U}(\epsilon, dom(F_r^\star), \omega)$.

Definition 11 (Entropy on the sample of uniform coverings of each term of $\star$-sequence of sets). *Since there are finitely many points in the sample of the uniform ε coverings of each term of $(dom(F_r))_{r\in\mathbb{N}}$ (def. 10), we:*

1. *Arrange the x-value of the points in the sample of uniform ε coverings from least to greatest. This is defined as:*
$$Ord(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi))$$
2. *Take the multi-set of the absolute differences between each consecutive pairs of elements in (1). This is defined as: $\triangledown Ord(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi))$*
3. *Normalize (2) into a discrete probability distribution. This is defined as:*

$$\mathbb{P}(\mathcal{S}(\mathscr{U}(\epsilon, {\scriptstyle dom}(F_r^\star), \omega), \psi)) = \left\{ y \Big/ \left(\sum_{z \in \triangledown Ord(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi))} z \right) : y \in \triangledown {\scriptstyle Ord}(\mathcal{S}(\mathscr{U}(\epsilon, {\scriptstyle dom}(F_r^\star), \omega), \psi)) \right\} \tag{3.1.3}$$

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4. *Take the entropy of (3), (for further reading, see [4, p.61-95]). This is defined as:*

$$E(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi)) = -\sum_{x \in \mathbb{P}(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi))} x \log_2 x$$

where (4) is the entropy on the sample of uniform coverings of $dom(F_r^\star)$.

Definition 12 (Starred-Sequence of sets converging Sublinearly, Linearly, or Superlinearly to A compared to that of another $\star$-Sequence). *Suppose we define starred-sequences of sets* $(dom(F_r^\star))_{r\in\mathbb{N}}$ *and* $(dom(F_j^{\star\star}))_{j\in\mathbb{N}}$*, where for every* $\varepsilon \in range(H^h)$*, we get* $\varepsilon > 0$ *and* $r \in \mathbb{N}$ *such that:*

(a) *From def. 10 and 11, suppose we have:*

$$\overline{|\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi)|} = \inf\left\{|\mathcal{S}(\mathscr{U}(\epsilon, dom(F_j^{\star\star}), \omega'), \psi')| : j \in \mathbb{N},\ \omega' \in \Omega,\ \psi' \in \Psi_\omega,\ E(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_j^{\star\star}), \omega'), \psi')) \geq E(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi))|\right\} \tag{3.1.4}$$

then (using $\overline{|\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi)|}$*) we have:*

$$\overline{\alpha}(\epsilon, r, \omega, \psi) = |\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi))| \,/\, \overline{|\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi)|} \tag{3.1.5}$$

(b) *Using def. 10 and 11, suppose we have:*

$$\underline{|\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi)|} = \sup\left\{|\mathcal{S}(\mathscr{U}(\epsilon, dom(F_j^{\star\star}), \omega'), \psi')| : j \in \mathbb{N},\ \omega' \in \Omega,\ \psi' \in \Psi_\omega,\ E(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_j^{\star\star}), \omega'), \psi')) \leq E(\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi))\right\} \tag{3.1.6}$$

then (using $\underline{|\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi)|}$*) we get*

$$\underline{\alpha}(\epsilon, r, \omega, \psi) = |\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi))| \,/\, \underline{|\mathcal{S}(\mathscr{U}(\epsilon, dom(F_r^\star), \omega), \psi)|} \tag{3.1.7}$$

1. *If using* $\overline{\alpha}(\epsilon, r, \omega, \psi)$ *and* $\underline{\alpha}(\epsilon, r, \omega, \psi)$ *we have that:*

$$\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{\alpha}(\epsilon, r, \omega, \psi) = \liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{\alpha}(\epsilon, r, \omega, \psi) = 0$$

we say $(dom(F_r^\star))_{r\in\mathbb{N}}$ *converges to* A *at a rate* ***superlinear rate*** *to that of* $(dom(F_j^{\star\star}))_{j\in\mathbb{N}}$.

2. *If using equations* $\overline{\alpha}(\epsilon, r, \omega, \psi)$ *and* $\underline{\alpha}(\epsilon, r, \omega, \psi)$ *we have either:*

(a) $0 \leq \liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{\alpha}(\epsilon, r, \omega, \psi) < +\infty$

$0 < \limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{\alpha}(\epsilon, r, \omega, \psi) \leq +\infty$

(b) $0 < \liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{\alpha}(\epsilon, r, \omega, \psi) \leq +\infty$

$0 \leq \limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{\alpha}(\epsilon, r, \omega, \psi) < +\infty$

(c) $0 \leq \limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{\alpha}(\epsilon, r, \omega, \psi) < +\infty$

$0 < \liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{\alpha}(\epsilon, r, \omega, \psi) \leq +\infty$

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(d) $0 < \limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{\alpha}(\epsilon,r,\omega,\psi) \le +\infty$
$0 \le \liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{\alpha}(\epsilon,r,\omega,\psi) < +\infty$

we then say $(dom(F_r^\star))_{r\in\mathbb{N}}$ *converges to A at a rate* ***linear*** *to that of* $(dom(F_j^{\star\star}))_{j\in\mathbb{N}}$.

3. *If using equations* $\overline{\alpha}(\epsilon,r,\omega,\psi)$ *and* $\underline{\alpha}(\epsilon,r,\omega,\psi)$ *we have that:*

$$\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{\alpha}(\epsilon,r,\omega,\psi) = \limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{\alpha}(\epsilon,r,\omega,\psi) = +\infty$$

we say $(dom(F_r^\star))_{r\in\mathbb{N}}$ *converges uniformly to A at a rate* ***sublinear*** *to that of* $(dom(F_j^{\star\star}))_{j\in\mathbb{N}}$.

4. Attempt to Answer Main Question Of Section 2.4

4.1. Choice Function

Suppose $\mathbb{S}'(A)$ is the set of the starred-sequences of sets that have finite *generalized expected values* (def. 5). We shall attempt to define a $\star$-sequence of sets (i.e. $(F_k^{\star\star\star})_{k\in\mathbb{N}}$) which satisfy (1), (2), and (3) of the main question §2.4 and include $\star$-sequence $(F_j^{\star\star})_{j\in\mathbb{N}}$ which is an element $\mathbb{S}'(A)$ but not an element of the set of equivalent starred-sequences of sets of $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ i.e. $\sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$, where $(F_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(A)\setminus \sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$.
Further note from def. 4, if we take:

$$\overline{|\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k^{\star\star\star}),\omega),\psi)|} = \inf\left\{|\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_j^{\star\star}),\omega'),\psi')| : j\in\mathbb{N},\ \omega'\in\Omega,\ \psi'\in\Psi_\omega,\ E(\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_j^{\star\star}),\omega'),\psi')) \ge E(\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k^{\star\star\star}),\omega),\psi))\right\} \tag{4.1.1}$$

and from def. 4, we take:

$$\underline{|\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k^{\star\star\star}),\omega),\psi)|} = \sup\left\{|\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_j^{\star\star}),\omega'),\psi')| : j\in\mathbb{N},\ \omega'\in\Omega,\ \psi'\in\Psi_\omega,\ E(\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_j^{\star\star}),\omega'),\psi')) \le E(\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k^{\star\star\star}),\omega),\psi))\right\} \tag{4.1.2}$$

Then, using def. 10 with equations $\overline{|\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k^{\star\star\star}),\omega),\psi)|}$ and $\underline{|\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k^{\star\star\star}),\omega),\psi)|}$, if:

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k^{\star\star\star}),\omega),\psi) = \mathcal{S}'(\varepsilon,\text{dom}(F_k^{\star\star\star})) = \mathcal{S}' \tag{4.1.3}$$

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\overline{|\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k^{\star\star\star}),\omega),\psi)|} = \overline{|\mathcal{S}'(\varepsilon,\text{dom}(F_k^{\star\star\star}))|} = \overline{|\mathcal{S}'|} \tag{4.1.4}$$

$$\sup_{\omega\in\Omega}\sup_{\psi\in\Psi_\omega}\underline{|\mathcal{S}(\mathscr{U}(\epsilon,\text{dom}(F_k''),\omega),\psi)|} = \underline{|\mathcal{S}'(\varepsilon,\text{dom}(F_k^{\star\star\star}))|} = \underline{|\mathcal{S}'|} \tag{4.1.5}$$

where, using absolute value function $||\cdot||$, we have:

$$\begin{aligned} S(k) = {} & (\sup(\text{dom}(F_{k+1}^{\star\star\star})) - \sup(\text{dom}(F_k^{\star\star\star})))\,(\inf(\text{dom}(F_k^{\star\star\star})) - \inf(\text{dom}(F_{k+1}^{\star\star\star}))) \\ & ||\,(\inf(\text{dom}(F_k^{\star\star\star})) - \inf(\text{dom}(F_{k+1}^{\star\star\star})))\,(\sup(\text{dom}(F_{k+1}^{\star\star\star})) - \sup(\text{dom}(F_k^{\star\star\star})) - 1)\,|| \end{aligned} \tag{4.1.6}$$

such that we define:

$$\begin{aligned} T(k) = {} & (\sup(\text{dom}(F_{k+1}^{\star\star\star}))\inf(\text{dom}(F_k^{\star\star\star})) - \sup(\text{dom}(F_k^{\star\star\star}))\inf(\text{dom}(F_{k+1}^{\star\star\star}))) \\ & \Big(\,(\inf(\text{dom}(F_k^{\star\star\star})) - \inf(\text{dom}(F_{k+1}^{\star\star\star}))) - (\sup(\text{dom}(F_{k+1}^{\star\star\star})) - \sup(\text{dom}(F_k^{\star\star\star}))) - 1\Big) \\ & (\inf(\text{dom}(F_k^{\star\star\star})) - \inf(\text{dom}(F_{k+1}^{\star\star\star})))\,(\sup(\text{dom}(F_{k+1}^{\star\star\star})) - \sup(\text{dom}(F_k^{\star\star\star}))) \end{aligned} \tag{4.1.7}$$

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then using equations $\mathcal{S}'$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, $S(k)$, $T(k)$ with the nearest integer function $[\cdot]$, we want:

$$K(\varepsilon, \text{dom}(F_k^{\star\star\star})) = ||1 - S(k)|| \left(\left\| \frac{|\mathcal{S}'| \left(1 + \left[\frac{|\mathcal{S}'|(\underline{|\mathcal{S}'|} + 2|\mathcal{S}')|}{(\underline{|\mathcal{S}'|} + |\mathcal{S}'|)(\underline{|\mathcal{S}'|} + |\mathcal{S}'| + \overline{|\mathcal{S}'|})} \right]\right) \left(1 + \left[\underline{|\mathcal{S}'|}/|\mathcal{S}'|\right]\right)}{\left(1 + \left[|\mathcal{S}'|/\overline{|\mathcal{S}'|}\right]\right)\left(1 + \left[\underline{|\mathcal{S}'|}/\overline{|\mathcal{S}'|}\right]\right)} - |\mathcal{S}'| \right\| + |\mathcal{S}'| \right) - T(k) \tag{4.1.8}$$

where using $K(\varepsilon, F_k^{\star\star\star})$, if set $\mathbb{S}''(A) \subseteq \mathbb{S}'(A)$ and $\mathcal{P}(\cdot)$ is the power-set, then set $C(A)$ is the largest element of:

$$\Big\{ \mathbb{S}''(A) \subseteq \mathbb{S}'(A) : \forall(\epsilon_1 > 0)\exists(M \in \mathbb{N})\forall(\varepsilon \in \text{range}(H^h))\exists(v \in \mathbb{N})\forall(k \in \mathbb{N})\forall(\{F_k^{\star\star\star}\} \in \mathbb{S}''(A)) \tag{4.1.9}$$
$$\Big(0 < \varepsilon \leq M,\ k \geq v \Rightarrow \Big|\mathcal{S}'(\varepsilon, \text{dom}(F_k^{\star\star\star})) - K(\varepsilon, \text{dom}(F_k^{\star\star\star})) - \inf_{\{F_g^{\star}\} \in \mathbb{S}'(A)} \Big(\mathcal{S}'(\varepsilon, \text{dom}(F_g^{\star})) - K(\varepsilon, \text{dom}(F_g^{\star}))\Big) \Big| < \epsilon_1 \Big) \Big\} \subseteq \mathcal{P}(\mathbb{S}'(A))$$

w.r.t to inclusion, such that the **choice function** is $C(A)$ if the following only contains sequences of sets equivelant to $F_k^{\star\star\star}$ (see original post).

Otherwise, for $k \in \mathbb{N}$, suppose we say $C^k(A)$ represents the k-th iteration of the choice function of A, e.g. $C^3(A) = C(C(C(A)))$, where the infinite iteration of $C(A)$ (if it exists) is $\lim_{k \to \infty} C^k(A) = C^{\infty}(A)$. Therefore, when taking the following:

$$C'(A) = \begin{cases} C(A) & \text{if } C(A) \text{ contains one element} \\ C^j(A) & \text{if } j \in \mathbb{N}, \text{ such for all } k \geq j,\ C^k(A) \text{ contains one element} \\ C^{\infty}(A) & \text{if it exists, and } C^{\infty}(A) \text{ contains one element} \end{cases} \tag{4.1.10}$$

we say $C'(A)$ is the *choice* function and the **chosen expected value**, using the generalized expected value in the original post, is $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$, i.e.:

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(k \in \mathbb{N}) \left(k \geq N \Rightarrow \left| \frac{1}{H^h\left(\text{dom}\left(F_k^{\star\star\star}\right)\right)} \int_{\text{dom}(F_k^{\star\star\star})} f_k^{\star}\, dH^h - \mathbb{E}^{**}[f, F_k^{\star\star\star}] \right| < \epsilon \right) \tag{4.1.11}$$

4.2. Increasing the Chances of A Unique Expected Value

In case $C'(A)$, in equation 4.1.10, does not exist; if there exists a unique and finite $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ where:

$$\forall\left((F_k^{\star\star\star})_{k \in \mathbb{N}} \in C(A)\right)\left(\mathbb{E}^{**}[f, F_k^{\star\star\star}] \text{ is unique \& finite}\right) \tag{4.2.1}$$

Then $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ is the **generalized expected value w.r.t choice function** C, which answers criteria (1), (2) and (3) of the question in the OP; however, there is still a chance that the equation above fails to give an unique $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$. Hence; if $s \in \mathbb{N}$, we take the s-th iteration of the choice function $C(A)$, such [that] there exists a $t \in \mathbb{N}$, where for all $s \geq t$, if $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ is unique and finite then the following is the **generalized expected value w.r.t finitely iterated** C.

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In other words, if the s-th iteration of C is represented as $C^{[s]}$ (where e.g. $C^3(A) = C(C(C(A)))$), we want a unique and finite $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ where:

$$\exists\,(t \in \mathbb{N})\,\forall(s \in \mathbb{N})\left(s \geq t \Rightarrow \forall\left((F_k^{\star\star\star})_{k\in\mathbb{N}} \in C^{[k]}\,(A)\right)(\mathbb{E}^{**}[f, F_k^{\star\star\star}] \text{ is unique \& finite})\right) \tag{4.2.2}$$

If this still does not give a unique and finite expected value, we then take the **most generalized expected value w.r.t an infinitely iterated** C where if the *infinite iteration* of C is stated as $\lim_{m\to\infty} C^{[m]}(f[A]) = C^{\infty}(f[A])$, we then want a unique $\mathbb{E}^{**}[f, F_k^{\star\star\star}]$ where:

$$\forall\left((F_k^{\star\star\star})_{k\in\mathbb{N}} \in C^{\infty}\,(A)\right)(\mathbb{E}^{**}[f, F_k^{\star\star\star}] \text{ is unique \& finite}) \tag{4.2.3}$$

4.3. Notes on Answer

If either of the attempts answer criteria 1., 2., 3. (or even 4.) of the question on the original post, we can apply either attempts when A has no dimension function, A is non-fractal, or the points on the graph of f cover an infinite expanse of space.

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