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Article

General Quantum Gravity

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Abstract: General Quantum Gravity (GQG) is a formalization of quantized gravity that emerges from General Relativity through Quantum Mechanics. GQG is developed as a sense of bosonic and fermionic fields that additionally provides us Dark Energy and Dark Matter. Here, we have developed two different aspects of GQG, such as: 'Four-velocity' Comprised Theory of GQG and 'Four-momentum' Comprised Theory of GQG. In the former one, Hilbert-Einstein field equation is developed in a quantum-Riemannian spacetime, whereas, the latter one gives us a Hilbert-Einstein field equation in a purely quantum non-Riemannian spacetime. In GQG, gravity is the bending of spacetime intermediated by gravitons in its GQG field, whose geometric part bends spacetime, whereas its quantum part interacts with spacetime by exchanging gravitons. In GQG's generalized forms, the bosonic and fermionic fields (which are emerged either from Hilbert-Einstein field equations or from line elements of Minkowski spacetime) argue that the Klein-Gordon equation is a subset of the second order equations of GQG, whereas, the Dirac equation is a subset of the first order equations of GQG.

Keywords: quantum gravity; general relativity; dark energy; dark matter

1. Introduction

Only at Planck scales ($\sim 10^{19}$ GeV), the quantum effects of gravity is believed to be showed up. The authors of Ref. [1] proposed that the quantum effects of gravity should be testable at laboratory scales without regarding Planck scales, and in this context, they also proposed in their paper that the possibility of looking for the effects of Dark Energy at atomic (i.e., laboratory) scales.

Cosmological observations are inconsistent with Einstein's equations of General Relativity in the absence of Dark Energy (or Λ) and Dark Matter. If we extend the idea of Ref. [2] as follows: "It is also theoretically possible that the cosmological constant problem could be resolved by replacing General Relativity with an alternative theory of gravity, with no dark components being imposed separately but comprised within the explanation of this alternative theory of gravity", then we can able to develop a formalism of General Quantum Gravity (GQG), where, the above idea of Ref. [1] is also being included.

In GQG, we describe gravity through Quantum Mechanics without considering Planck scales in general. But, if we consider Planck scales in this quantum gravity formalism, our proposed scenarios immediately develop a sense of bosonic and fermionic fields for both Dark Energy and Dark Matter quite naturally without presuming any additional conditions, such as supersymmetry, superstrings, etc.

2. General Quantum Gravity

General Quantum Gravity (GQG) is a formalization of quantized gravity that emerges from General Relativity through Quantum Mechanics. In the formalism of GQG, we are going to develop two different aspects of GQG, such as:

1. 'Four-velocity' Comprised Theory of GQG, and
2. 'Four-momentum' Comprised Theory of GQG.

In the former one, GQG Hilbert-Einstein field equation is a classical-like Hilbert-Einstein field equation in quantum-Riemannian spacetime, whereas, the latter one gives us a purely Quantum Mechanical

Hilbert-Einstein field equation in a non-Riemannian quantum spacetime. But in both cases, we always get the classical Schrödinger equation as a byproduct, though it is now in a $(3+1)D$ quantum spacetime.

No one ever ask whether Klein-Gordon and Dirac equations are the part of any large scenario. Here, we generalizes that the bosonic and fermionic fields (which are emerged either from Hilbert-Einstein field equations or from line elements of Minkowski spacetime, which is an impossible thing in conventional physics) argue that the Klein-Gordon equation is a subset of the second order equations of our quantum gravity, whereas, the Dirac equation is a subset of the first order equations of our quantum gravity. All basic bosonic and fermionic fields, including baryonic matters, Dark Energy and Dark Matter, which are obtainable from GQG are listed in the Table 1.

Table 1. In General Quantum Gravity, we have twelve staple bosonic and fermionic field equations in two different orders.

'Four-velocity' Comprised:	First Order Equations
	$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t)+i\,\hbar\,\mathrm{v}^{j}\,\vec{\nabla}_{i}\,\psi(\vec{r},t)=0$ $i\,\hbar\,\mathrm{v}^{\nu}\,\vec{\Delta}_{\mu}\,\psi(\vec{r},t)+i\,\hbar\,g_{\mu\nu}\,\mathrm{v}^{\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t)=0$ $i\,\hbar\,\gamma^{\mu}\left[\mathrm{R}_{\zeta\eta}-\frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{(1/2)}\vec{\nabla}_{\mu}\,\psi(\vec{r},t)-\mathcal{U}^{(1/2)}\,\mathcal{P}\,\psi(\vec{r},t)=0$
	Second Order Equation
	$\hbar^{2}\left[\mathrm{R}_{\zeta\eta}-\frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]\Box\,\psi(\vec{r},t)+\mathcal{U}\,\mathcal{P}^{2}\,\psi(\vec{r},t)=0$
'Four-momentum' Comprised:	First Order Equation
	$i\,\hbar\left(\frac{\ell_{P}^{\zeta}}{m_{P}^{3}}\right)^{(1/2)}\gamma^{\zeta}\,\vec{\nabla}_{P\,\zeta}\,\psi(\vec{r}_{P},t_{P})-\mathscr{C}\,\psi(\vec{r}_{P},t_{P})=0$ $i\,\hbar\,\gamma^{\mu}\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t)-m_{0}\,\psi(\vec{r},t)=0$ $i\,\hbar\,\sqrt{2\Lambda}\,\gamma^{\mu}\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t)-\langle T_{0}\rangle^{(1/2)}\,\psi(\vec{r},t)=0$
	Second Order Equations
	$-\,\hbar^{2}\,\vec{\nabla}_{0}\,\vec{\nabla}^{0}\,\psi(\vec{r},t)+\hbar^{2}\,\vec{\nabla}_{i}\,\vec{\nabla}^{i}\,\psi(\vec{r},t)=0$ $-\,\hbar^{2}\,\vec{\Delta}_{\mu}\,\vec{\Delta}_{\nu}\,\psi(\vec{r},t)+\hbar^{2}\,g_{\mu\nu}\,\vec{\nabla}_{\mu}\,\vec{\nabla}_{\nu}\,\psi(\vec{r},t)=0$ $\hbar^{2}\,\frac{\ell_{P}^{\zeta}}{m_{P}^{3}}\,\Box_{P}\,\psi(\vec{r}_{P},t_{P})+\mathscr{C}^{2}\,\psi(\vec{r}_{P},t_{P})=0$ $\hbar^{2}\,\mathrm{g}_{\mu\nu}^{2}\,\Box\,\psi(\vec{r},t)+m_{0}^{2}\,\psi(\vec{r},t)=0$ $2\,\hbar^{2}\,\Lambda\,\mathrm{g}_{\mu\nu}^{2}\,\Box\,\psi(\vec{r},t)+\langle T_{0}\rangle\,\psi(\vec{r},t)=0$

Two different aspects of GQG are as follows.

2.1. *'Four-velocity' Comprised Theory of GQG*

Let the line element of Minkowski spacetime,

$$\mathrm{d}s^{2}=c^{2}\,\mathrm{d}t^{2}-\sum\mathrm{d}x^{i}\,\mathrm{d}x^{j}=g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\equiv\frac{\mathrm{d}t^{2}}{m}\,g_{\mu\nu}\,P^{\mu}\,\mathrm{v}^{\nu}\,, \tag{1}$$

where $P^{\mu}=m\,\mathrm{v}^{\mu}$ is the 'Four-momentum', hence $P^{\mu}\,\mathrm{v}^{\nu}\equiv p^{0}\,\mathrm{v}^{0}+p^{i}\,\mathrm{v}^{j}$ for $i,j=1,2,3$, and $\mu,\nu=0,1,2,3$, whereas v^{μ} is the 'four-velocity'. (Be careful that it is not $P^{\mu}\,\mathrm{v}^{\nu}\equiv p^{0}\,\mathrm{v}^{0}-p^{i}\,\mathrm{v}^{j}$, because $g_{\mu\nu}$ takes the '$-$' sign.) Note it that $m\neq m_0$ in Eq. (1) for the rest mass m_0. Thus, Eq. (1) gives us an energy-momentum invariant line element as,

$$
\begin{aligned}
m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{2} &= mc^{2} - m\sum\frac{\mathrm{d}x^{i}}{\mathrm{d}t}\,\frac{\mathrm{d}x^{j}}{\mathrm{d}t} = E - \sum p^{i}\,\mathrm{v}^{j} = p^{0}\,\mathrm{v}^{0} - \sum p^{i}\,\mathrm{v}^{j} \\
&= g_{\mu\nu}\,P^{\mu}\,\mathrm{v}^{\nu}\,, \qquad (2)
\end{aligned}
$$

that is, we have a new line element as,

$$
\mathrm{d}\,\mathcal{S}^{2} = m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{2} = p^{0}\,\mathrm{v}^{0} - \sum p^{i}\,\mathrm{v}^{j} = g_{\mu\nu}\,P^{\mu}\,\mathrm{v}^{\nu}\,, \qquad (3)
$$

then, the rearrangement of Eq. (2) gives the following equation by using Eq. (1) as,

$$
E = p^{i}\,\mathrm{v}^{j} + m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{2} = p^{i}\,\mathrm{v}^{j} + P^{\mu}\,\mathrm{v}^{\nu}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\mu}}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\nu}} = p^{i}\,\mathrm{v}^{j} + P^{\mu}\,\mathrm{v}^{\nu}\,g_{\mu\nu}\,. \qquad (4)
$$

Let us consider the representation of a wave field $\psi(\vec{r},t)$ by superposition of a free particle (de Broglie wave) for Eq. (4) and by using $\mathrm{g} = g_{\mu\nu}$ as follows,

$$
\begin{aligned}
\psi(\vec{r},t) &= \frac{1}{(2\pi\,\hbar)^{3/2}}\,\exp\left[\frac{i}{\hbar}\,(\vec{p}\cdot\vec{r} - Et)\right] = \frac{1}{(2\pi\,\hbar)^{3/2}}\,\exp\left[\frac{i}{\hbar}\left\{\left(Et - \mathrm{g}\,\vec{P}\cdot\vec{R}\right) - Et\right\}\right] \\
&\equiv \frac{1}{(2\pi\,\hbar)^{3/2}}\,\exp\left[\frac{i}{\hbar}\left(-\,\mathrm{g}\,\vec{P}\cdot\vec{R}\right)\right], \qquad (5)
\end{aligned}
$$

where $\vec{P} \rightarrow \vec{P}^{\,\mu}$ and $\vec{R} \rightarrow \vec{r}^{\,\nu}$, as $\vec{r}^{\,\nu} = (\mathrm{v}^{\nu}t)$. Using the first two terms of Eq. (5), we can generate the following wave equation for Eq. (4) as,

$$
i\,\hbar\,\mathrm{v}^{0}\,\vec{\nabla}_{0}\,\psi(\vec{r},t) + i\,\hbar\,\mathrm{v}^{j}\,\vec{\nabla}_{i}\,\psi(\vec{r},t) + i\,\hbar\,g_{\mu\nu}\,\mathrm{v}^{\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t) = 0\,, \qquad (6)
$$

where the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}} \rightarrow\; -\,i\,\hbar\,\vec{\nabla}_{\mu}$, i.e.,

$$
\hat{\boldsymbol{\mathcal{P}}} \rightarrow\; -\,i\,\hbar\,\vec{\nabla}_{\mu} \rightarrow \left[\hat{\boldsymbol{p}}_{0}\,,\;\hat{\boldsymbol{p}}_{i}\right]^{\mathrm{T}} \rightarrow \left[i\,\hbar\,\vec{\nabla}_{0}\,,\;-\,i\,\hbar\,\vec{\nabla}_{i}\right]^{\mathrm{T}}, \qquad (7)
$$

and $\vec{\nabla}_{0} = \left(\partial/\partial\,(\mathrm{v}^{0}\,t)\right)$ for $x^{0} = (\mathrm{v}^{0}\,t)$, here, $\hat{\boldsymbol{p}}_{i} \rightarrow\; -\,i\,\hbar\,\vec{\nabla}_{i}$ is the three momentum operator.

Remark 1. *The signature of the metric $g_{\mu\nu}$, i.e., $(+,-,-,-)$, has been absorbed and retained unaltered by the last term of Eq. (6), as long as it satisfies Eq. (3) and Eq. (4). Thus, readers are requested to be careful not to presume space and time separately in Eq. (6), what we usually assume in the conventional Quantum Mechanics.*

Remark 2. *The wave field $\psi(\vec{r},t)$ itself in Eq. (5) is now relativistic due to $\mathrm{g} = g_{\mu\nu}$ in its last term. So, we can say that Quantum Mechanics and Relativity are correlated in wave field $\psi(\vec{r},t)$. Equally, Eq. (6) also assures us that Quantum Mechanics and Relativity are must be correlated for $g_{\mu\nu}$ in the last term of wave equation Eq. (6).*

Prescription 1. *From Remark 2, we have a sufficient reason to use the relativistic to quantum relations, and vice versa, as,*

$$
\begin{aligned}
& P^{\mu} &&\Longleftrightarrow\quad \hat{\boldsymbol{\mathcal{P}}} \\
\textit{i.e.,}\quad & m\,\mathrm{v}^{\mu} &&\Longleftrightarrow\quad -\,i\,\hbar\,\vec{\nabla}_{\mu}\,.
\end{aligned}
$$

We will use this trick throughout our work. (This prescription is quite straightforward than some commonly used textbook procedures, for example, Ref. [3].)

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Again rearranging Eq. (6) by using Eq. (7), we may get,

$$i\hbar\,\mathrm{v}^0\left(1-g_{00}\right)\vec{\nabla}_0\,\psi(\vec{r},t)+i\hbar\,\mathrm{v}^j\left(1+g_{ij}\right)\vec{\nabla}_i\,\psi(\vec{r},t)=0\,,$$

or, simply discarding $\left(1-g_{00}\right)=\left(1+g_{ij}\right)=0$, we can have the 'Four-velocity' Comprised First Variance of the First Order Equation of GQG as,

$$\begin{aligned} i\hbar\,\mathrm{v}^0\,\vec{\nabla}_0\,\psi(\vec{r},t)+i\hbar\,\mathrm{v}^j\,\vec{\nabla}_i\,\psi(\vec{r},t)&=0\,,\\ \therefore\quad i\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t)+i\hbar\,\mathrm{v}^j\,\vec{\nabla}_i\,\psi(\vec{r},t)&=0\,. \end{aligned}\tag{8}$$

Here, $\psi(\vec{r},t)$ is definitely a fermionic field.

Evidently, Eq. (8) may take the form $\hat{E}\,\psi(\vec{r},t)=-\,i\hbar\,\mathrm{v}^j\,\vec{\nabla}_i\,\psi(\vec{r},t)$ for the energy operator $\hat{E}\rightarrow i\hbar\,\partial_t$, and setting the Hamiltonian operator as $\hat{H}\,\psi(\vec{r},t)=-\,i\hbar\,\mathrm{v}^j\,\vec{\nabla}_i\,\psi(\vec{r},t)\equiv\hat{\boldsymbol{p}}_i\,\mathrm{v}^j\,\psi(\vec{r},t)$, where the three momentum operator $\hat{\boldsymbol{p}}_i\rightarrow-\,i\hbar\,\vec{\nabla}_i$, we can therefore have, $\hat{E}\,\psi(\vec{r},t)=\hat{H}\,\psi(\vec{r},t)$. Interested readers can easily check it that Eq. (8) is nothing but the Classical Schrödinger equation, which has been written with v^j and with the signature of the metric $(+,-,-,-)$.

It is also possible to develop a 'Four-velocity' Comprised Second Variance of the First Order Equation of GQG from Eq. (6) as follows,

$$i\hbar\,\mathrm{v}^\nu\,\vec{\Delta}_\mu\,\psi(\vec{r},t)+i\hbar\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)=0\,,$$

where $i\hbar\,\vec{\Delta}_\mu\rightarrow\left[\hat{\boldsymbol{p}}_0\,,\,-\hat{\boldsymbol{p}}_i\right]^{\mathrm{T}}\rightarrow\left[i\hbar\,\vec{\nabla}_0\,,\,i\hbar\,\vec{\nabla}_i\right]^{\mathrm{T}}$.

Now, returning to the main purpose of the present work, let us multiply both sides of Eq. (6) by $(\mathrm{d}t^2/m)$, so as,

$$i\hbar\,\frac{\mathrm{d}t^2}{m}\,\mathrm{v}^0\,\vec{\nabla}_0\,\psi(\vec{r},t)+i\hbar\,\frac{\mathrm{d}t^2}{m}\,\mathrm{v}^j\,\vec{\nabla}_i\,\psi(\vec{r},t)=-\,i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)\,,\tag{9}$$

which has the form of a general inhomogeneous Lorentz transformation (or Poincaré transformation).

Note it that Eq. (9) is exactly equivalent to Eq. (1), i.e., $\mathrm{d}s^2\equiv(\mathrm{d}t^2/m)\,g_{\mu\nu}\,P^\mu\,\mathrm{v}^\nu$, for the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}}\rightarrow-\,i\hbar\,\vec{\nabla}_\mu$. In other words, we can say that the quantum line element is,

$$\begin{aligned} \mathrm{d}s^2\,\psi(\vec{r},t)\;&\equiv\;-\,i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)\\ &=\;-\,\frac{\mathrm{v}^\mu}{\mathrm{v}^\mu}\left(i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t)\\ &=\;-\,\frac{1}{\mathrm{v}^\mu}\left(i\hbar\,\frac{\mathrm{d}x^\mu}{m}\,g_{\mu\nu}\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t)\\ &=\;-\,\frac{1}{(\mathrm{v}^\mu\,m)}\left(i\hbar\,g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t)\,, \end{aligned}$$

hence, by considering $\mathcal{E}=\left(\mathrm{v}^\mu\,m\right)^{-1}$, we have,

$$\mathrm{d}s^2\,\psi(\vec{r},t)=-\,i\hbar\,\mathcal{E}\,g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)\,.\tag{10}$$

Remark 3. *Technically, $\mathcal{E}$ and $\left(-\,i\hbar\,\vec{\nabla}_\mu\right)$ cancel each other for the Prescription 1, leaving behind a classical-like line element. More explicitly we can say that the quantum line element, $ds^2\,\psi(\vec{r},t)=-\,i\hbar\,\mathcal{E}\,g_{\mu\nu}\,dx^\mu\,dx^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)$, can transform into the classical line element, $ds^2\,\psi(\vec{r},t)=g_{\mu\nu}\,dx^\mu\,dx^\nu\,\psi(\vec{r},t)$, in a quantum spacetime, as $\mathcal{E}$ and $\left(-\,i\hbar\,\vec{\nabla}_\mu\right)$ are canceling each other.*

Let us consider $\vec{\nabla}'_{\mu} = \left(\delta/\delta x^{\mu}\right)$, etc., and let us also consider that $g_{\mu\nu}$ would transform as,

$$g_{\mu\nu} = g_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\right) \longmapsto g_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\right)\left(\frac{i\hbar\,\vec{\nabla}'_{\alpha}}{i\hbar\,\vec{\nabla}'_{\mu}}\right) = g^{(\mu)}_{\mu\nu} \longmapsto f^{(\mu)}_{\mu\nu}\,, \tag{11}$$

where $f^{(\mu)}_{\mu\nu}$ is a 'semi-quantum metric tensor' in a quantum-Riemannian spacetime, i.e.,

$$f^{(\mu)}_{\mu\nu} = f^{(\alpha)}_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\,\frac{i\hbar\,\vec{\nabla}'_{\alpha}}{i\hbar\,\vec{\nabla}'_{\mu}}\right),$$

so as the 'semi-quantum metric tensor' $f^{(\mu)}_{\mu\nu}$ and the Riemannian metric tensor $g_{\mu\nu}$ should establish a relation as follows,

$$\begin{aligned}
-\,i\hbar\,\vec{\nabla}_{\mu}\otimes f^{(\mu)}_{\mu\nu} &= -\,i\hbar\,\vec{\nabla}_{\mu}\otimes f^{(\alpha)}_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\,\frac{i\hbar\,\vec{\nabla}'_{\alpha}}{i\hbar\,\vec{\nabla}'_{\mu}}\right)\\
&= -\,i\hbar\,\vec{\nabla}_{\mu}\otimes g_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\right)\left(\frac{i\hbar\,\vec{\nabla}'_{\alpha}}{i\hbar\,\vec{\nabla}'_{\mu}}\right)\\
&= -\,i\hbar\,g_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\right)\vec{\nabla}_{\alpha} = -\,i\hbar\,g_{\mu\nu}\,\vec{\nabla}_{\alpha}\,,
\end{aligned} \tag{12}$$

thus, for Eq. (11),

$$f^{(\mu)}_{\mu\nu}\left(\frac{i\hbar\,\vec{\nabla}'_{\mu}}{i\hbar\,\vec{\nabla}'_{\alpha}}\right) = g_{\mu\nu}\,. \tag{13}$$

Without any loss of generality, we may assume that the 'quantum metric tensor' is symmetric: $f^{(\mu)}_{\mu\nu} = f^{(\mu)}_{\nu\mu}$, and $det\left(f^{(\mu)}_{\mu\nu}\right) \neq 0$. It has an inverse matrix f^{-1} whose components are themselves the components of matrix f, as their product gives: $f^{-1}f =$ identity matrix, i.e., in terms of components, $f^{(\mu)}_{\mu\nu}\,f^{\mu\gamma}_{(\mu)} = f^{\gamma\mu}_{(\mu)}\,f^{(\mu)}_{\mu\nu} = \delta^{\gamma}_{\nu}$, where, δ^{γ}_{ν} is the Kronecker delta.

Hence, Eq. (10) should be rewritten as,

$$\begin{aligned}
\mathrm{d}s^{2}\,\psi(\vec{r},t) &= -\,\mathcal{E}\,f^{(\alpha)}_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\mathrm{d}x^{\mu}\right)\left(\frac{\partial x^{\beta}}{\partial x^{\nu}}\,\mathrm{d}x^{\nu}\right)\left(\frac{i\hbar\,\vec{\nabla}'_{\alpha}}{i\hbar\,\vec{\nabla}'_{\mu}}\,i\hbar\,\vec{\nabla}_{\mu}\right)\left(\frac{i\hbar\,\vec{\nabla}'_{\mu}}{i\hbar\,\vec{\nabla}'_{\alpha}}\right)\psi(\vec{r},t)\\
&= -\,i\hbar\,\mathcal{E}\,f^{(\alpha)}_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\,\frac{i\hbar\,\vec{\nabla}'_{\alpha}}{i\hbar\,\vec{\nabla}'_{\mu}}\right)\left(\frac{i\hbar\,\vec{\nabla}'_{\mu}}{i\hbar\,\vec{\nabla}'_{\alpha}}\right)\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t)\\
&= -\,i\hbar\,\mathcal{E}\,f^{(\mu)}_{\mu\nu}\left(\frac{i\hbar\,\vec{\nabla}'_{\mu}}{i\hbar\,\vec{\nabla}'_{\alpha}}\right)\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t)\\
&\equiv -\,i\hbar\,\mathcal{E}\,g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t)\,.
\end{aligned}$$

Let us vary the length of a curve [4–7] as,

$$\begin{aligned}\delta\, L|\gamma|\,\psi(\vec{r},t) &\equiv \int \delta \left\{ -\, i\,\hbar\,\mathcal{E}\, f^{(\mu)}_{\mu\nu} \left(\frac{i\,\hbar\,\vec{\nabla}'_{\mu}}{i\,\hbar\,\vec{\nabla}'_{\alpha}} \right) \dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} \right\}^{(1/2)} \mathrm{d}\tau\,\psi(\vec{r},t) \\ &= \frac{1}{2}\int \Bigg\{ -\, i\,\hbar\,\mathcal{E}\,\partial_{\epsilon}\, f^{(\mu)}_{\mu\nu} \left(\frac{i\,\hbar\,\vec{\nabla}'_{\mu}}{i\,\hbar\,\vec{\nabla}'_{\alpha}} \right) \dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} - \\ &\qquad -\, 2\,\frac{\mathrm{d}}{\mathrm{d}\tau} \left(-\, i\,\hbar\,\mathcal{E}\, f^{(\mu)}_{\epsilon\nu} \left(\frac{i\,\hbar\,\vec{\nabla}'_{\mu}}{i\,\hbar\,\vec{\nabla}'_{\alpha}} \right) \dot{x}^{\nu}\,\vec{\nabla}_{\mu} \right) \Bigg\}\,\delta x^{\epsilon}\,\mathrm{d}\tau\,\psi(\vec{r},t)\,.\end{aligned}$$

This gives,

$$\begin{aligned}&\left\{ -\, i\,\hbar\,\mathcal{E}\,\partial_{\epsilon}\, f^{(\mu)}_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} + i\,\hbar\,\mathcal{E}\,\partial_{\mu}\, f^{(\mu)}_{\epsilon\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} + i\,\hbar\,\mathcal{E}\,\partial_{\nu}\, f^{(\mu)}_{\epsilon\mu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} \right\} \psi(\vec{r},t) \\ &\qquad\qquad = -\,2\,i\,\hbar\,\mathcal{E}\, f^{(\mu)}_{\epsilon\delta}\,\ddot{x}^{\delta}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t)\,,\end{aligned}$$

then the Christoffel symbol $\Gamma^{\delta}_{\mu\nu}$ should be defined by,

$$\Gamma^{\delta}_{\mu\nu}\,\psi(\vec{r},t) = \frac{1}{2}\, f^{\delta\epsilon}_{(\mu)} \left(\partial_{\mu}\, f^{(\mu)}_{\epsilon\nu} + \partial_{\nu}\, f^{(\mu)}_{\epsilon\mu} - \partial_{\epsilon}\, f^{(\mu)}_{\mu\nu} \right) \psi(\vec{r},t)\,,$$

such that the Christoffel symbols are symmetric in the lower indices: $\Gamma^{\delta}_{\mu\nu} = \Gamma^{\delta}_{\nu\mu}$.

After a little exercise, we can yield the tensor,

$$\mathcal{R}^{\sigma}_{\ \nu\gamma\delta}\,\psi(\vec{r},t) = \left(\frac{\partial\Gamma^{\sigma}_{\nu\delta}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\sigma}_{\nu\gamma}}{\partial x^{\delta}} + \Gamma^{\sigma}_{\gamma\epsilon}\,\Gamma^{\epsilon}_{\nu\delta} - \Gamma^{\sigma}_{\delta\epsilon}\,\Gamma^{\epsilon}_{\nu\gamma} \right) \psi(\vec{r},t)\,,$$

thus we find,

$$\mathcal{R}_{\lambda\nu\gamma\delta}\,\psi(\vec{r},t) = \frac{1}{2} \left(f^{(\lambda)}_{\lambda\delta,\nu\gamma} + f^{(\lambda)}_{\nu\gamma,\lambda\delta} - f^{(\lambda)}_{\delta\nu,\lambda\gamma} - f^{(\lambda)}_{\lambda\gamma,\nu\delta} \right) \psi(\vec{r},t)\,,$$

which satisfies the properties like symmetry, antisymmetry and cyclicity as usual. Without much ado, we can easily obtain the 'Four-velocity' Comprised GQG Hilbert-Einstein field equations as,

$$\left[\mathcal{R}_{\zeta\eta} - \frac{1}{2}\, f^{(\zeta)}_{\zeta\eta}\,\mathcal{R} + \Lambda\, f^{(\zeta)}_{\zeta\eta} \right] \psi(\vec{r},t) = 8\pi\, G\,\mathcal{T}_{\zeta\eta}\,\psi(\vec{r},t)\,, \tag{14}$$

where $\mathcal{T}_{\zeta\eta}$ is the quantum energy momentum tensor, it is what the graviton field couples to, and G is the gravitational coupling. Let us develop an unusual gravitational coupling G in Planck scale using Eq. (13) as follows,

$$\begin{aligned} G &= \frac{(\mathrm{d}\ell_P)^2}{m_P}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2} = \frac{m_P}{m_P^2}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2}\,(\mathrm{d}\ell_P)^2 = \frac{F_P}{m_P^2}\,(\mathrm{d}\ell_P)^2 = \frac{F_P}{m_P^2}\,g_{\zeta\eta}\,\mathrm{d}\ell_P^{\zeta}\,\mathrm{d}\ell_P^{\eta} \\ &= \frac{F_P}{m_P^2}\, f^{(\zeta)}_{\zeta\eta} \left(\frac{i\,\hbar\,\vec{\nabla}'_{\zeta}}{i\,\hbar\,\vec{\nabla}'_{\alpha}} \right) \mathrm{d}\ell_P^{\zeta}\,\mathrm{d}\ell_P^{\eta}\,, \end{aligned} \tag{15}$$

where $F_P = m_P \left\{ \mathrm{d}^2\ell_P^{\zeta} / (\mathrm{d}t_P)^2 \right\}$.

Since $\mathcal{R}_{\zeta\eta} = f^{\lambda\nu}_{(\lambda)}\,\mathcal{R}_{\lambda\zeta\eta\nu}$ and $f^{(\lambda)}_{\lambda\nu}\, f^{\lambda\nu}_{(\lambda)} = g_{\lambda\nu}\, g^{\lambda\nu}$ and since the quantum metric $f^{(\zeta)}_{\zeta\eta}$ has constant Ricci curvature if $\mathcal{R}_{\zeta\eta} = \Lambda\, f^{(\zeta)}_{\zeta\eta}$ for cosmological constant Λ, the LHS of Eq. (14) may give us the purely Einsteinian form, i.e., $\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\, g_{\zeta\eta}\,\mathrm{R} + \Lambda\, g_{\zeta\eta} \right]$, if we use Eq. (12) as follows,

$$\left(-i\hbar\,\vec{\nabla}_\zeta\right)\otimes\left(-i\hbar\,\vec{\nabla}^\zeta\right)\otimes\left[\mathcal{R}_{\zeta\eta}-\frac{1}{2}f^{(\zeta)}_{\zeta\eta}\,\mathcal{R}+\Lambda\,f^{(\zeta)}_{\zeta\eta}\right]\Leftrightarrow$$
$$-\hbar^2\left[\mathrm{R}_{\zeta\eta}-\frac{1}{2}g_{\zeta\eta}\,\mathrm{R}+\Lambda\,g_{\zeta\eta}\right]\vec{\nabla}_\mu\,\vec{\nabla}^\mu\,.$$

Similarly, we can get the Einsteinian energy momentum tensor $\mathrm{T}_{\zeta\eta}$ from the quantum energy momentum tensor $\mathcal{T}_{\zeta\eta}$ of Eq. (14), if we assume $\mathcal{T}_{\zeta\eta}$ depends on quantum metric tensor (for example, $\mathcal{T}_{\zeta\eta}=-\mathcal{F}_{\zeta\mu}\,\mathcal{F}^{\mu}_{\eta}+\frac{1}{4}f^{(\zeta)}_{\zeta\eta}\,\mathcal{F}^{\mu\nu}\,\mathcal{F}_{\mu\nu}=-f^{(\zeta)}_{\zeta\eta}\,\mathcal{F}_{\zeta\mu}\,\mathcal{F}^{\zeta\mu}+\frac{1}{4}f^{(\zeta)}_{\zeta\eta}\,\mathcal{F}^{\mu\nu}\,\mathcal{F}_{\mu\nu}$, etc.). By using Eq. (15) in Eq. (14), we can get the modified GQG field equation as,

$$\left(-i\hbar\,\vec{\nabla}_\zeta\right)\otimes\left(-i\hbar\,\vec{\nabla}^\zeta\right)\otimes\left[\mathcal{R}_{\zeta\eta}-\frac{1}{2}f^{(\zeta)}_{\zeta\eta}\,\mathcal{R}+\Lambda\,f^{(\zeta)}_{\zeta\eta}\right]\psi(\vec{r},t)$$
$$=\left(-i\hbar\,\vec{\nabla}_\zeta\right)\otimes\left(-i\hbar\,\vec{\nabla}^\zeta\right)\otimes\left(8\pi\,G\,\mathcal{T}_{\zeta\eta}\right)\psi(\vec{r},t)\,,$$

$$\therefore\quad -\hbar^2\left[\mathrm{R}_{\zeta\eta}-\frac{1}{2}g_{\zeta\eta}\,\mathrm{R}+\Lambda\,g_{\zeta\eta}\right]\vec{\nabla}_\mu\,\vec{\nabla}^\mu\,\psi(\vec{r},t)$$
$$=-\hbar^2\,8\pi\,G\,\mathrm{T}_{\zeta\eta}\,\vec{\nabla}_\mu\,\vec{\nabla}^\mu\,\psi(\vec{r},t)\,. \tag{16}$$

Note it that Eq. (14) and Eq. (16) are exactly the same thing despite their different appearances.

Remark 4. *It is necessary to remember that,*

1. *Eq. (16) tells us that gravitation is an interaction in orthogonal curvilinear coordinates* x^ζ *(i.e., outer surface) of point* $P(x^\mu)$ *rather than at that very spacetime* x^μ *(i.e., core). Since* $-i\hbar\,\vec{\nabla}_\mu\nrightarrow 0$ *for a particle field, then the Einstein tensor must be unity, i.e.,* $\left[\mathrm{R}_{\zeta\eta}-\frac{1}{2}g_{\zeta\eta}\,\mathrm{R}+\Lambda\,g_{\zeta\eta}\right]\to 1$*, in Eq. (16) at the core* dx^μ*, and there, spacetime* dx^μ *behaves strongly Quantum Mechanical so as the other particle interactions (i.e., quantum chromodynamics and electroweak) are prioritized there locally. On the other hand, gravitational effects only start effective beyond the core* dx^μ*, i.e., in orthogonal curvilinear coordinates* dx^ζ *outside the core* dx^μ*, in other words, the outer surface of* dx^μ*.*
2. *If we consider a cutoff energy* $E_\star$*, then we can say that gravitons only appear in an energy zone* $\mathrm{E}_{\mathrm{IR}}\leq\mathrm{E}_\star\ll E_{\mathrm{PLANCK}}$*, and beyond that state, i.e.,* $\mathrm{E}_{\mathrm{UV}}>\mathrm{E}_\star$*, other particle interactions (i.e., Quantum Chromodynamics and Electroweak) are prioritized, where* E_{IR} *is the infra red energy zone, whereas* $\mathrm{E}_{\mathrm{UV}}\lessapprox E_{\mathrm{PLANCK}}$ *is the ultra violet energy zone. Thus, for gravity in Eq. (16), ultra violet zone is automatically ignored, i.e., the sum over* $\mathrm{E}_{\mathrm{IR}}\leq\mathrm{E}_\star<\mathrm{E}_{\mathrm{UV}}$ *intends the Feynman graphs to be finite. In the energy zone* $\mathrm{E}_{\mathrm{IR}}\leq\mathrm{E}_\star<\mathrm{E}_{\mathrm{UV}}$*, all gravitons behave as real particles. Let us assume additionally that* E_{PLANCK} *is not an external energy state, but the kernel of all energy states of a particle, then we can assume the energy states of a particle from the kernel energy of the core* x^μ *to its outer surface energy for* x^ζ *as:*

$$\begin{tikzcd}
\left(E_{\mathrm{PLANCK}}\right)_{\text{KERNEL OF THE CORE }x^\mu} \arrow[r] & \left(E_{\mathrm{UV}}\right)_{\text{CORE }x^\mu} \arrow[r] & \left(E_\star\right)_{\text{OUTER SURFACE }x^\zeta} \arrow[d] \\
 & & \left(E_{\mathrm{IR}}\right)_{\text{BEYOND THE PARTICLE}}
\end{tikzcd}$$

   Between core and outer surface energies, i.e., at $\mathrm{E}_{\mathrm{UV}}>\mathrm{E}_\star$*, Electroweak and Quantum Chromodynamic interactions take place, whereas, outside these states (i.e., at* $\mathrm{E}_{\mathrm{IR}}\leq\mathrm{E}_\star$*) gravity starts being prioritized.*
3. *For the* RHS *factor* $\hbar^2\,G$ *of Eq. (16), the gravitational coupling* G*, which has the dimension of a negative power of mass, now has lost its mass dimension due to* $\hbar^2$*. Consequently, if divergences are to be present, they could now be disposed of by the technique of renormalization (though, this will not play a role in our present discussion).*

Now, considering d'Alembertian operator $\Box = \vec{\nabla}_{\mu}\,\vec{\nabla}^{\,\mu}$, as well as $\mathcal{U} = \left(8\,\pi\,G\,\mathrm{T}_{\zeta\eta}\right)$, and inputting the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}} \to\ -\,i\,\hbar\,\vec{\nabla}_{\mu}$ into Eq. (16), we can get the 'Four-velocity' Comprised Second Order Equation of GQG as,

$$\hbar^{2}\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R} + \Lambda\,g_{\zeta\eta}\right]\Box\,\psi(\vec{r},t) + \mathcal{U}\,\mathcal{P}^{2}\,\psi(\vec{r},t) = 0\,. \tag{17}$$

The wavefunction $\psi(\vec{r},t)$ in Eq. (17) is emphatically defining a bosonic field. Thus, we can immediately develop a fermionic field (or the 'Four-velocity' Comprised Third Variance of the First Order Equation of GQG) out of Eq. (17) as,

$$i\,\hbar\,\gamma^{\mu}\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R} + \Lambda\,g_{\zeta\eta}\right]^{(1/2)}\vec{\nabla}_{\mu}\,\psi(\vec{r},t) - \mathcal{U}^{\,(1/2)}\,\mathcal{P}\,\psi(\vec{r},t) = 0\,, \tag{18}$$

where, $\gamma^{\,\mu}$ are Dirac's gamma matrices.

Dividing Eq. (18) either by $\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R} + \Lambda\,g_{\zeta\eta}\right]^{(1/2)}$ or by $\mathcal{U}^{\,(1/2)}$ gives us $\left(i\,\hbar\,\gamma^{\,\mu}\,\vec{\nabla}_{\mu} - \mathcal{P}\right)\psi(\vec{r},t) = 0$, from which the classical Dirac's equation should be derivable, but here, instead of $\partial/(\partial\,t)$, we have considered $\partial/(\partial x^{0})$ by absorbing v^{0} and similarly, $\mathcal{P}$ is not intended here to have a factor of rest mass, since $m \neq m_{0}$ in Eq. (1). Thus, we can say that Dirac's equation is a subset of the 'Four-velocity' Comprised Third Variance of the First Order Equation of GQG, i.e., Eq. (18). Similarly, we can also say that the Klein-Gordon equation is a subset of the 'Four-velocity' Comprised Second Order Equation of GQG, i.e., Eq. (17), and it should be derivable from $\left(\hbar^{2}\,\Box + \mathcal{P}^{2}\right)\psi(\vec{r},t) = 0$. An analogous formalism is equally applicable for the following Section 2.2.

2.2. 'Four-momentum' Comprised Theory of GQG

Let the line element of Minkowski spacetime,

$$\begin{aligned}
\mathrm{d}s^{2} &= g_{\mu\nu}\,\mathrm{d}x^{\,\mu}\,\mathrm{d}x^{\,\nu} \equiv \left(\frac{\mathrm{d}t}{m}\right)^{2} g_{\mu\nu}\,P^{\,\mu}\,P^{\,\nu} \\
\therefore\quad m^{2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{2} &= m^{2}c^{2} - m^{2}\sum\frac{\mathrm{d}x^{\,i}}{\mathrm{d}t}\,\frac{\mathrm{d}x^{\,j}}{\mathrm{d}t} = m\,E - \sum p^{\,i}\,p^{\,j} = p^{\,0}\,p^{\,0} - \sum p^{\,i}\,p^{\,j} \\
&= g_{\mu\nu}\,P^{\,\mu}\,P^{\,\nu}\,, \\
\text{and,}\quad m^{2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{2} &= m^{2}\left(1 - \frac{\mathrm{v}^{2}}{c^{2}}\right)c^{2} = m_{0}^{2}\,c^{2}\,,
\end{aligned} \tag{19}$$

for the rest mass m_{0}, when,

$$\mathrm{d}\,\mathcal{S}'^{\,2} = m^{2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{2} = p^{\,0}\,p^{\,0} - \sum p^{\,i}\,p^{\,j} = g_{\mu\nu}\,P^{\,\mu}\,P^{\,\nu}\,,$$

then, rearrangement of Eq. (19) gives,

$$m\,E = p^{\,i}\,p^{\,j} + m^{2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{2} = p^{\,i}\,p^{\,j} + P^{\,\mu}\,P^{\,\nu}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\,\mu}}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\,\nu}} = p^{\,i}\,p^{\,j} + P^{\,\mu}\,P^{\,\nu}\,g_{\mu\nu}\,. \tag{20}$$

Then, considering the representation of a wave field $\psi(\vec{r},t)$ by superposition of a free particle (de Broglie wave) for Eq. (20) and by using $\mathrm{g} = g_{\mu\nu}$ as follows,

$$\begin{aligned}\psi(\vec{r},t) &= \frac{1}{(2\pi\,\hbar)^{3/2}}\,\exp\left[\frac{i}{\hbar\,m}\,(m\,\vec{p}\cdot\vec{r}-m\,Et)\right]\\ &= \frac{1}{(2\pi\,\hbar)^{3/2}}\,\exp\left[\frac{i}{\hbar\,m}\left\{m\,\left(Et-\mathrm{g}\,\vec{P}\cdot\vec{R}\right)-m\,Et\right\}\right]\\ &\equiv \frac{1}{(2\pi\,\hbar)^{3/2}}\,\exp\left[\frac{i}{\hbar\,m}\left(-\,\mathrm{g}\,m\,\vec{P}\cdot\vec{R}\right)\right],\end{aligned}$$

we can generate the following wave equation using Eq. (20) as,

$$-\,\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t)+\hbar^2\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)+\hbar^2\,g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)=0\,, \tag{21}$$

which may give,

$$-\,\hbar^2\left(1-g_{00}\right)\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t)+\hbar^2\left(1+g_{ij}\right)\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0\,,$$

or, simply discarding $(1-g_{00})=(1+g_{ij})=0$, we can get the ‘Four-momentum’ Comprised First Variance of the Second Order Equation of GQG as,

$$\begin{aligned}&-\,\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t)+\hbar^2\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0\,,\\ \therefore\quad &-\,\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}^0\,\psi(\vec{r},t)+\hbar^2\,\vec{\nabla}_i\,\vec{\nabla}^i\,\psi(\vec{r},t)=0\,.\end{aligned} \tag{22}$$

Here, $\psi(\vec{r},t)$ is definitely a bosonic field. But the uppermost equation of Eq. (22) may give us the Classical Schrödinger equation by using the Prescription 1 (the exercise is left for the readers),

$$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t)+\frac{\hbar^2}{m}\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0\,.$$

Putting differently,

$$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t)+\frac{\hbar^2}{m}\,g_{ij}\,\vec{\nabla}_i\,\vec{\nabla}^i\,\psi(\vec{r},t)=0\,.$$

Hence, we get the classical Schrödinger equation, though it is now in a $(3+1)D$ quantum spacetime.

Now, the ‘Four-momentum’ Comprised Second Variance of the Second Order Equation of GQG from Eq. (21) should be,

$$-\,\hbar^2\,\vec{\Delta}_\mu\,\vec{\Delta}_\nu\,\psi(\vec{r},t)+\hbar^2\,g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)=0\,,$$

where $i\,\hbar\,\vec{\Delta}_\mu\rightarrow\left[\hat{\boldsymbol{p}}_0\,,\,-\,\hat{\boldsymbol{p}}_i\right]^{\mathrm{T}}\rightarrow\left[i\,\hbar\,\vec{\nabla}_0\,,\,i\,\hbar\,\vec{\nabla}_i\right]^{\mathrm{T}}$.

Multiplying both sides of Eq. (21) by $(\mathrm{d}t/m)^2$ and comparing it with Eq. (19), we have the quantum line element for the ‘Four-momentum’ operator $\hat{\boldsymbol{\mathcal{P}}}\rightarrow\,-\,i\,\hbar\,\vec{\nabla}_\mu$ as follows,

$$\mathrm{d}s^2\,\psi(\vec{r},t)=\,-\,\hbar^2\left(\frac{\mathrm{d}t}{m}\right)^2\,g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)\,. \tag{23}$$

Let us now prescribe $g_{\mu\nu}$ as follows by using Prescription 1 as,

$$\begin{aligned} g_{\mu\nu} &= g_{\alpha\beta}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\right) = g_{\alpha\beta}\left(\frac{m}{\mathrm{d}t}\right)^{2}\left(\frac{\mathrm{d}t}{m}\right)^{2}\left(\frac{\partial x^{\alpha}}{\partial x^{\mu}}\,\frac{\partial x^{\beta}}{\partial x^{\nu}}\right) \\ &\to g_{\alpha\beta}\left(\frac{\hat{\mathcal{P}}^{\alpha}}{\hat{\mathcal{P}}^{\mu}}\,\frac{\hat{\mathcal{P}}^{\beta}}{\hat{\mathcal{P}}^{\nu}}\right) \to g_{\alpha\beta}\left(\frac{i\hbar\,\vec{\nabla}_{\alpha}}{i\hbar\,\vec{\nabla}_{\mu}}\,\frac{i\hbar\,\vec{\nabla}_{\beta}}{i\hbar\,\vec{\nabla}_{\nu}}\right). \end{aligned}$$

To avoid any confusion between the Riemannian metric tensor $g_{\mu\nu}$ and the above prescription of quantum metric tensor, let us assume that,

$$\mathrm{g}_{\mu\nu} = g_{\alpha\beta}\left(\frac{i\hbar\,\vec{\nabla}_{\alpha}}{i\hbar\,\vec{\nabla}_{\mu}}\,\frac{i\hbar\,\vec{\nabla}_{\beta}}{i\hbar\,\vec{\nabla}_{\nu}}\right) = g_{\mu\nu}\,. \tag{24}$$

This approach is quantizing gravity. The 'quantum metric tensor' $\mathrm{g}_{\mu\nu}$ is symmetric, i.e., $\mathrm{g}_{\mu\nu} = \mathrm{g}_{\nu\mu}$, and $det\left(\mathrm{g}_{\mu\nu}\right) \neq 0$. Components of its inverse matrix g^{-1} are themselves the components of matrix g, i.e., $\mathrm{g}_{\mu\nu}\,\mathrm{g}^{\mu\gamma} = \mathrm{g}^{\gamma\mu}\,\mathrm{g}_{\mu\nu} = \delta^{\gamma}_{\nu}$, where, δ^{γ}_{ν} is the Kronecker delta.

Then, Eq. (23) becomes as,

$$\mathrm{d}s^{2}\,\psi(\vec{r},t) = -\hbar^{2}\left(\frac{\mathrm{d}t}{m}\right)^{2}\mathrm{g}_{\mu\nu}\,\vec{\nabla}_{\mu}\,\vec{\nabla}_{\nu}\,\psi(\vec{r},t)\,. \tag{25}$$

Let us vary the length of a curve [4–7] as,

$$\begin{aligned} \delta\,L|\gamma|\,\psi(\vec{r},t) &\equiv \int\delta\left\{-\hbar^{2}\left(\frac{\mathrm{d}t}{m}\right)^{2}\mathrm{g}_{\mu\nu}\,\vec{\nabla}_{\mu}\,\vec{\nabla}_{\nu}\right\}^{(1/2)}\mathrm{d}\tau\,\psi(\vec{r},t) \\ &= \frac{1}{2}\int\Bigg\{-\hbar^{2}\left(\frac{\mathrm{d}t}{m}\right)^{2}\partial_{\epsilon}\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_{\mu}\,\vec{\nabla}_{\nu} - \\ &\qquad -2\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\hbar^{2}\left(\frac{\mathrm{d}t}{m}\right)^{2}\mathrm{g}_{\mu\nu}\,\vec{\nabla}_{\nu}\right)\Bigg\}\,\delta\,\vec{\nabla}_{\epsilon}\,\mathrm{d}\tau\,\psi(\vec{r},t)\,. \end{aligned}$$

Similar to the Section 2.1, after a little exercise, we can develop,

$$\mathcal{R}_{\lambda\nu\gamma\delta}\,\psi(\vec{r},t) = \frac{1}{2}\left(\mathrm{g}_{\lambda\delta,\nu\gamma} + \mathrm{g}_{\nu\gamma,\lambda\delta} - \mathrm{g}_{\delta\nu,\lambda\gamma} - \mathrm{g}_{\lambda\gamma,\nu\delta}\right)\psi(\vec{r},t)\,,$$

and then, we can obtain the 'Four-momentum' Comprised GQG Hilbert-Einstein field equations as,

$$\left[\mathcal{R}_{\zeta\eta} - \frac{1}{2}\,\mathrm{g}_{\zeta\eta}\,\mathcal{R} + \Lambda\,\mathrm{g}_{\zeta\eta}\right]\psi(\vec{r},t) = 8\pi\,G\,\mathcal{T}_{\zeta\eta}\,\psi(\vec{r},t)\,. \tag{26}$$

Let us develop another unusual gravitational coupling G in Planck scale using Prescription 1 as follows,

$$\begin{aligned} G &= \frac{(\mathrm{d}\ell_{P})^{2}}{m_{P}}\,\frac{\mathrm{d}^{2}\ell_{P}}{(\mathrm{d}t_{P})^{2}} = \frac{m_{P}}{m_{P}^{2}}\,\frac{\mathrm{d}^{2}\ell_{P}}{(\mathrm{d}t_{P})^{2}}\,(\mathrm{d}\ell_{P})^{2} = \frac{F_{P}}{m_{P}^{2}}\,g_{\zeta\eta}\,\mathrm{d}\ell_{P}^{\zeta}\,\mathrm{d}\ell_{P}^{\eta} = \frac{F_{P}}{m_{P}^{2}}\,\mathrm{d}\ell_{P}^{\zeta}\,\mathrm{d}\ell_{P\,\zeta} \\ &\to -\hbar^{2}\,F_{P}\left(\frac{\mathrm{d}t_{P}}{m_{P}^{2}}\right)^{2}\vec{\nabla}_{P\,\zeta}\,\vec{\nabla}_{P}^{\zeta} = -\hbar^{2}\,\frac{\ell_{P}^{\zeta}}{m_{P}^{3}}\,\vec{\nabla}_{P\,\zeta}\,\vec{\nabla}_{P}^{\zeta}\,, \end{aligned} \tag{27}$$

for $F_{P} = m_{P}\left\{\mathrm{d}^{2}\ell_{P}^{\zeta}/(\mathrm{d}t_{P})^{2}\right\}$.

Interested readers can easily check that Eq. (14) and Eq. (26) are exactly the same thing but comprised with different components: the earlier one with 'Four-velocity' components and the later one with 'Four-momentum' components. Another noticeable difference between them is that Eq. (14) has a mixed expression of classical and quantum geometric expressions for $f_{\mu\nu}^{(\mu)}$, whereas, Eq. (26) has a purely quantum geometric expression for $g_{\mu\nu}$. In other words, we can say that Eq. (14) is in a quantum-Riemannian spacetime, whereas, Eq. (26) is in a purely quantum non-Riemannian spacetime.

If we transform our spacetime into Planck scale, i.e., $x^\mu \to \ell_P^\mu$ and $t \to t_P$, then we can rewrite Eq. (26) using Eq. (27) as,

$$\hbar^2 \frac{\ell_P^\zeta}{m_P^3} \vec{\nabla}_{P\zeta} \vec{\nabla}_P^\zeta \psi(\vec{r}_P, t_P) + \frac{1}{8\pi \mathcal{T}_{P\zeta\eta}} \left[\mathcal{R}_{P\zeta\eta} - \frac{1}{2} g_{\zeta\eta} \mathcal{R}_P + \Lambda g_{\zeta\eta} \right] \psi(\vec{r}_P, t_P) = 0\,.$$

Let,

$$\mathscr{C}^2 = \frac{1}{8\pi \mathcal{T}_{P\zeta\eta}} \left[\mathcal{R}_{P\zeta\eta} - \frac{1}{2} g_{\zeta\eta} \mathcal{R}_P + \Lambda g_{\zeta\eta} \right] = G\,, \tag{28}$$

then we have,

$$\hbar^2 \frac{\ell_P^\zeta}{m_P^3} \vec{\nabla}_{P\zeta} \vec{\nabla}_P^\zeta \psi(\vec{r}_P, t_P) + \mathscr{C}^2 \psi(\vec{r}_P, t_P) = 0\,.$$

Considering d'Alembertian operator $\Box_P = \vec{\nabla}_{P\zeta} \vec{\nabla}_P^\zeta$, we can get the 'Four-momentum' Comprised Third Variance of the Second Order Equation of GQG as,

$$\hbar^2 \frac{\ell_P^\zeta}{m_P^3} \Box_P \psi(\vec{r}_P, t_P) + \mathscr{C}^2 \psi(\vec{r}_P, t_P) = 0\,. \tag{29}$$

Here, $\psi(\vec{r}_P, t_P)$ is definitely a bosonic field. Thus, we can immediately develop a fermionic field (or the 'Four-momentum' Comprised First Order Equation of GQG) out of Eq. (29) as,

$$i\hbar \left(\frac{\ell_P^\zeta}{m_P^3} \right)^{(1/2)} \gamma^\zeta \vec{\nabla}_{P\zeta} \psi(\vec{r}_P, t_P) - \mathscr{C} \psi(\vec{r}_P, t_P) = 0\,, \tag{30}$$

where, γ^ζ are Dirac's gamma matrices. Considering $\left(\ell_P^\zeta\right)^{-1} = m_P$, let Eq. (30) be for Eq. (28) as,

$$\begin{aligned} i\hbar \gamma^\zeta \vec{\nabla}_{P\zeta} \psi(\vec{r}_P, t_P) - m_P^2 \mathscr{C} \psi(\vec{r}_P, t_P) &= 0 \\ \therefore \quad i\hbar \gamma^\zeta \vec{\nabla}_{P\zeta} \psi(\vec{r}_P, t_P) - m_P^2 G^{(1/2)} \psi(\vec{r}_P, t_P) &= 0 \\ \Rightarrow \quad (8\pi)^{(1/2)} i\hbar \gamma^\zeta \vec{\nabla}_{P\zeta} \psi(\vec{r}_P, t_P) - m_P \psi(\vec{r}_P, t_P) &= 0\,, \end{aligned} \tag{31}$$

since $8\pi G = m_P^{-2}$. Now, Eq. (31) is quite handy to use. But Eq. (31) also suggests us that whatever matter satisfies such a fermionic relation is definitely originated (clustered) as matter from fundamentally different physics at the Planck scale, maybe at very different cosmological epochs. Moreover, the first term of the last equation of Eq. (31) is almost five-times larger than any Dirac-like term for baryonic matters, which is quite unusual. At this characteristic Planck scale, the matter that satisfies Eq. (31) must not provide a natural mechanism of the electroweak symmetry breaking, thus the matter must be non-baryonic. The only possible candidate having such characteristics is Dark Matter, which accounts for 26.8% of the critical density in the Universe against 4.9% of the critical density of baryonic matters, in other words, the critical density of Dark Matter is almost $(8\pi)^{(1/2)}$ times higher than the critical density of baryonic matters – it exactly matches with Eq. (31).

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Again, returning to Eq. (25) and using $m^2\left(\mathrm{d}s^2/\mathrm{d}t^2\right)$ of Eq. (19) so as $m^2\left(\mathrm{d}s^2/\mathrm{d}t^2\right)=m^2\,(1-v^2)\equiv m_0^2$ for the rest mass m_0, we can get,

$$\begin{aligned} & m_0^2\,\psi(\vec{r},t) = -\hbar^2\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t) \\ \therefore\quad & \hbar^2\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)+m_0^2\,\psi(\vec{r},t)=0\,. \end{aligned}$$

Then, considering $\vec{\nabla}_\nu=\mathrm{g}_{\mu\nu}\,\vec{\nabla}^\mu$ and d'Alembertian operator $\Box=\vec{\nabla}_\mu\,\vec{\nabla}^\mu$, we have,

$$\hbar^2\,\mathrm{g}_{\mu\nu}^2\,\Box\,\psi(\vec{r},t)+m_0^2\,\psi(\vec{r},t)=0\,.$$

Thus, we can immediately develop a fermionic field equation as,

$$i\hbar\,\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t)-m_0\,\psi(\vec{r},t)=0\,. \tag{32}$$

Here, the real fermions exist only in temporal dimension. Thus, Eq. (32) gives us the Dirac equations for fermions, but with an extension that antifermions, those are exist in spatial dimensions, are thrice in number than real fermions in nature. As the motion in temporal dimension is the basic consideration of relativity, the '+'-ve signature of $\mathrm{g}_{\mu\nu}$ in Eq. (32) explains us the reason of the forward expansion of the Universe in temporal dimension.

Now, let us replace m of Eq. (19) with the Planck mass m_P, when m_P satisfies as [8]: $m_P^2\,\Lambda=\frac{1}{2}\,\langle T\rangle$, where the cosmological constant $\Lambda=8\pi\,G\,\rho_\Lambda$, so as we have $m_P^2\left(\mathrm{d}s^2/\mathrm{d}t^2\right)=m_P^2\,(1-v^2)\equiv m_{P0}^2$ for the Planck rest mass m_{P0}, thus, for,

$$m_P^2\,\Lambda\,(1-v^2)=\frac{1}{2}\,\langle T\rangle\,(1-v^2)\qquad\Longleftrightarrow\qquad m_{P0}^2\,\Lambda=\frac{1}{2}\,\langle T_0\rangle\,,$$

where, $\langle T\rangle\,(1-v^2)\equiv\langle T_0\rangle$, and following the argument cited above,

$$\begin{aligned} \hbar^2\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)+m_{P0}^2\,\psi(\vec{r},t) &= 0 \\ \therefore\quad 2\hbar^2\,\Lambda\,\mathrm{g}_{\mu\nu}^2\,\Box\,\psi(\vec{r},t)+\langle T_0\rangle\,\psi(\vec{r},t) &= 0 \\ \text{and,}\quad \hbar^2\,\Lambda\,\mathrm{g}_{\mu\nu}^2\,\Box\,\psi(\vec{r},t)+\rho_{\Lambda 0}\,\psi(\vec{r},t) &= 0\,, \end{aligned} \tag{33}$$

by replacing $\frac{1}{2}\,\langle T\rangle$ with ρ_Λ. So, for the cosmological constant $\Lambda=8\pi\,G\,\rho_\Lambda=m_P^{-2}\,\rho_\Lambda$ using $8\pi\,G=m_P^{-2}$ and by switching $(1-v^2)^{(1/2)}$ right to left in the term: $m_P\,\langle T_0\rangle^{(1/2)}=m_P\,\langle T\rangle^{(1/2)}\,(1-v^2)^{(1/2)}\equiv m_{P0}\,\langle T\rangle^{(1/2)}$, we can develop a fermionic field equation as follows,

$$\begin{aligned} i\hbar\,\sqrt{2\Lambda}\,\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t)-\langle T_0\rangle^{(1/2)}\,\psi(\vec{r},t) &= 0 \\ \therefore\quad i\hbar\,(2\,\rho_\Lambda)^{(1/2)}\,\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t)-m_{P0}\,\langle T\rangle^{(1/2)}\,\psi(\vec{r},t) &= 0\,. \end{aligned} \tag{34}$$

The interesting thing in Eq. (34) is that Dark Energy has a direct relationship with gravity. In other words, Dark Energy would be obtainable from the breaking of particle symmetry where gravity counts (though, this will not play a role in our present discussion).

The last equation of Eq. (33) is definitely applicable simultaneously whether the matter is baryonic or non-baryonic.

Again, either from the first equation of Eq. (33), or by placing $\frac{1}{2}\,\langle T\rangle=\rho_\Lambda$ in Eq. (34), we can find,

$$i\hbar\,\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t)-m_{P0}\,\psi(\vec{r},t)=0\,, \tag{35}$$

which is the Planck scale counterpart of Eq. (32), in other words, Eq. (32) and Eq. (35) counterbalance each other's actions of the forward expansion of the Universe in temporal dimension due to their $\mathrm{g}_{\mu\nu}$.

3. Conclusion and Discussion

In this work, we have quantized the classical theory of General Relativity and contributing a very natural geometric way, we have wrote a fundamental theory of quantum gravity coupled to matter.

Present physics is unable to provide us a more acceptable scenario of Hilbert-Einstein field equation which is developed in a quantum spacetime. Moreover, it is commonly believed in contemporary physics that gravity is the bending of spacetime, but in this work, Hilbert-Einstein field equations Eq. (16) and Eq. (26) assure us that, *gravity is the bending of spacetime intermediated by gravitons in its quantum gravity field, whose geometric part bends spacetime, whereas its quantum part interacts with the spacetime by exchanging gravitons*.

In our two different aspects of quantum gravity: one Hilbert-Einstein field equation Eq. (16) is developed in a quantum-Riemannian spacetime, whereas, another Hilbert-Einstein field equation Eq. (26) is developed in a purely quantum non-Riemannian spacetime. This is a remarkable achievement of GQG.

Except this work, there has no evidence of bosonic and fermionic fields that provides us Dark Energy and Dark Matter. The 'Four-momentum' Comprised GQG fields give us a complete understanding of Dark Matter through Eq. (31), which will assist us to reveal the properties of Dark Matter in quantum spacetime. Similarly, we have seen that Dark Energy also appears in the same GQG fields quite naturally. In other words, contrary to Einsteinian General Relativity, we can say that LHS of Eq. (16) or Eq. (26) is strongly depended upon different matter fields that exchange different types of vector bosons causing either positive pressure or negative pressure by bending spacetime.

The grate difference between Eq. (31) and Eq. (35) is that the nature of the former one is non-baryonic, whereas, the later one is independent of matter's constructive property, i.e., its effects can be observable simultaneously both in the cases of baryonic and non-baryonic matters. Another difference is that Eq. (31) is effective at m_P scale, whereas, Eq. (35) is effective at m_{P0} scale, i.e., Dark Energy had originated at much earlier cosmological epochs than Dark Matter. Similarly, Dark Matter had originated at much earlier cosmological epochs than baryonic matters of Eq. (32) at m_0 scale. Thus, we have a quite fair chronology of the formation of cosmological matters in the Universe. Note it here that gravity was not observable at the cosmological epochs at m_{P0} scale where Dark Energy had originated. At this scale, gravitons just behaved as energy states rather than real particles due to Remark 4. Gravity was also not observable at the next cosmological epochs started out at m_P scale where Dark Matter had originated. Gravity became observable first time only in the energy zone $E_{\mathrm{IR}} \leq E_\star \ll E_{\mathrm{PLANCK}}$ as we had claimed in Remark 4.

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