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Functional Donoho-Stark Approximate Support Uncertainty Principle

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Abstract: Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ be two p-orthonormal bases for a finite dimensional Banach space $\mathcal{X}$. If $x \in \mathcal{X} \setminus \{0\}$ is such that $\theta_f x$ is ε-supported on $M \subseteq \{1, \dots, n\}$ w.r.t. p-norm and $\theta_g x$ is δ-supported on $N \subseteq \{1, \dots, n\}$ w.r.t. p-norm, then we show that (1) $o(M)^{\frac{1}{p}} o(N)^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1 \leq j,k \leq n} |f_j(\omega_k)|} \max\{1 - \varepsilon - \delta, 0\}$, (2) $o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} \geq \frac{1}{\max\limits_{1 \leq j,k \leq n} |g_k(\tau_j)|} \max\{1 - \varepsilon - \delta, 0\}$, where $\theta_f : \mathcal{X} \ni x \mapsto (f_j(x))_{j=1}^n \in \ell^p([n])$; $\theta_g : \mathcal{X} \ni x \mapsto (g_k(x))_{k=1}^n \in \ell^p([n])$ and q is the conjugate index of p. We call Inequalities (1) and (2) as **Functional Donoho-Stark Approximate Support Uncertainty Principle**. Inequalities (1) and (2) improve the finite approximate support uncertainty principle obtained by Donoho and Stark *[SIAM J. Appl. Math., 1989]*.

Keywords: uncertainty principle; orthonormal basis; Hilbert space; Banach space

Mathematics Subject Classification (2020): 42C15, 46B03, 46B04

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1. Introduction

Let $0 \leq \varepsilon < 1$. Recall that a function $f \in \mathcal{L}^2(\mathbb{R}^d)$ is said to be ε**-supported on a measurable subset** $E \subseteq \mathbb{R}^d$ (also known as ε**-approximately supported** as well as ε**-essentially supported**) [1,9] if

$$\left(\int\limits_{E^c} |f(x)|^2 \, dx \right)^{\frac{1}{2}} \leq \varepsilon \left(\int\limits_{\mathbb{R}^d} |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Let $d \in \mathbb{N}$ and $\widehat{\cdot} : \mathcal{L}^2(\mathbb{R}^d) \to \mathcal{L}^2(\mathbb{R}^d)$ be the unitary Fourier transform obtained by extending uniquely the bounded linear operator

$$\widehat{\cdot} : \mathcal{L}^1(\mathbb{R}^d) \cap \mathcal{L}^2(\mathbb{R}^d) \ni f \mapsto \widehat{f} \in C_0(\mathbb{R}^d); \quad \widehat{f} : \mathbb{R}^d \ni \xi \mapsto \widehat{f}(\xi) := \int\limits_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} \, dx \in \mathbb{C}.$$

In 1989, Donoho and Stark derived the following uncertainty principle on approximate supports of function and its Fourier transform [1].

Theorem 1.1. [1] *(**Donoho-Stark Approximate Support Uncertainty Principle**) If $f \in \mathcal{L}^2(\mathbb{R}^d) \setminus \{0\}$ is ε-supported on a measurable subset $E \subseteq \mathbb{R}^d$ and $\widehat{f}$ is δ-supported on a measurable subset $F \subseteq \mathbb{R}^d$, then*

$$m(E)m(F) \geq (1 - \varepsilon - \delta)^2.$$

Ultimate result in [1] is the finite dimensional Heisenberg uncertainty principle known today as Donoho-Stark uncertainty principle. It is then natural to seek a finite dimensional version of Theorem 1.1. For this, first one needs the notion of approximate support in finite dimensions. Donoho and Stark defined this notion as follows. For $h \in \mathbb{C}^d$, let $\|h\|_0$ be the number of nonzero entries in h. Let $\widehat{\cdot} : \mathbb{C}^d \to \mathbb{C}^d$ be the Fourier transform. Given a subset $M \subseteq \{1, \dots, n\}$, the number of elements in M is denoted by $o(M)$.

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Definition 1.2. *[1] Let $0 \leq \varepsilon < 1$. A vector $(a_j)_{j=1}^d \in \mathbb{C}^d$ is said to be **ε-supported on a subset** $M \subseteq \{1, \dots, d\}$ if*

$$\left(\sum_{j \in M^c} |a_j|^2\right)^{\frac{1}{2}} \leq \varepsilon \left(\sum_{j=1}^d |a_j|^2\right)^{\frac{1}{2}}.$$

Finite dimensional version of Theorem 1.1 then reads as follows.

Theorem 1.3. *[1] (**Finite Donoho-Stark Approximate Support Uncertainty Principle**) If $h \in \mathbb{C}^d \setminus \{0\}$ is ε-supported on $M \subseteq \{1, \dots, d\}$ and $\widehat{h}$ is δ-supported on $N \subseteq \{1, \dots, d\}$, then*

$$o(M)o(N) \geq d(1 - \varepsilon - \delta)^2.$$

In 1990, Smith [8] generalized Theorem 1.3 to Fourier transforms defined on locally compact abelian groups. Recently, Banach space version of finite Donoho-Stark uncertainty principle has been derived in [2]. Therefore we seek a Banach space version of Theorem 1.3. This we obtain in this paper.

2. Functional Donoho-Stark Approximate Support Uncertainty Principle

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{X}$ denotes a finite dimensional Banach space over $\mathbb{K}$. Identity operator on $\mathcal{X}$ is denoted by $I_{\mathcal{X}}$. Dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. Whenever $1 < p < \infty$, q denotes the conjugate index of p. For $d \in \mathbb{N}$, the standard finite dimensional Banach space $\mathbb{K}^d$ over $\mathbb{K}$ equipped with standard $\|\cdot\|_p$ norm is denoted by $\ell^p([d])$. Canonical basis for $\mathbb{K}^d$ is denoted by $\{e_j\}_{j=1}^d$ and $\{\zeta_j\}_{j=1}^d$ be the coordinate functionals associated with $\{e_j\}_{j=1}^d$.

Definition 2.1. [3] *Let $\mathcal{X}$ be a finite dimensional Banach space over $\mathbb{K}$. Let $\{\tau_j\}_{j=1}^n$ be a basis for $\mathcal{X}$ and let $\{f_j\}_{j=1}^n$ be the coordinate functionals associated with $\{\tau_j\}_{j=1}^n$. The pair $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be a **p-orthonormal basis** $(1 < p < \infty)$ for $\mathcal{X}$ if the following conditions hold.*

(i) $\|f_j\| = \|\tau_j\| = 1$ *for all* $1 \leq j \leq n$.
(ii) *For every* $(a_j)_{j=1}^n \in \mathbb{K}^n$,

$$\left\|\sum_{j=1}^n a_j \tau_j\right\| = \left(\sum_{j=1}^n |a_j|^p\right)^{\frac{1}{p}}.$$

Given a p-orthonormal basis $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ for $\mathcal{X}$, we get the following two invertible isometries:

$$\theta_f : \mathcal{X} \ni x \mapsto (f_j(x))_{j=1}^n \in \ell^p([n]), \quad \theta_\tau : \ell^p([n]) \ni (a_j)_{j=1}^n \mapsto \sum_{j=1}^n a_j \tau_j \in \mathcal{X}.$$

Then we have the following proposition.

Proposition 2.2. *Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ be a p-orthonormal basis for $\mathcal{X}$. Then*

(i) θ_f *is an invertible isometry.*
(ii) θ_τ *is an invertible isometry.*
(iii) $\theta_\tau \theta_f = I_{\mathcal{X}}$.

It is natural to guess the following version of Definition 1.2 for $\ell^p([n])$.

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Definition 2.3. *Let $0 \leq \varepsilon < 1$. A vector $(a_j)_{j=1}^n \in \ell^p([n])$ is said to be ε-supported on a subset $M \subseteq \{1.\dots,n\}$ w.r.t. p-norm if*

$$\left(\sum_{j\in M^c}|a_j|^p\right)^{\frac{1}{p}} \leq \varepsilon\left(\sum_{j=1}^{n}|a_j|^p\right)^{\frac{1}{p}}.$$

With the above definition we have following theorem.

Theorem 2.4. *(Functional Donoho-Stark Approximate Support Uncertainty Principle) Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ be two p-orthonormal bases for a finite dimensional Banach space $\mathcal{X}$. If $x \in \mathcal{X}\setminus\{0\}$ is such that $\theta_f x$ is ε-supported on $M \subseteq \{1,\dots,n\}$ w.r.t. p-norm and $\theta_g x$ is δ-supported on $N \subseteq \{1,\dots,n\}$ w.r.t. p-norm, then*

$$o(M)^{\frac{1}{p}}o(N)^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1\leq j,k\leq n}|f_j(\omega_k)|}\max\{1-\varepsilon-\delta,0\}, \tag{3}$$

$$o(M)^{\frac{1}{q}}o(N)^{\frac{1}{p}} \geq \frac{1}{\max\limits_{1\leq j,k\leq n}|g_k(\tau_j)|}\max\{1-\varepsilon-\delta,0\}. \tag{4}$$

Proof. For $S \subseteq \{1,\dots,n\}$, define $P_S : \ell^p([n]) \ni (a_j)_{j=1}^n \mapsto \sum_{j\in S}a_je_j \in \ell^p([n])$ be the canonical projection onto the coordinates indexed by S. Now define $V := P_M\theta_f\theta_\omega P_N : \ell^p([n]) \to \ell^p([n])$. Then for $z \in \ell^p([n])$,

$$\begin{aligned}
\|Vz\|^p &= \|P_M\theta_f\theta_\omega P_N z\|^p = \left\|P_M\theta_f\theta_\omega P_N\left(\sum_{k=1}^n\zeta_k(z)e_k\right)\right\|^p = \left\|P_M\theta_f\theta_\omega\left(\sum_{k=1}^n\zeta_k(z)P_Ne_k\right)\right\|^p \\
&= \left\|P_M\theta_f\theta_\omega\left(\sum_{k\in N}\zeta_k(z)e_k\right)\right\|^p = \left\|P_M\theta_f\left(\sum_{k\in N}\zeta_k(z)\theta_\omega e_k\right)\right\|^p = \left\|P_M\theta_f\left(\sum_{k\in N}\zeta_k(z)\omega_k\right)\right\|^p \\
&= \left\|\sum_{k\in N}\zeta_k(z)P_M\theta_f\omega_k\right\|^p = \left\|\sum_{k\in N}\zeta_k(z)P_M\left(\sum_{j=1}^n f_j(\omega_k)e_j\right)\right\|^p = \left\|\sum_{k\in N}\zeta_k(z)\sum_{j=1}^n f_j(\omega_k)P_Me_j\right\|^p \\
&= \left\|\sum_{k\in N}\zeta_k(z)\sum_{j\in M}f_j(\omega_k)e_j\right\|^p = \left\|\sum_{j\in M}\left(\sum_{k\in N}\zeta_k(z)f_j(\omega_k)\right)e_j\right\|^p = \sum_{j\in M}\left|\sum_{k\in N}\zeta_k(z)f_j(\omega_k)\right|^p \\
&\leq \sum_{j\in M}\left(\sum_{k\in N}|\zeta_k(z)f_j(\omega_k)|\right)^p \leq \left(\max_{1\leq j,k\leq n}|f_j(\omega_k)|\right)^p\sum_{j\in M}\left(\sum_{k\in N}|\zeta_k(z)|\right)^p \\
&= \left(\max_{1\leq j,k\leq n}|f_j(\omega_k)|\right)^p o(M)\left(\sum_{k\in N}|\zeta_k(z)|\right)^p \leq \left(\max_{1\leq j,k\leq n}|f_j(\omega_k)|\right)^p o(M)\left(\sum_{k\in N}|\zeta_k(z)|^p\right)^{\frac{p}{p}}\left(\sum_{k\in N}1^q\right)^{\frac{p}{q}} \\
&= \left(\max_{1\leq j,k\leq n}|f_j(\omega_k)|\right)^p o(M)\left(\sum_{k\in N}|\zeta_k(z)|^p\right)^{\frac{p}{p}}o(N)^{\frac{p}{q}} \leq \left(\max_{1\leq j,k\leq n}|f_j(\omega_k)|\right)^p o(M)\left(\sum_{k=1}^n|\zeta_k(z)|^p\right)^{\frac{p}{p}}o(N)^{\frac{p}{q}} \\
&= \left(\max_{1\leq j,k\leq n}|f_j(\omega_k)|\right)^p o(M)\|z\|^p o(N)^{\frac{p}{q}}.
\end{aligned}$$

Therefore

$$\|V\| \leq \left(\max_{1\leq j,k\leq n}|f_j(\omega_k)|\right)o(M)^{\frac{1}{p}}o(N)^{\frac{1}{q}}. \tag{5}$$

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We now wish to find a lower bound on the operator norm of V. For $x \in \mathcal{X}$, we find

$$\begin{aligned}\|\theta_f x - V\theta_g x\| &\leq \|\theta_f x - P_M\theta_f x\| + \|P_M\theta_f x - V\theta_g x\| \leq \varepsilon\|\theta_f x\| + \|P_M\theta_f x - V\theta_g x\| \\ &= \varepsilon\|\theta_f x\| + \|P_M\theta_f x - P_M\theta_f\theta_\omega P_N\theta_g x\| = \varepsilon\|\theta_f x\| + \|P_M\theta_f(x - \theta_\omega P_N\theta_g x)\| \\ &\leq \varepsilon\|\theta_f x\| + \|x - \theta_\omega P_N\theta_g x\| = \varepsilon\|\theta_f x\| + \|\theta_\omega\theta_g x - \theta_\omega P_N\theta_g x\| \\ &= \varepsilon\|\theta_f x\| + \|\theta_\omega(\theta_g x - P_N\theta_g x)\| = \varepsilon\|\theta_f x\| + \|\theta_g x - P_N\theta_g x\| \\ &\leq \varepsilon\|\theta_f x\| + \delta\|\theta_g x\| = \varepsilon\|x\| + \delta\|x\| = (\varepsilon + \delta)\|x\|.\end{aligned}$$

Using triangle inequality, we then get

$$\|x\| - \|V\theta_g x\| = \|\theta_f x\| - \|V\theta_g x\| \leq \|\theta_f x - V\theta_g x\| \leq (\varepsilon + \delta)\|x\|, \quad \forall x \in \mathcal{X}.$$

Since θ_g is an invertible isometry,

$$(1 - \varepsilon - \delta)\|x\| \leq \|V\theta_g x\|, \quad \forall x \in \mathcal{X} \implies (1 - \varepsilon - \delta)\|y\| = (1 - \varepsilon - \delta)\|\theta_g^{-1}y\| \leq \|Vy\|, \quad \forall y \in \ell^p([n]),$$

i.e.,

$$\max\{1 - \varepsilon - \delta, 0\} \leq \|V\|. \tag{6}$$

Using Inequalities (5) and (6) we get

$$\max\{1 - \varepsilon - \delta, 0\} \leq \left(\max_{1\leq j,k\leq n} |f_j(\omega_k)|\right) o(M)^{\frac{1}{p}} o(N)^{\frac{1}{q}}.$$

To prove second inequality, define $W := P_N\theta_g\theta_\tau P_M : \ell^p([n]) \to \ell^p([n])$. Then for $z \in \ell^p([n])$,

$$\begin{aligned}\|Wz\|^p &= \|P_N\theta_g\theta_\tau P_M z\|^p = \left\|P_N\theta_g\theta_\tau P_M\left(\sum_{j=1}^n \zeta_j(z)e_j\right)\right\|^p = \left\|P_N\theta_g\theta_\tau\left(\sum_{j=1}^n \zeta_j(z)P_M e_j\right)\right\|^p \\ &= \left\|P_N\theta_g\theta_\tau\left(\sum_{j\in M} \zeta_j(z)e_j\right)\right\|^p = \left\|P_N\theta_g\left(\sum_{j\in M} \zeta_j(z)\theta_\tau e_j\right)\right\|^p = \left\|P_N\theta_g\left(\sum_{j\in M} \zeta_j(z)\tau_j\right)\right\|^p \\ &= \left\|\sum_{j\in M} \zeta_j(z)P_N\theta_g\tau_j\right\|^p = \left\|\sum_{j\in M} \zeta_j(z)P_N\left(\sum_{k=1}^n g_k(\tau_j)e_k\right)\right\|^p = \left\|\sum_{j\in M} \zeta_j(z)\sum_{k=1}^n g_k(\tau_j)P_N e_k\right\|^p \\ &= \left\|\sum_{j\in M} \zeta_j(z)\sum_{k\in N} g_k(\tau_j)e_k\right\|^p = \left\|\sum_{k\in N}\left(\sum_{j\in M} \zeta_j(z)g_k(\tau_j)\right)e_k\right\|^p = \sum_{k\in N}\left|\sum_{j\in M} \zeta_j(z)g_k(\tau_j)\right|^p \\ &\leq \sum_{k\in N}\left(\sum_{j\in M} |\zeta_j(z)g_k(\tau_j)|\right)^p \leq \left(\max_{1\leq j,k\leq n} |g_k(\tau_j)|\right)^p \sum_{k\in N}\left(\sum_{j\in M} |\zeta_j(z)|\right)^p \\ &= \left(\max_{1\leq j,k\leq n} |g_k(\tau_j)|\right)^p o(N)\left(\sum_{j\in M} |\zeta_j(z)|\right)^p \leq \left(\max_{1\leq j,k\leq n} |g_k(\tau_j)|\right)^p o(N)\left(\sum_{j\in M} |\zeta_j(z)|^p\right)^{\frac{p}{p}}\left(\sum_{j\in M} 1^q\right)^{\frac{p}{q}} \\ &= \left(\max_{1\leq j,k\leq n} |g_k(\tau_j)|\right)^p o(N)\left(\sum_{j\in M} |\zeta_j(z)|^p\right)^{\frac{p}{p}} o(M)^{\frac{p}{q}} \leq \left(\max_{1\leq j,k\leq n} |g_k(\tau_j)|\right)^p o(N)\left(\sum_{j=1}^n |\zeta_j(z)|^p\right)^{\frac{p}{p}} o(M)^{\frac{p}{q}} \\ &= \left(\max_{1\leq j,k\leq n} |g_k(\tau_j)|\right)^p o(N)\|z\|^p o(M)^{\frac{p}{q}}.\end{aligned}$$

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Therefore

$$\|W\| \leq \left(\max_{1\leq j,k\leq n} |g_k(\tau_j)|\right) o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}}. \tag{7}$$

Now for $x \in \mathcal{X}$,

$$\begin{aligned}\|\theta_g x - W\theta_f x\| &\leq \|\theta_g x - P_N\theta_g x\| + \|P_N\theta_g x - W\theta_f x\| \leq \delta\|\theta_g x\| + \|P_N\theta_g x - W\theta_f x\| \\ &= \delta\|\theta_g x\| + \|P_N\theta_g x - P_N\theta_g\theta_\tau P_M\theta_f x\| = \delta\|\theta_g x\| + \|P_N\theta_g(x - \theta_\tau P_M\theta_f x)\| \\ &\leq \delta\|\theta_g x\| + \|x - \theta_\tau P_M\theta_f x\| = \delta\|\theta_g x\| + \|\theta_\tau\theta_f x - \theta_\tau P_M\theta_f x\| \\ &= \delta\|\theta_g x\| + \|\theta_\tau(\theta_f x - P_M\theta_f x)\| = \delta\|\theta_g x\| + \|\theta_f x - P_M\theta_f x\| \\ &\leq \delta\|\theta_g x\| + \varepsilon\|\theta_f x\| = \delta\|x\| + \varepsilon\|x\| = (\delta+\varepsilon)\|x\|.\end{aligned}$$

Using triangle inequality and the fact that θ_f is an invertible isometry we then get

$$\max\{1-\varepsilon-\delta, 0\} \leq \|W\|. \tag{8}$$

Using Inequalities (7) and (8) we get

$$\max\{1-\varepsilon-\delta, 0\} \leq \left(\max_{1\leq j,k\leq n} |g_k(\tau_j)|\right) o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}}.$$

□

Corollary 2.5. *Let $\{\tau_j\}_{j=1}^n$ and $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Set*

$$\theta_\tau : \mathcal{H} \ni h \mapsto (\langle h, \tau_j\rangle)_{j=1}^n \in \mathbb{C}^n, \quad \theta_\omega : \mathcal{H} \ni h \mapsto (\langle h, \omega_j\rangle)_{j=1}^n \in \mathbb{C}^n.$$

If $h \in \mathcal{H} \setminus \{0\}$ is such that $\theta_\tau h$ is ε-supported on $M \subseteq \{1,\dots,n\}$ and $\theta_\omega h$ is δ-supported on $N \subseteq \{1,\dots,n\}$, then

$$o(M)o(N) \geq \frac{1}{\max_{1\leq j,k\leq n} |\langle \tau_j, \omega_k\rangle|^2}(1-\varepsilon-\delta)^2.$$

In particular, Theorem 1.3 follows from Theorem 2.4.

Proof. Define

$$f_j : \mathcal{H} \ni h \mapsto \langle h, \tau_j\rangle \in \mathbb{K}; \quad g_j : \mathcal{H} \ni h \mapsto \langle h, \omega_j\rangle \in \mathbb{K}, \quad \forall 1 \leq j \leq n.$$

Then $p = q = 2$ and $|f_j(\omega_k)| = |\langle \omega_k, \tau_j\rangle|$ for all $1 \leq j,k \leq n$. Theorem 1.3 follows by taking $\{\tau_j\}_{j=1}^n$ as the standard basis and $\{\omega_j\}_{j=1}^n$ as the Fourier basis for $\mathbb{C}^n$. □

Corollary 2.6. *Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ be two p-orthonormal bases for a finite dimensional Banach space $\mathcal{X}$. Let $x \in \mathcal{X} \setminus \{0\}$ is such that $\theta_f x$ is ε-supported on $M \subseteq \{1,\dots,n\}$ w.r.t. p-norm and $\theta_g x$ is δ-supported on $N \subseteq \{1,\dots,n\}$ w.r.t. p-norm. If $\varepsilon+\delta \leq 1$, then*

$$o(M)^{\frac{1}{p}} o(N)^{\frac{1}{q}} \geq \frac{1}{\max_{1\leq j,k\leq n} |f_j(\omega_k)|}(1-\varepsilon-\delta),$$

$$o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} \geq \frac{1}{\max_{1\leq j,k\leq n} |g_k(\tau_j)|}(1-\varepsilon-\delta).$$

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Corollary 2.7. *Let* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *and* $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ *be two p-orthonormal bases for a finite dimensional Banach space* $\mathcal{X}$*. If* $x \in \mathcal{X} \setminus \{0\}$ *is such that* $\theta_f x$ *is 0-supported on* $M \subseteq \{1, \dots, n\}$ *w.r.t. p-norm and* $\theta_g x$ *is 0-supported on* $N \subseteq \{1, \dots, n\}$ *w.r.t. p-norm (saying differently,* $\theta_f x$ *is supported on M and* $\theta_g x$ *is supported on N), then*

$$o(M)^{\frac{1}{p}} o(N)^{\frac{1}{q}} \geq \frac{1}{\max\limits_{1 \leq j,k \leq n} |f_j(\omega_k)|}, \quad o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} \geq \frac{1}{\max\limits_{1 \leq j,k \leq n} |g_k(\tau_j)|}.$$

Corollary 2.7 is not the Theorem 2.3 in [2] (it is a particular case) because Theorem 2.3 in [2] is derived for p-Schauder frames which is general than p-orthonormal bases. Theorem 2.4 promotes the following question.

Question 2.8. *Given p and a Banach space* $\mathcal{X}$ *of dimension n, for which pairs of p-orthonormal bases* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$, $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ *for* $\mathcal{X}$*, subsets* M, N *and* ε, δ*, we have equality in Inequalities (3) and (4)?*

Observe that we used $1 < p < \infty$ in the proof of Theorem 2.4. Therefore we have the following problem.

Question 2.9. *Whether there are Functional Donoho-Stark Approximate Support Uncertainty Principle (versions of Theorem 2.4) for 1-orthonormal bases and* ∞*-orthonormal bases?*

Keeping ℓ^p-spaces for $0 < p < 1$ as a model space equipped with

$$\|(a_j)_{j=1}^n\|_p := \sum_{j=1}^n |a_j|^p, \quad \forall (a_j)_{j=1}^n \in \mathbb{K}^n,$$

we set following definitions.

Definition 2.10. *Let* $\mathcal{X}$ *be a vector space over* $\mathbb{K}$*. We say that* $\mathcal{X}$ *is a* ***disc-Banach space*** *if there exists a map called as* ***disc-norm*** $\|\cdot\| : \mathcal{X} \to [0, \infty)$ *satisfying the following conditions.*

(i) *If* $x \in \mathcal{X}$ *is such that* $\|x\| = 0$*, then* $x = 0$.
(ii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathcal{X}$.
(iii) $\|\lambda x\| \leq |\lambda| \|x\|$ *for all* $x \in \mathcal{X}$ *and for all* $\lambda \in \mathbb{K}$ *with* $|\lambda| \geq 1$.
(iv) $\|\lambda x\| \geq |\lambda| \|x\|$ *for all* $x \in \mathcal{X}$ *and for all* $\lambda \in \mathbb{K}$ *with* $|\lambda| \leq 1$.
(v) $\mathcal{X}$ *is complete w.r.t. the metric* $d(x, y) := \|x - y\|$ *for all* $x, y \in \mathcal{X}$.

Definition 2.11. *Let* $\mathcal{X}$ *be a finite dimensional disc-Banach space over* $\mathbb{K}$*. Let* $\{\tau_j\}_{j=1}^n$ *be a basis for* $\mathcal{X}$ *and let* $\{f_j\}_{j=1}^n$ *be the coordinate functionals associated with* $\{\tau_j\}_{j=1}^n$*. The pair* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *is said to be a* ***p-orthonormal basis*** $(1 < p < \infty)$ *for* $\mathcal{X}$ *if the following conditions hold.*

(i) $\|f_j\| = \|\tau_j\| = 1$ *for all* $1 \leq j \leq n$.
(ii) *For every* $(a_j)_{j=1}^n \in \mathbb{K}^n$,

$$\left\|\sum_{j=1}^n a_j \tau_j\right\| = \sum_{j=1}^n |a_j|^p.$$

Then we also have the following question.

Question 2.12. *Whether there are versions of Theorem 2.4 for p-orthonormal bases* $0 < p < 1$*?*

We wish to mention that in [2] the functional uncertainty principle was derived for p-Schauder frames which is general than p-orthonormal bases. Thus it is desirable to derive Theorem 2.4 or a variation of it for p-Schauder frames, which we can't.

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We end by asking the following curious question whose motivation is the recently proved Balian-Low theorem (which is also an uncertainty principle) for Gabor systems in finite dimensional Hilbert spaces [5–7].

Question 2.13. ***Whether there is a Functional Balian-Low Theorem (which we like to call Functional Balian-Low-Lammers-Stampe-Nitzan-Olsen Theorem) for Gabor-Schauder systems in finite dimensional Banach spaces (Gabor-Schauder system is as defined in [4])?***

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