Article

Not peer-reviewed version

Particle in a Markov Cube by the Non-Classical Information Entropy Space

Elias Koorambas *

Posted Date: 26 June 2023

doi: 10.20944/preprints202306.1760.v1

Keywords: Quantum confinement; Markov numbers; Quantum potential; Quantum jumps; Quantum–classical correspondence

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 June 2023 doi:10.20944/preprints202306.1760.v1

Article

Particle in a Markov Cube by the Non-Classical Information Entropy Space

E Koorambas

Computational Applications, Group, Division of Applied Technologies, National Center for Science and Research 'Demokritos', Aghia Paraskevi-Athens, Greece
* Correspondence: elias.koor@gmail.com

Abstract: Starting with a particle in the cube system, we investigate the exotic possibility that the quantum numbers (n_x,n_y,n_z) satisfy the Markov equation. We find that the proposed particle in the Markov-cube system follows from a non-classical information entropic space by the quantum potential Q. We study the classical correspondents of the proposed particle in the Markov-cube system. According to our proposition, this particle identified by detailed measurements of the confinement particle-energy levels, is an artefact or occurs in nature.

Keywords: quantum confinement; Markov numbers; quantum potential; quantum jumps; quantum–classical correspondence

1. Introduction

The particle-in-the-box model it is commonly used as an approximation for more complicated quantum systems in which a particle is trapped in a narrow region of low electric potential between two high-potential barriers. These quantum well systems are important in optoelectronics [1–9]; they are used in devices such as the quantum well laser, the quantum well-infrared photodetector, and the quantum-confined Stark effect modulator [1–9]. The particle-in the-box model is also used to model a lattice in the Kronig-Penney model and for a finite metal with the free electron approximation [1–9]. Recently, researchers in China (C.Geng, X.Di, X.Tang, H.Han, 2023: [10]) developed a way to improve the quality of Cherenkov images using a flexible, non-toxic sheet of carbon quantum dots (cQDs) attached to the patient [11].

A Quantum Dot (QD) is a crystal of semiconductor material whose diameter is on the order of several nanometers. At this size, the free charge carriers experience "quantum confinement" in all three spatial dimensions [5]. QD particles are present inside the semiconductors used in microelectronics.These particles may contain an electron and a "hole" (absence of an electron); therefore, they are real-world "particle in a box" systems. Indeed, the electrons will never leave the interior of the QD particle of the semiconductor [5]. We note that, in QDs, the effects of particle size on the energy levels of the system can be easily investigated.

Santamato (1980 [12]) formulated de Broglie-Bohm quantum mechanics by relating the quantum potential to fundamental geometrical properties in Weyl's geometry [13,14]. Based on the Weyl picture, J.T. Wheeler (1990 [15]) proposed a new picture of quantum theory in the context of the standard interpretation. Based on Weyl space, R. Wood and G. Papini (1995[4]16) developed a modified Weyl-Dirac theory that joins the particle aspects of matter and Weyl symmetry breaking. Novello, Salim and Falciano (2011[24]17) showed that there is a close connection between the de Broglie-Bohm interpretation of quantum mechanics and the Weyl integrable space. They demonstrated that Bohm's quantum potential Q can be identified with the curvature scalar of the Weyl integrable space.

More recently, G.Resconi, I.Licata, and D.Fiscaletti (2013 [18,19]) showed that Bohm's quantum potential Q can be derived from the minimum condition of Fisher information connected to the entropy S of a quantum system. In the non-classical entropic picture, the curvature of space indicates the Weyl-like gauge potential B_μ connected with the Fisher metric. Since Bohm's quantum potential

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 June 2023 doi:10.20944/preprints202306.1760.v1

Q is a consequence of the Schrödinger Equation (SE) and vice versa, one can give a thermodynamic justification of the quantum mechanics modeled by the SE [18,19].

The Markov numbers are the positive integers that appear in the solutions of the equation: $m_1^2 + m_2^2 + m_3^2 = 3m_1 m_2 m_3$, where $(m_1, \mathrm{m}_2, \mathrm{m}_3) \in \mathbb{N}^3$ (Markov1879 [20]). These numbers are a classical subject in number theory, and have important ramifications in hyperbolic geometry, algebraic geometry, and combinatorics [21–24]. D.Zagier (1982 [25]) showed that the number M(x) of Markov numbers below the real number x is:

$$M(x) = c(\log x)^2 + O(\log x(\mathrm{loglogx})^2), \tag{1}$$

where $c = 0.180717104711507 \ldots$ is an explicit constant [25].

In the present article, we are investigating the exotic possibility that the quantum numbers (n_x,n_y,n_z) satisfy the Markov equation for a particle in the cube system, which we call the Markov cube. Based on Resconi,Licata, and Fiscaletti (2013 [18,19]), we find that the proposed particle follows from a non-classical information entropic space by the quantum potential Q. We also study the classical correspondents for this particle.

2. Particle in a Markov cube

The energy levels of the particle in the box system are given by:

$$E_{n_x,n_y,n_z} = \frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right), \tag{2}$$

where

$$n_x = 1, 2, 3, \ldots \infty, \qquad n_y = 1, 2, 3, \ldots \infty, \qquad n_z = 1, 2, 3, \ldots \infty \tag{3}$$

are the triplet of quantum numbers $(n_x, n_y, n_z) \in \mathbb{N}^3$. In the case of cubic symmetry $(L = L_x = L_y = L_z)$, Equation (2) becomes:

$$(E_{n_x,n_y,n_z})_{Cube} = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right). \tag{4}$$

The value of energy levels with the corresponding combinations and sum of the squares of the quantum numbers

$$n^2 = n_x^2 + n_y^2 + n_z^2, \tag{5}$$

is shown in Table 1.

Table 1. The value of the energy levels given by Equation (4), with the corresponding combinations and sum of the squares of the quantum numbers for an electron inside a cube of length ($L = 10^{-10} m$). E_n is the energy of the cube system, m_e is the electron mass ($m_e = 9.1093837015 \cdot 10^{-31} Kg$), and h is the reduced Planck constant ($h = 1.055 \cdot 10^{-34} J \cdot s$).

(n_x, n_y, n_z)	$n_x^2 + n_y^2 + n_z^2$	E_n (in eV)	Degree of Degeneracy
(1,1,1)	3	113	1
(2,1,1)	6	226	3
(2,2,1)	9	339	3
(3,1,1)	11	414	3
(2,2,2)	12	452	1

(3,1,2)	14	527	6
(3,2,2)	17	640	3
(4,1,1)	18	677	3
(4,1,2)	21	790	6

The energy levels of the particle in a Markov-Cube system is given by:

$$(E_{m_x,\mathrm{m}_y,\mathrm{m}_z})_{Markov-Cube} = \frac{h^2}{8mL^2}(\mathrm{m}_x^2 + m_y^2 + m_z^2) = \frac{3h^2}{8mL^2}(\mathrm{m}_x\, m_y m_z)\,, \quad (6)$$

where

$$m_x^2 + m_y^2 + m_z^2 = 3m_x m_y m_z\,, \quad (7)$$

is the Markov equation satisfied by the triplet of quantum numbers $\left(m_x,\mathrm{m}_y,\mathrm{m}_z\right) \in \mathbb{N}^3$. The first few solutions of Equation (7) are the following quantum numbers:

$$\left(m_x,\mathrm{m}_y,\mathrm{m}_z\right) = \{(1,1,1),(1,1,2),(1,2,5),(1,5,13),(2,5,29)\}\ .\ [26] \quad (8)$$

The first two of these solutions {(1, 1, 1), (1, 1, 2)} can generate all other solutions: since Equation (7) is a quadratic in each of the variables, one integer solution leads to a second, and all solutions other than the first two singular ones have distinct values of m_x, m_y, and m_z, and share two of their three values with three other solutions (Guy 1994, p. 166 [27]). The Markov quantum numbers are then given by 1, 2, 5, 13, 29, 34 ... (OEIS A002559) [28]. The value of energy levels for a particle in a Markov Cube (see Equation (6)), with the corresponding combinations and sum of the squares of the quantum numbers are shown in Table 2.

Table 2. The value of energy levels for a particle in a Markov Cube (see Equation (6)), with the corresponding combinations and sum of the squares of the quantum numbers. For an electron inside a cube of length $L = 10^{-10}m$, E_m is the energy of the cube system (6), m_e is the electron mass ($m_e = 9.1093837015 \cdot 10^{-31} Kg$), and h is the reduced Planck constant $h = 1.055 \cdot 10^{-34} J \cdot s$).

$\left(m_x,\mathrm{m}_y,\mathrm{m}_z\right)$	$m_x^{\ 2} + m_y^{\ 2} + m_z^{\ 2}$	E_m **(in eV)**	**Degree of Degeneracy**
(1,1,1)	3	113	1
(2,1,1)	6	226	3
(2,5,1)	30	1130	6
(13,5,1)	195	7338	6
(2,5,29)	870	32741	6
(13,34,1)	1326	49902	6
(2,169,29)	29406	1106652	6
(13,5,194)	37830	1423677	6
(433,5,29)	18835	7088468	6

Following Ref [26,29], we introduce the Quantum Vieta Jumping (QVJ) involutions as follows: If (m_x,m_y,m_z) is a Markov triple of quantum numbers, then, by the QVJ proposition, we have (m_x,m_y,$3m_xm_y - m_z$). Applying this quantum involution twice returns the same triple we started with. Joining each normalised Markov triple to the first, second, or third normalised triples we obtain from QVJ involution gives a graph starting from the ground state (1,1,1) of the quantum system (Figure 1). This graph is connected; in other words, every Markov triple can be connected to the (1,1,1) ground state of the quantum system by a sequence of QVJ involutions . For instance, starting with the (1,5,13) quantum state, we get its three neighbours, the (5,13,194), (1,13,34) and (1,2,5) quantum states, in the Markov quantum tree (Figure 1).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 June 2023 doi:10.20944/preprints202306.1760.v1

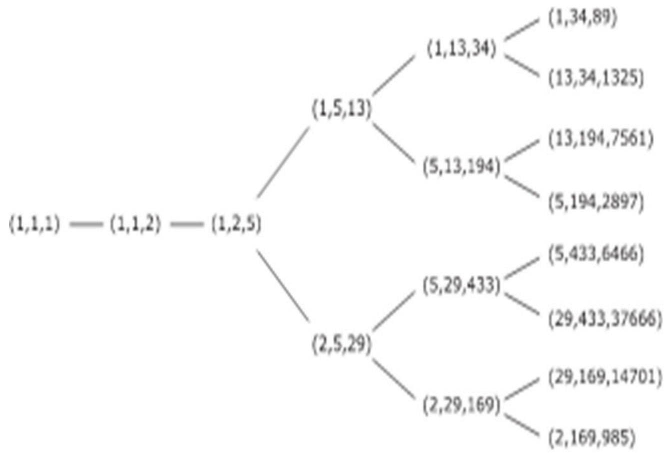

Figure 1. The Markov quantum tree (Matheus, and Moreira, 2021 [26]).

From Tables 1 and 2, we observe that the Markov cube system and the ordinary cube system have the (1,1,1) ground state and the first excitation (2,1,1) in common, while all the other excitations are strongly divergent: for example, the difference between the nine excitation levels of the Markov and ordinary cube systems is in the order of O(MeV). This suggests that, at the O(eV) energy scale, in the Markov cube system, the particle is trapped in the first two energy levels; as we know, however, in the ordinary cube system, the particle can be at higher excitation energy levels. This strong divergence between the excitation energy levels may be an experimental signature of the proposed Markov cube system.

3. The Markov particle by non-classical information entropy space

Following G. Resconi, I.Licata, and D.Fiscaletti (2013 [18,19]), in the de Broglie-Bohm interpretation of quantum mechanics, the quantum potential emerges as an information medium determined by the vector of the superposition of entropies:

$$\begin{aligned} S_1 &= k_B \log W_1\left(\theta_1,\theta_2,\ldots,\theta_p\right) \\ S_2 &= k_B \log W_2\left(\theta_1,\theta_2,\ldots,\theta_p\right) \\ &\ldots \\ S_n &= k_B \log W_n\left(\theta_1,\theta_2,\ldots,\theta_p\right) \end{aligned} \quad [18,19], \tag{9}$$

where W is the number of the microstates for the same parameters, θ (temperature, pressure, etc.), and $k_B = 1.38\times10^{-23} J/K$ is the Boltzmann constant. In this picture, the quantum effects are equivalent to the geometry described by the following equation:

$$\frac{\partial}{\partial x^k} + \frac{\partial^2 S_j}{\partial x^k \partial x^p}\frac{\partial x^i}{\partial S_j} = \frac{\partial}{\partial x^k} + k_B\frac{\partial W_j}{\partial x_h} = \frac{\partial}{\partial x^k} + B_{j,h}, \tag{10}$$

where

$$B_{j,h} = \frac{\partial S_j}{\partial x_h} = k_B\frac{\partial \log W_j}{\partial x_h} = \frac{k_B}{W_j}\partial_h W_j, \tag{11}$$

is a Weyl-like gauge potential [30,31]. A deformation of the moments p for the non-local change of this geometry is represented by the following expression:

$$A=\int\rho\left[\frac{\partial A}{\partial t}+\frac{1}{2m}\left(p_i+B_{k,i}\right)\left(p_j+B_{k,\mathrm{j}}\right)+V\right]dtd^nx\,, \tag{12}$$

which gives

$$A=\int\rho\left[\frac{\partial A}{\partial t}+\frac{1}{2m}\left(p_ip_j+B_{k,i}B_{k,\mathrm{j}}\right)+V\right]dtd^nx\quad, \tag{13}$$

namely

$$A=\int\rho\left[\frac{\partial A}{\partial t}+\frac{1}{2m}\left(p_ip_j+\frac{\partial\log W_k}{\partial x_i}\frac{\partial\log W_k}{\partial x_j}\right)+V\right]dtd^nx\,, \tag{14}$$

that is,

$$A=\int\rho\left[\frac{\partial A}{\partial t}+\frac{1}{2m}p_ip_j+V+\frac{1}{2m}\left(\frac{\partial\log W_k}{\partial x_i}\frac{\partial\log W_k}{\partial x_j}\right)\right]dtd^nx\,. \tag{15}$$

The quantum action assumes the minimum value when $\delta A=0$, namely:

$$\delta\int\rho\left[\frac{\partial A}{\partial t}+\frac{1}{2m}p_ip_j+V\right]dtd^nx+\delta\int\frac{\rho}{2m}\left(\frac{\partial\log W_k}{\partial x_i}\frac{\partial\log W_k}{\partial x_j}\right)dtd^nx=0\,, \tag{16}$$

and thus

$$\begin{aligned}&\frac{\partial A}{\partial t}+\frac{1}{2m}p_ip_j+V(x,y,z)+\frac{1}{2m}\left(\frac{1}{W_k^2}\frac{\partial W_k}{\partial x_i}\frac{\partial W_k}{\partial x_j}-\frac{2}{W_k}\frac{\partial^2 W_k}{\partial x_i\partial x_j}\right)\\&=\frac{\partial S_k}{\partial t}+\frac{1}{2m}p_ip_j+V(x,y,z)+Q\end{aligned}\,, \tag{17}$$

where

$$Q=\frac{1}{2m}\left(\frac{1}{W_k^2}\frac{\partial W_k}{\partial x_i}\frac{\partial W_k}{\partial x_j}-\frac{2}{W_k}\frac{\partial^2 W_k}{\partial x_i\partial x_j}\right)\text{[18]}, \tag{18}$$

Q is the Bohm quantum potential that is a consequence for the extreme condition of Fisher information [18]. On the basis of Ref [18], we interpret Q as an information channel determined by the functions W_k that define the number of microstates of the physical system under consideration. The latter depend on the parameters θ of the distribution probability. Now, by the definition of the quantum potential as an informational entity that pertains to the geometry (18), the quantum Hamilton-Jacobi equation is given by:

$$\frac{|\nabla S|^2}{2m}+V(x,y,z)+Q=-\frac{\partial S}{\partial t}\,. \tag{19}$$

Equation (19) provides a new way to read the energy conservation law in quantum mechanics in the non-classical entropic picture, where the curvature of space indicates the Weyl-like gauge potential B_μ connected with the Fisher metric [18,19]. For the particle in the cube system, it is:

$$V(x,y,z)=\begin{Bmatrix} 0,\, 0<x<L, 0<y<L, 0<z<L \\ \infty, elsewhere \end{Bmatrix}. \tag{20}$$

Physically, we set $V(x,y,z)=\infty$, so there is no chance that the particles penetrate the cube. This means that the wave function $\psi(x,y,z)$ must be exactly zero in the region where $V(x,y,z)=\infty$, making it impossible for the particle to occur beyond the cube.

Following D.Bohm (1951 [32,33]), and using Equation (18), we evaluate the quantum potential Q for the particle of mass m inside the cube, where $0<x<L$, $0<y<L$, $0<z<L$, as follows:

$$\begin{aligned} Q_{n_x,n_y,n_z} &= \frac{1}{2m}\left(\left(\frac{\nabla W}{W}\right)^2 - 2\frac{\nabla^2 W}{W}\right) = -\frac{\hbar^2}{2m}\frac{\nabla^2\psi(x,y,z)}{\psi(x,y,z)} \\ &= \frac{1}{2m}\left[\left(\frac{\pi n_x \hbar}{L}\right)^2 + \left(\frac{\pi n_y \hbar}{L}\right)^2 + \left(\frac{\pi n_z \hbar}{L}\right)^2\right] = \frac{\pi^2\hbar^2}{2mL^2}\left(n_x^2+n_y^2+n_z^2\right) \end{aligned}, \tag{21}$$

where

$$\psi(x,y,z)=\sin\left(\frac{n_x\pi}{L}x\right)\sin\left(\frac{n_y\pi}{L}y\right)\sin\left(\frac{n_z\pi}{L}z\right) [33], \tag{22}$$

and the triplet of quantum numbers $(n_x,n_y,n_z)\in\mathbb{N}^3$. Now, following D.Zagier (1982 [25]), for a particle in the cube system, the number $M(W_j)$ of the quantum Markov triplet (m_x,m_y,m_z) below the number of microstates W_j of the physical system is:

$$M(\mathrm{W}_j)=c(\log \mathrm{W}_j)^2 + O(\log \mathrm{W}_j(\log\log \mathrm{W}_j)^2), \tag{23}$$

where $c = 0.180717104711507\ldots$ is an explicit constant [25,26]. Using Equation (9), the number $M(W_j)$ of quantum the Markov triplet (m_x,m_y,m_z) below the number of microstates W_j of the physical system can be written in terms of the quantum entropy S_j:

$$M(\mathrm{W}_j)\approx cS^2_{(Markov)}(\mathrm{W}_j). \tag{24}$$

The Weyl-like gauge potential $B_{j,h}$ (Equation (11)) can be written in terms of the number $M(W_j)$ of the quantum Markov triplet (m_x,m_y,m_z) below the number of microstates W_j of the physical system, as follows:

$$B^{(Markov)}_{j,h}=\frac{\partial S^{(Markov)}_j}{\partial x_h}=\frac{k_B}{2\sqrt{cM(W_j)}}\partial_h M(W_j). \tag{25}$$

By the Equation (11), the following condition holds:

$$B_{j,h}=B^{(Markov)}_{j,h}; \tag{26}$$

namely,

$$\frac{k_B}{W_j}\partial_h W_j = \frac{k_B \partial_h M(W_j)}{2\sqrt{cM(W_j)}}. \tag{27}$$

Thus

$$\partial_h W_j = \frac{W_j}{2\sqrt{cM(W_j)}}\partial_h M(W_j). \tag{28}$$

Using Equation (28), the quantum potential Q given by Equation (21) can be written in terms of the number $M(W_j)$ of the quantum Markov triplet (m_x, m_y, m_z) below the number of microstates W_j of the physical system, as follows:

$$\begin{aligned} Q_{m_x,\mathrm{m}_y,\mathrm{m}_z} &= \frac{1}{2m}\left(\frac{(\nabla M)^2}{4cM} - \frac{\nabla^2 M}{\sqrt{cM}}\right) = -\frac{\hbar^2}{2m}\frac{\nabla^2 \psi^{(Markov)}(x,y,z)}{\psi^{(Markov)}(x,y,z)} \\ &= \frac{1}{2m}\left[\left(\frac{\pi m_x \hbar}{L}\right)^2 + \left(\frac{\pi m_y \hbar}{L}\right)^2 + \left(\frac{\pi m_z \hbar}{L}\right)^2\right] = \frac{\pi^2\hbar^2}{2mL^2}\left(m_x^2 + m_y^2 + m_z^2\right) = \frac{3\pi^2\hbar^2}{2mL^2} m_x m_y m_z \end{aligned}, \tag{29}$$

where

$$\psi^{(Markov)}(x,y,z) = \sin\left(\frac{m_x \pi}{L}x\right)\sin\left(\frac{m_y \pi}{L}y\right)\sin\left(\frac{m_z \pi}{L}z\right), \tag{30}$$

and the Markov triplet of the quantum numbers $(m_x, \mathrm{m}_y, \mathrm{m}_z) \in \mathbb{N}^3$ satisfies the Markov Equation (7).

4. The quantum–classical correspondence

The classical limit of quantum mechanics remains an unresolved problem [34]. Planck's h → 0 limit [35] and Bohr's n →∞ limit [36] are the oldest proposals for the formulation of the classical limit of quantum theory. For a particle of mass m in the cube system, from Equation (4) we note that the energy between adjacent levels is not constant [37]:

$$E_{n+1} - E_n = 2n + 1. \tag{31}$$

This leads to

$$\frac{E_{n+1} - E_n}{E_n} = \frac{(n+1)^2 - n^2}{n^2} = \frac{2n+1}{n^2}. \tag{32}$$

In the Bohr's limit n →∞,

$$\lim_{n\to\infty}\frac{E_{n+1} - E_n}{E_n} = \lim_{n\to\infty}\frac{2n+1}{n^2} = 0. \tag{33}$$

The adjacent energy levels become so close as to be practically indistinguishable.For a particle of mass m in the proposed Markov cube system, the situation is very different than that in the cube system due to the QVJ involutions between adjacent levels. Let the QVJ involution acting on the Markov quantum triplet $(m_x, \mathrm{m}_y, \mathrm{m}_z) \in \mathbb{N}^3$ be as follows:

$$QVJ:\left(m_x,m_y,m_z\right)\to\left(m'_x,m'_y,m'_z\right)=\left(m_x,m_y,3m_xm_y-m_z\right), \quad (34)$$

where

$$m'_x=m_x\,,\;\; m'_y=m_y\;\; m'_z=3m_xm_y-m_z\,. \quad (35)$$

We calculate:

$$\begin{aligned} &m'^2-m^2=m'^2_x+m'^2_y+m'^2_z-\left(m_x^2+m_y^2+m_z^2\right)=(m_x^2-m'^2_x)+(m_y^2-m'^2_y)+(m_z^2-m'^2_z)\\ &=m_z^2-\left(3m_xm_y-m_z\right)^2=-9m_x^2m_y^2+6m_xm_ym_z \end{aligned} \quad (36)$$

Therefore, we have

$$\begin{aligned} &\left(\frac{E_{m'}-E_m}{E_m}\right)_z=\left(\frac{m'^2-m^2}{m^2}\right)_z=\frac{-9m_x^2m_y^2+6m_xm_ym_z}{m_x^2+m_y^2+m_z^2}\\ &=\frac{-9m_x^2m_y^2+6m_xm_ym_z}{3m_xm_ym_z}=-\frac{3m_xm_y}{m_z}+2 \end{aligned}, \quad (37)$$

similarly:

$$\left(\frac{E_{m'}-E_m}{E_m}\right)_y=\left(\frac{m'^2-m^2}{m^2}\right)_y=-\frac{3m_xm_z}{m_y}+2\,, \quad (38)$$

$$\left(\frac{E_{m'}-E_m}{E_m}\right)_x=\left(\frac{m'^2-m^2}{m^2}\right)_x=-\frac{3m_ym_z}{m_x}+2\,. \quad (39)$$

In the Bohr's limit n →∞,

$$\lim_{m_z\to\infty}\left(\frac{E_{m'}-E_m}{E_m}\right)_z=\lim_{m_y\to\infty}\left(\frac{E_{m'}-E_m}{E_m}\right)_y=\lim_{m_x\to\infty}\left(\frac{E_{m'}-E_m}{E_m}\right)_x=2\,. \quad (40)$$

Thus, for a Markov quantum state in the proposed Markov cube system, a classical limit n →∞ does not exist.

5. Conclusions

G.Resconi,, I.Licata,, and D.Fiscaletti (2013 [18,19]) derived Bohm potential Q from the minimum condition of Fisher information connected to the entropy S of a quantum system. In the non-classical entropic picture, the curvature of space indicates the Weyl-like gauge potential B_μ connected with the Fisher metric. Based on this interpretation of the Bohm potential Q, we show that the Markov cube system follows from a non-classical information entropic space by the quantum potential. We find that a classical limit n →∞ does not exist for a Markov quantum state due to the quantum Vieta jumping involutions between adjacent levels.

References

1. Bransden, B. H, Joachain, C. J. (2000). Quantum mechanics . Essex: Pearson Education. ISBN 978-0-582-35691-7.
2. Davies, John H. (2006). The Physics of Low-Dimensional Semiconductors: An Introduction (6th reprint ed.). Cambridge University Press.
3. Griffiths, David J. (2004). Introduction to Quantum Mechanics (2nd ed.). Prentice Hall. ISBN 978-0-13-111892-8.
4. Rice, C.V; Griffin, G.A. (2008). "Simple Syntheses of CdSe Quantum Dots". Journal of Chemical Education. **85** (6): 842.doi:10.1021/ed085p842.
5. Lodovico.Lappetito, Quantum Dots: a True "Particle in a Box" System". Physics Open Lab. (20 November 2015).
6. Overney, René M. "Quantum Confinement" .University of Washington. November 2016.
7. Zahn, Dietrich R.T. "Surface and Interface Properties of Semiconductor Quantum Dots by Raman Spectroscopy" . Technische Universität Chemnitz. Retrieved 5 November 2016.
8. Bentolila, Laurent A.; Ebenstein, Yuval (2009). "Quantum Dots for In Vivo Small-Animal Imaging". Journal of Nuclear Medicine. **50** (4): 49496. doi:10.2967/jnumed.108.053561.
9. Alberto, P, Fiolhais, C; Gil, V M S (1996). "Relativistic particle in a box" . European Journal of Physics. **17** (1): 19–24 doi:10.1088/0143-0807/17/1/004.
10. C.Geng, X.Di, X.Tang, H.Han Med. Phys. (2023) doi:10.1002/mp.16121.
11. C.E Keen, A sheet of quantum dots enhances Cherenkov imaging of radiotherapy dose Physics World, (12 Jan 2023) IOP Publishing.
12. D.Fiscaletti,The Geometry of Quantum Potential ISBN: 9789813227972 (Print) 19 pages, 9789813227989 (eBook) Published: 2018 by WSP, doi: 10.1142/10653.
13. Weyl, H.: Sitzber. Preuss. Akad. Wiss. Berlin, 465 (1918).
14. Weyl, H.: Space, Time, Matter. Dover, New York (1922).
15. Wheeler, J.T.: Quantum measurement and geometry. Phys. Rev. D **41**(2), 431 (1990).
16. Wood, W.R., Papini, G.: A geometric approach to the quantum mechanics of de-Broglie–Bohm and Vigier. In: The Present Status of Quantum Theory of Light, Proc. of Symposium in Honor of J.P. Vigier,York University, 27–30 Aug. (1995).
17. Novello, M., Salim, J.M., Falciano, F.T.: On a geometrical description of quantum mechanics. Int. J. Geom. Methods Mod. Phys. **8**(1), 87–98 (2011).
18. Fiscaletti, D.; Licata, I. Weyl geometries, Fisher information and quantum entropy in quantum mechanics. Int. J. Theor. Phys. 2012, **51**, 3587–3595.
19. G.Resconi,I.Licata,D.Fiscaletti, Entropy 2013, **15**, 3602-3619.
20. A. Markoff. Sur les formes quadratiques binaires ind'efinies. Math. Ann. **15**, (1879) 381–406.
21. R C. Reutenauer, From Christoffel words to Markoff numbers. Oxford University Press, Oxford, (2019).
22. A M. Aigner, Markov's theorem and 100 years of the uniqueness conjecture. A mathematical journey from irrational numbers to perfect matchings. Springer, Cham, (2013).
23. K.Lee, L.Li, M.Rabideau, R.Schiffler, On the ordering of the Markov numbers, arXiv:2010.13010 [math.NT] (11 Oct 2022).
24. G. Frobenius, Uber die Markoffschen Zahlen, S. B. Preuss. Akad. Wiss. Berlin (1913) 458–487; available ¨ in Gesammelte Abhandlungen Band III, Springer (1968) 598–627.
25. D .Zagier, On the number of Markoff numbers below a given bound, Math. Comp. 39 (1982), no. **160**, 709–723. MR669663.
26. C.Matheus ,C.G.Moreira, Diophantine Approximation, Lagrange and Markov Spectra, and Dynamical Cantor Sets, Notices of the American mathematical society68,**8**,(2021),1301-1311.
27. Guy, R. K. "Markoff Numbers." §D12 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 166-168, 1994.
28. Sloane, N. J. A. Sequences A002559/M1432 and A030452 in "The On-Line Encyclopedia of Integer Sequences.".
29. Weisstein, Eric W. "Markov Number." From MathWorld—AWolfram.
30. Castro, C., Mahecha, J.: On nonlinear quantum mechanics, Brownian motion, Weyl geometry and Fisher information. Prog. Phys. **1**, 38–45 (2006).
31. Falciano, F.T., Novello, M., Salim, J.M.: Geometrizing relativistic quantum mechanics. Found. Phys. **40**, 1885–1901 (2010).
32. Bohm, David (1952). "A Suggested Interpretation of the Quantum Theory in Terms of "Hidden Variables" I". Physical Review. **85** (2): 166–179.
33. Bohm, David (1952). "A Suggested Interpretation of the Quantum Theory in Terms of "Hidden Variables", II". Physical Review. **85** (2): 180–193.
34. Choi JR (2021) Classical Limit of Quantum Mechanics for Damped Driven Oscillatory Systems: Quantum–Classical Correspondence. Front. Phys. 9:670750. doi: 10.3389/fphy.2021.670750.

 doi:10.20944/preprints202306.1760.v1

35. Planck M. Vorlesungen uber die Theorie der Wärmestralhung. 978- 0270103878. New York: Dover Publications (1959).
36. Bohr N. The Theory of Spectra and Atomic Constitution. 978-1107669819. London: Cambridge University Press (1922).
37. N.Zettili,Quantum mechanics: concepts and applications, John Wiley and Sons Ltd, England (2001).