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On Certain Modular Equations of Degree Three

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Abstract: We establish certain modular equations of degree three based on some results of Ramanujan's theta functions.

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MSC: 11F11; 11E25; 11F27; 33E05

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1. Introduction

Throughout this paper we assume that $|ab| < 1$ and $|q| < 1$. The Ramanujan's general theta function is defined by ([2], p. 34)

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}.$$

The Jacobi triple product identity ([6], 1.6.1) can be written in the notation of Ramanujan

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,$$

where $(z;q)_\infty$ is the q-shifted factorial given by

$$(z;q)_\infty := \prod_{n=0}^{\infty}(1-zq^n).$$

Ramanujan recorded many identities involving $f(a,b)$ and its special cases $\varphi(q) := f(q,q)$ and $\psi(q) := f(q,q^3)$. Some of them are

$$\varphi(q) = \frac{(-q;-q)_\infty}{(q;-q)_\infty} \tag{1.1}$$

and

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.2}$$

In Ramanujan's Notebook" and "Ramanujan's Lost Notebook", many beautiful identities involving modular equations of degree three and five are recorded. These formulas play important roles in the study of Ramanujan's theta functions. See [1,7–15] for more modular equations of degree three and five and various modular equations with degree seven. In [16,17], some modular equations of degree three were obtained. More related research please refer to [3–5]. In this paper, we will establish certain new modular equations of degree three.

In this paper we adopt the following notation for multiple q-shifted factorials:

$$(a_1,a_2,\ldots,a_m;q)_n := (a_1;q)_n(a_2;q)_n\ldots(a_m;q)_n,$$

where

$$n \in \mathbb{N} \cup \{\infty\}.$$

2. Auxiliary results

We recall several auxiliary results which are useful to prove our main results.

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Lemma 2.1. *(See [2, p. 46, Entry 30, (ii) and (iii)]) We have*

$$f(a,b)+f(-a,-b)=2f(a^3b,ab^3),$$
$$f(a,b)-f(-a,-b)=2af\left(\frac{b}{a},\frac{a}{b}a^4b^4\right).$$

Lemma 2.2. *(See [2, p. 45, Entry 29]) Let $ab=cd$. Then*

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)=2f(ac,bd)f(ad,bc),$$
$$f(a,b)f(c,d)-f(-a,-b)f(-c,-d)=2af\left(\frac{b}{c},\frac{c}{b}abcd\right)f\left(\frac{b}{d},\frac{d}{b}abcd\right).$$

Taking $c=a$ and $d=b$ in Lemma 2.2 and using the identity $f(1,q)=2\psi(q)$, we obtain

Corollary 2.1. *We have*

$$f^2(a,b)+f^2(-a,-b)=2f(a^2,b^2)\varphi(ab),$$
$$f^2(a,b)-f^2(-a,-b)=4af\left(\frac{b}{a},\frac{a}{b}a^2b^2\right)\psi(a^2b^2).$$

The following facts are also very important.

$$(a^2;q^2)_\infty=(a,-a;q)_\infty,\quad (a,q)_\infty=(a,aq,aq^2,\cdots,aq^{k-1};q^k)_\infty.$$

3. Modular equations of degree three

We begin this section with a pair of identities, which are refinements of ([16], (26) and (27)).

Theorem 3.1. *We have*

$$\sum_{n=-\infty}^{\infty}q^{n^2}+\sum_{n=-\infty}^{\infty}q^{3n^2}=2\frac{(q^2,q^6,q^{10},q^{12};q^{12})_\infty}{(q,q^3,q^9,q^{11};q^{12})_\infty},$$
$$\sum_{n=-\infty}^{\infty}q^{n^2}-\sum_{n=-\infty}^{\infty}q^{3n^2}=2q\frac{(q^2,q^6,q^{10},q^{12};q^{12})}{(q^3,q^5,q^7,q^9;q^{12})_\infty}.$$

Proof. It follows from (1.1) and the Jacobi triple product identity that

$$\frac{\varphi(-q)}{\varphi(-q^3)}=\frac{(q;q)_\infty}{(-q;q)_\infty}\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}=\frac{f(-q,-q^2)}{f(q,q^2)}. \tag{3.1}$$

By Lemma 2.1,

$$f(-q,-q^2)+f(q,q^2)=2f(q^5,q^7), \tag{3.2}$$
$$f(-q,-q^2)-f(q,q^2)=-2qf(q,q^{11}). \tag{3.3}$$

Combining (3.1)–(3.3), we find

$$\frac{\varphi(-q)}{\varphi(-q^3)}+1=\frac{f(-q,-q^2)}{f(q,q^2)}+1=2\frac{f(q^5,q^7)}{f(q,q^2)},$$
$$\frac{\varphi(-q)}{\varphi(-q^3)}-1=\frac{f(-q,-q^2)}{f(q,q^2)}-1=-2q\frac{f(q,q^{11})}{f(q,q^2)}.$$

Multiplying the above two identities by $\varphi(-q^3)$ and then replacing $-q$ with q, we establish the results. □

We now give a new proof of ([16], (26) and (27)).

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Proposition 3.1. *We have*

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 + \left(\sum_{n=-\infty}^{\infty} q^{3n^2}\right)^2 = 2\frac{(q^6;q^{12})_\infty^4 (q^{12};q^{12})_\infty^2 (q^3;q^6)_\infty}{(q,-q^2,-q^4,q^5;q^6)_\infty},$$
$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 - \left(\sum_{n=-\infty}^{\infty} q^{3n^2}\right)^2 = 4q\frac{(q^2,q^6,q^{10},q^{12};q^{12})_\infty^2}{(q,q^5,q^7,q^{11};q^{12})_\infty (q^3,q^9;q^{12})_\infty^2}.$$

Proof. It is easily seen from Corollary 2.1 that

$$f^2(-q,-q^2) + f^2(q,q^2) = 2f(q^2,q^4)\varphi(q^3),$$
$$f^2(-q,-q^2) - f^2(q,q^2) = -4qf(q,q^5)\psi(q^6).$$

Then

$$\frac{\varphi^2(-q)}{\varphi^2(-q^3)} + 1 = \frac{f^2(-q,-q^2)}{f^2(q,q^2)} + 1 = 2\frac{f(q^2,q^4)\varphi(q^3)}{f^2(q,q^2)},$$
$$\frac{\varphi^2(-q)}{\varphi^2(-q^3)} - 1 = \frac{f^2(-q,-q^2)}{f^2(q,q^2)} - 1 = -4q\frac{f(q,q^5)\psi(q^6)}{f^2(q,q^2)}.$$

Multiplying the above two identities by $\varphi^2(-q^3)$ and replacing $-q$ with q, we easily obtain the results. □

In [16], the proofs of the above identities in Proposition 3.1 are given by using the Jacobi theta functions. The second identity of Proposition 3.1 can also be obtained by multiplying the two identities of Theorem 3.1.

The following identities do not seem to be observed in the literature. We give an elementary proof without resorting to the Jacobi theta functions.

Theorem 3.2. *We have*

$$\varphi(q) + i\sqrt{3}\varphi(q^3) = 2\omega\varphi(q)\frac{f(-q\omega,-q^3\omega^2)}{f(\omega,-q\omega^2)},$$
$$\varphi(q) - i\sqrt{3}\varphi(q^3) = 2\varphi(q)\frac{f(-q\omega^2,-q^3\omega)}{f(\omega,-q\omega^2)},$$
$$\sqrt{3}\varphi(q^3) = i\varphi(q)\frac{f(-\omega,q\omega^2)}{f(\omega,-q\omega^2)},$$
$$\psi(q^4) + i\sqrt{3}q\psi(q^{12}) = (1-\omega)\psi(q^{12})\frac{f(q\omega,-q\omega^2)}{f(-\omega,-q^8\omega^2)},$$
$$\psi(q^4) - i\sqrt{3}q\psi(q^{12}) = (1-\omega)\psi(q^{12})\frac{f(-q\omega,q\omega^2)}{f(-\omega,-q^8\omega^2)},$$

where $\omega = \exp(2\pi i/3)$.

Proof. In the derivation we will appeal to the following identities: $1+\omega+\omega^2 = 0$, $\omega^3 = 1$, $\omega - \omega^2 = i\sqrt{3}$ and $(a,q)_\infty(a\omega,q)_\infty(a\omega^2,q)_\infty = (a^3,q^3)_\infty$.

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It follows from (1.1) and the second identity of Lemma 2.1 that

$$\begin{aligned}
\varphi(-q)+i\sqrt{3}\varphi(-q^3) &= \frac{(q;q)_\infty}{(-q;q)_\infty}+i\sqrt{3}\frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left(1+(\omega-\omega^2)\frac{(q\omega;q)_\infty(q\omega^2;q)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty}\right) \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left(1+\frac{\omega(1-\omega)(1+\omega)(q\omega;q)_\infty(q\omega^2;q)_\infty}{(1+\omega)(-q\omega;q)_\infty(-q\omega^2;q)_\infty}\right) \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left(1-\frac{(\omega;q)_\infty(q\omega^2;q)_\infty(q;q)_\infty}{(-\omega;q)_\infty(-q\omega^2;q)_\infty(q;q)_\infty}\right) \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left(\frac{f(\omega,q\omega^2)-f(-\omega,-q\omega^2)}{f(\omega,q\omega^2)}\right) \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left(2\omega\frac{f(q\omega,q^3\omega^2)}{f(\omega,q\omega^2)}\right).
\end{aligned}$$

Replacing $-q$ by q, we obtain the first identity. The proof of the second identity is similar to that of the first one and we omit it here.

We now give the proof of the third identity. Let $\alpha=\exp(\pi i/6)$, then

$$\begin{aligned}
\sqrt{3}\varphi(-q^3) &= \sqrt{3}\frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty} \\
&= (\alpha+1/\alpha)\frac{(q\omega;q)_\infty(q\omega^2;q)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty} \\
&= \frac{(1+\alpha^2)}{\alpha}\frac{(q\omega;q)_\infty(q\omega^2;q)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty} \\
&= \frac{(1+\omega)(1-\omega)}{\alpha}\frac{(q\omega;q)_\infty(q\omega^2;q)_\infty}{(-q\omega;q)_\infty(-q\omega^2;q)_\infty} \\
&= \frac{(1+\omega)^2}{\alpha}\frac{(\omega;q)_\infty(q\omega^2;q)_\infty}{(-\omega;q)_\infty(-q\omega^2;q)_\infty} \\
&= \frac{\omega}{\alpha}\frac{f(-\omega;-q\omega^2)}{f(\omega;q\omega^2)}=i\frac{f(-\omega;-q\omega^2)}{f(\omega;q\omega^2)}.
\end{aligned}$$

Replacing $-q$ by q, we get the third identity.

Finally, we prove the fourth and fifth identities. It can be seen from (1.2) that

$$\begin{aligned}
\frac{\psi(q^4)}{\psi(q^{12})} &= \frac{(q^8;q^8)_\infty(q^{12};q^{24})_\infty}{(q^4;q^8)_\infty(q^{24};q^{24})_\infty}=\frac{(1-\omega)}{(1-\omega)}\frac{(q^4\omega,q^4\omega^2;q^8)_\infty}{(q^8\omega,q^8\omega^2;q^8)_\infty} \\
&= (1-\omega)\frac{f(-q^4\omega,-q^4\omega^2)}{f(-\omega,-q^8\omega^2)}.
\end{aligned} \tag{3.4}$$

Choosing $a=q\omega$ and $b=-q\omega^2$ in Lemma 2.1 gives

$$\begin{aligned}
f(q\omega,-q\omega^2)+f(-q\omega,q\omega^2) &= 2f(-q^4\omega^2,-q^4\omega), \\
f(q\omega,-q\omega^2)-f(-q\omega,q\omega^2) &= 2q\omega f(-\omega,-q^8\omega^8).
\end{aligned}$$

Then we have

$$f(-q^4\omega^2,-q^4\omega)+q\omega f(-\omega,-q^8\omega^2)=f(q\omega,-q\omega^2)$$

and

$$f(-q^4\omega^2,-q^4\omega)-q\omega f(-\omega,-q^8\omega^2)=f(-q\omega,q\omega^2).$$

Multiplying the above two identities by $(1-\omega)/f(-\omega,-q^8\omega^2)$ and applying (3.4), we obtain

$$\frac{\psi(q^4)}{\psi(q^{12})}+i\sqrt{3}q=\frac{\psi(q^4)}{\psi(q^{12})}+q\omega(1-\omega)=(1-\omega)\frac{f(q\omega,-q\omega^2)}{f(-\omega,-q^8\omega^2)}$$

and

$$\frac{\psi(q^4)}{\psi(q^{12})}-i\sqrt{3}q=\frac{\psi(q^4)}{\psi(q^{12})}-q\omega(1-\omega)=(1-\omega)\frac{f(-q\omega,q\omega^2)}{f(-\omega,-q^8\omega^2)}.$$

We multiply the above two identities by $\psi(q^{12})$ to yield the fourth and fifth identities. This ends the proof of Theorem 3.2. □

Theorem 3.3. *We have*

$$\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^2-3\left(\sum_{n=-\infty}^{\infty}q^{3n^2}\right)^2=-2\frac{(q^2;q^2)_\infty^2(-q^6;q^6)_\infty(-q;q^2)_\infty}{(-q^3;q^3)_\infty(q;q^2)_\infty(-q^2;q^2)_\infty^2},$$

$$\left(\sum_{n=-\infty}^{\infty}q^{n^2}\right)^2+3\left(\sum_{n=-\infty}^{\infty}q^{3n^2}\right)^2=4\frac{(-q,q^2;q^2)_\infty^2(q^3;q^{12})_\infty(q^9;q^{12})_\infty}{(q,-q^2;q^2)_\infty^2(q^3;q^3)_\infty^2},$$

$$\left(\sum_{n=1}^{\infty}q^{2n(n-1)}\right)^2+3q^2\left(\sum_{n=1}^{\infty}q^{6n(n-1)}\right)^2=-3\omega\frac{(q^6;q^{12})_\infty}{(-q;q^2)_\infty(q^{24};q^{24})_\infty}.$$

Proof. We see from the third identity of Theorem 3.2 that

$$\varphi(-q)+\sqrt{3}\varphi(-q^3)=\frac{(q;q)_\infty}{(-q;q)_\infty}\left(1-i\frac{f(-\omega;-q\omega^2)}{f(\omega;q\omega^2)}\right),$$

$$\varphi(-q)-\sqrt{3}\varphi(-q^3)=\frac{(q;q)_\infty}{(-q;q)_\infty}\left(1+i\frac{f(-\omega;-q\omega^2)}{f(\omega;q\omega^2)}\right).$$

Multiplying the above two identities and then applying Corollary 2.1, we get the first identity.

The second identity is obtained from the product of the first two identities of Theorem 3.2 and the third one follows by multiplying the last two identities of Theorem 3.2. □

Conflicts of Interest: The authors declare no conflict of interest.

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