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The Normalizer of a Lie Group. Applications and Challenges

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Article

The normalizer of a Lie Group. Applications and Challenges

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Abstract: Let G be a connected Lie group with Lie algebra $\mathbf{g}$. This review is devoted to studying the fundamental dynamic properties of elements in the normalizer $\mathfrak{N}_G$ of G. Through an algebraic characterization of $\mathfrak{N}_G$, we analyze the different dynamics inside the normalizer. $\mathfrak{N}_G$ contains the well-known left-invariant vector fields and the linear and affine vector fields on G. In any case, we show the shape of the solutions of these ordinary differential equations on G. We give examples in low-dimensional Lie groups. It is worth saying that these dynamics generate the linear and bilinear control systems on Euclidean spaces and the invariant and linear control systems on Lie groups. Moreover, the Jouan Equivalence Theorem shows how to extend this theory to control systems on manifolds.

Keywords: lie groups; normalizer; invariant; linear and affine vector fields; algebraic control systems

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1. Introduction

Let G be a connected Lie group with Lie algebra $\mathbf{g}$, and denotes by $\mathbb{X}^\infty(G)$ the Lie algebra of all C^∞ (smooth) vector fields on G. By definition, the normalizer $\mathfrak{N}_G$ of $\mathbf{g}$ is the Lie sub-algebra of $\mathbb{X}^\infty(G)$, which leaves invariant $\mathbf{g}$ under the Lie brackets. Precisely,

$$\mathfrak{N}_G = \{X \in \mathbb{X}^\infty(G) \mid [X, Y] \in \mathbf{g}, \text{ for all } Y \in \mathbf{g}\}.$$

In this review, we start to show a characterization of the normalizer of G, both, when the group is just connected and when G is connected and simply connected. We also mention the relationship between the group with its universal covering group through their corresponding normalizers.

Our first goal is to give basic properties of elements inside $\mathfrak{N}_G$, i.e., the different classes of vector fields in the normalizer, the corresponding associated differential equations, and their solutions. Elements in the normalizer generate well known classes of control systems on Lie groups. As a second goal, we invite the readers to research this area through a challenge and a list of specific related open problems for a general class of control systems in $\mathfrak{N}_G$.

Here, we follows [11]. Assume that group G is connected and simply connected. The normalizer is isomorphic to the semi-direct product of $\mathbf{g}$ with the Lie algebra of all $\mathbf{g}$-derivations, i.e.,

$$\mathfrak{N}_G = \mathbf{g} \otimes_s \partial\mathbf{g}. \tag{1}$$

There are three kinds of dynamics in the normalizer. At the first place, we consider $\mathbf{g}$ as the set of left-invariant vector fields on G.

On the other hand, a vector field $\mathcal{X}$ is called *linear vector field* if its flow $\{\mathcal{X}_t : t \in \mathbb{R}\}$ is a 1-parameter group of $Aut(G)$, the group of G-automorphisms. Associated to $\mathcal{X}$, there exists a $\mathbf{g}$-derivation $\mathcal{D} : \mathbf{g} \to \mathbf{g}$, i.e., a linear transformation which respects the Leibniz rule. Thus, the linear vector field $\mathcal{X}$ is associated to $\mathfrak{N}_G$ through the derivation $\mathcal{D}$.

Finally, a general element in $\mathfrak{N}_G$ reads

$$Y + \mathcal{X}, Y \in \mathbf{g}, \text{and } \mathcal{X} = \mathcal{X}^{\mathcal{D}}, \mathcal{D} \in \partial\mathbf{g}. \tag{2}$$

These general members of $\mathfrak{N}_G$ are called affine vector fields.

Our approach to studying the normalizer came from a generalization of the notion of Linear Control Systems on Euclidean spaces, from $\mathbb{R}^n$, [27], to a connected Lie group G ,[11]. Moreover, it is worth mentioning that the normalizer contains the dynamic of every control system with some algebraic structure-property. It includes the class of linear and bilinear control systems on Euclidean spaces, [15,27] respectively. The class of invariant [16], and linear control systems on Lie groups [11], see also [22]. The first three of them deeply developed from the earlies 60. The linear ones on G was introduced in 1999. Moreover, all these classes are models for real applications [16,17,21,23,28].

Furthermore, the Jouan Equivalence Theorem [19], shows that for any non-linear affine control system on a differential manifold, such that the Lie algebra generated by its vector fields is finite-dimensional, it is equivalent to a linear control system on a Lie group or a homogeneous space.

Equivalent systems share their main properties. Therefore, the knowledge of control systems inside the normalizer can be applied to any non-linear equivalent control system. And mainly analyzed just through numerical analysis or other techniques. Therefore, it is relevant to classify linear control systems on Lie groups for any relevant property of control systems, such as controllability, control sets, and optimality, [4,6–8].

We also mention that $\mathfrak{N}_G$ is related with the notion of Almost Riemannian Structures, [1,2,9].

In Section 2, we describe the tangent bundle to introduce the definition of normalizer. We start with the Euclidean Abelian group $\mathbb{R}^n$. Then, we proceed with a general n-dimensional connected Lie group. Section 3 explains why we decided to introduce $\mathfrak{N}_G$. Then, we show the normalizer's algebraic characterization, which allows us to understand the dynamics inside of this algebraic structure, its vector fields, and the shape of their solutions. Section 4 contains examples of the dynamics of elements in $\mathfrak{N}_G$ on low-dimension nilpotent, solvable, and semi-simple Lie groups. Section 5 recalls some classes of control systems with dynamics inside of $\mathfrak{N}_G$. We establish the Jouan Equivalence Theorem, and concludes with a challenge to start studying the affine control systems generated by general affine vector fields.

For facts on Lie theory and control systems, see [3,16,18,25,26].

2. Preliminaires

Roughly speaking, a vector field on a domain M is defined by the selection of a tangent vector at any state of M. To define this notion we need to introduce the concept of tangent bundle TM of the domain, [13]. At first place, consider the Euclidian space $G = \mathbb{R}^n$, and $x \in \mathbb{R}^n$. The tangent space of $\mathbb{R}^n$ at the state x is defined by the n-dimensional vector space

$$T_x\mathbb{R}^n = Span\left\{ \left(\frac{\partial}{\partial x_i}\right)_x \mid i = 1, 2, \ldots, n \right\}.$$

Where, $e_i = \left(\frac{\partial}{\partial x_i}\right)_0 \in T_0\mathbb{R} = \mathbb{R}^n$, denotes the canonical vector. And, for any $i = 1, 2, .., n$, the vector $\left(\frac{\partial}{\partial x_i}\right)_x$ initializing at the point $x \in \mathbb{R}^n$, denotes the canonical vector translated to the state x.

The tangent bundle $T\mathbb{R}^n$ of $\mathbb{R}^n$ is given by $T\mathbb{R}^n = \cup_{x\in\mathbb{R}^n} T_x\mathbb{R}^n$. Since the translation of $\mathbb{R}^n$ by x generate the full space $T_x\mathbb{R}^n$, it follows that $T\mathbb{R}^n$ is isomorphic to the direct product $\mathbb{R}^n \times \mathbb{R}^n$.

A vector field X on $\mathbb{R}^n$ is determined by the map $X : \mathbb{R}^n \to T\mathbb{R}^n$ through the selection $X(x)$, $x \in \mathbb{R}^n$.

$\mathbb{X}^\infty(\mathbb{R}^n)$ and $\mathbb{C}(\mathbb{R}^n)$, the vector space of all smooth applications from $\mathbb{R}^n$ to $\mathbb{R}^n$, are isomorphics. Any $f : \mathbb{R}^n \to \mathbb{R}^n$ induce the vector field

$$X^f(x) = \sum_{i=1}^{n} f_i(x)\left(\frac{\partial}{\partial x_i}\right)_x \in T_x\mathbb{R}^n, x \in \mathbb{R}^n$$

where $f = (f_1, f_2, \ldots, f_n)$, and reciprocally. Geometrically, $f \to X^f$ is determined by translation of the vector $f(x)$ at the point x, $x \in \mathbb{R}^n$.

In a more general set up, let G be a n-dimensional connected Lie group with Lie algebra

$$\mathbf{g} = Span\left\{Y^1, Y^2, ..., Y^n\right\}, \tag{3}$$

generated by the basis $Y^j, j = 1, 2, ..., n$, as a vector space.

The group G is a differential manifold, actually, an analytical manifold,[26]. The tangent space of G at the point g, is given by

$$T_g G = Span\left\{Y^1(g), Y^2(g), ..., Y^n(g)\right\}. \tag{4}$$

The tangent bundle TG, which is the disjoint union of $T_g G$ with $g \in G$, is also well defined. Actually, $TG = G \times \mathbf{g}$ is a special model for conservative mechanics, involving the parameters $(\text{position, momentum})$.

The triaviallity of TG is essentialy a property of Lie groups. For instance, the tangent bundle of the sphere $\mathbb{S}^2$, i.e., the homogeneous space of the rotational group $SO_3(\mathbb{R})$, is not trivial, since any continuous vector field on the sphere has a singularity. This happens because the Euler characteristic of $\mathbb{S}^2$ is two and non zero. Thus, $T\mathbb{S}^2$ can non be written as a global direct product. However, this property is always locally true.

The notion of Lie algebra depends on the existence of a vector space $\mathbf{g}$ with a Lie bracket bilinear map $[,] : \mathfrak{g} \times \mathbf{g} \to \mathbf{g}$, which must satisfy the following properties. It is skew symmetric, i.e., for any $X, Y \in \mathbf{g}$, $[X, Y] = -[Y, X]$. And, also satisfy the Jacobi identity, i.e., for each triple $X, Y, Z \in \mathbf{g}$,

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0. \tag{5}$$

Recall that for a matrix group, the Lie brackets is nothing more than the usual conmutator, i.e., $[A, B] = AB - BA$.

Finally, we introduce some special Lie algebras, [25].

- $\mathbf{g}$ is said to be *Abelian*, if

$$\mathbf{ad}^1 = [\mathbf{g}, \mathbf{g}] = 0, i.e., X, Y \in \mathbf{g} \Longrightarrow [X, Y] = 0. \tag{6}$$

- $\mathbf{g}$ is called nilpotent, if

$$\exists k \geq 1 : \mathbf{ad}^1 = [\mathbf{g}, \mathbf{g}] \supset \ldots \supset \mathbf{ad}^{k+1} = \left[\mathbf{ad}^k, \mathbf{g}\right] = 0. \tag{7}$$

- $\mathbf{g}$ is say to be solvable, if

$$\exists k \geq 1 : \mathbf{ad}^1 \supset \ldots \supset \mathbf{ad}^{(k)} = \left[\mathbf{ad}^{(k-1)}, \mathbf{ad}^{(k-1)}\right] = 0. \tag{8}$$

- $\mathbf{g}$ is said to be semi-simple if the only solvable ideal is trivial

A vector space $V \subset \mathbf{g}$ is an ideal if $[V, \mathbf{g}] \subset \mathbf{g}$.

Any Abelian Lie algebra is nilpotent and solvable. Each nilpotent Lie algebra is solvable. And, semi-simple Lie algebras goes in a complementary direction.

A Lie group is called Abelian, nilpotent, solvable or semi-simple, if its Lie algebra has the corresponding property, respectively.

For instance, a general Abelian Lie group has the form $G = \mathbb{T}^m \times \mathbb{R}^n$, $m, n \in \mathbb{N}$, where $T^m = \mathbb{S}^1 \times \mathbb{S}^1 \times ... \times \mathbb{S}^1$ is the m-dimensional torus. The 3-dimensional Heisenberg Lie group is nilpotent. The affine group $Aff(2)$ of plane movements is a solvable non nilpotent Lie group. On the other hand, the orthogonal group $SO_n(\mathbb{R})$, and $SL_n(\mathbb{R})$, the matrix group of order n and determinant 1, are semi-simples.

3. The normalizer

In this section we start to show the reason why we decided to introduce the definition of normalizer for any connected Lie group G.

The classical linear control systems on the Euclidean group $\mathbb{R}^n$, reads

$$\Sigma_{\mathbb{R}^n} : \dot{x}(t) = Ax(t) + Bu, u \in \mathcal{U},$$

where $\mathcal{U}$ is the class of admissible piecewise constant control functions, with values in a closed set $\Omega \subset \mathbb{R}^m$. Here, A and B are matrices of order n and $n \times m$ respectively, [27].

A is a linear vector field with flow $\left\{e^{tA} : t \in \mathbb{R}\right\} \subset Aut(\mathbb{R}^n)$. And, for any constant control $u \in \mathcal{U}$, the vector $Bu \in \mathbb{R}^n$ determines a left-invariant vector field. Just observe that $Bu = \sum\limits_{j=1}^{m} u_j b^j$, where $b^1, \ldots, b^m$, are the column vectors of B, and $u = (u_1, ..., u_m)$.

According to this notion, in [11], the authors introduce the following generalization, see also Markus [22].

Definition 1. *A linear control system Σ_G is determined by the family of differential equations,*

$$\Sigma_G : \dot{g(t)} = \mathcal{X}(g(t)) + \sum\limits_{j=1}^{m} u_j(t) Y^j(g(t)), g(t) \in G, u \in \mathcal{U},$$

parametrized by $u \in \mathcal{U}$, as before.

Here, $\mathcal{X}$ is a linear vector field with flow inside of $Aut(G)$. And, for any $j = 1, \ldots, m$, the control vector $Y^j \in \mathbf{g}$ is a left invariant vector field.

Therefore, it is clear that Σ_G is a perfect extension of $\Sigma_{\mathbb{R}^n}$. We introduce,

Definition 2. *Let G be a connected Lie group with Lie algebra* $\mathbf{g}$. *The normalizer $\mathfrak{N}_G$ of $\mathfrak{g}$ is given by*

$$\mathfrak{N}_G = \{X \in \mathbb{X}^\infty(G) \mid [X, Y] \in \mathbf{g}, \text{ for all } Y \in \mathbf{g}\}.$$

In the sequel, we show a characterization of $\mathfrak{N}_G$, both, when the group G is connected and also when G is connected and simply connected. We also show the relationship between the group with its universal covering group through their corresponding normalizers.

In [11] the authors prove that the algebraic structure of $\mathfrak{N}_G$, reads

Theorem 3. *If G is just connected, then $\mathfrak{N}_G \cong \mathbf{g} \otimes_s \mathfrak{aut}(G)$.*
If G is also simply connected, then $\mathfrak{N}_G \cong \mathbf{g} \otimes_s \partial\mathbf{g}$.

Here, $\mathfrak{aut}(G) \subset \partial\mathbf{g}$, denotes the Lie algebra of $Aut(G)$, the Lie group of G-automorphism; $\partial\mathbf{g}$ is the Lie algebra of all $\mathbf{g}$-derivation, and $\otimes_s$ is the semi-direct product between algebras.

Through these isomorphisms any vector field in $\mathfrak{N}_G$ is associated to an element $Y + \mathcal{D}$, with $Y \in \mathbf{g}$, and $\mathcal{D} \in \mathfrak{aut}(G)$. In particular, if $\mathcal{D} = 0$, we get $Y \in \mathbf{g}$ is a left-invariant vector field. Moreover, if $Y = 0$, we obtain a linear vector field $\mathcal{X} = \mathcal{X}^{\mathcal{D}}$ determined by the derivation $\mathcal{D}$. It turns out that, [20],

$$\mathcal{X} \text{ is linear} \Leftrightarrow \mathcal{X} \in \mathfrak{N}_G, \text{ and } \mathcal{X}_e = 0. \tag{9}$$

If G is simply connected, the homomorphism $Aut(G) \to Aut(\mathbf{g})$, which send Φ into its differential map at the identity element $(d\Phi)_e$, is an isomorphism. And, it is well known that the Lie algebra of $Aut(\mathbf{g})$ is $\partial\mathbf{g}$. Thus, it is possible to identify $\partial\mathbf{g}$ with the Lie algebra of the group $Aut(G)$, [26]. Precisely,

$$\mathcal{D} \in \partial\mathbf{g} \Leftrightarrow e^{t\mathcal{D}} \in Aut(\mathbf{g}) \Leftrightarrow \exp(e^{t\mathcal{D}}) \in Aut(G).$$

In this case, $\mathcal{D}$ induces the vector field $\mathcal{X}^{\mathcal{D}}$ with flow,

$$\mathcal{X}_t^{\mathcal{D}}(\exp(Y)) = \exp(e^{t\mathcal{D}}Y), \text{for each } Y \in \mathbf{g}.$$

Next, we follow [12]. Let us denote by $\widetilde{G}$ the universal covering of G. By the standard classification of Lie groups, we know that G is isomorphic to a homogeneous space of $\widetilde{G}$ by a discrete central subgroup Γ of $\widetilde{G}$. Therefore, $Aut(G)$ identifies with a subgroup of $Aut(\widetilde{G})$, which leaves invariant Γ. It turns out that,

$$\mathfrak{aut}(G) = \left\{\mathcal{D} \in \partial\mathbf{g} : \mathcal{X}^{\mathcal{D}}(g) = 0, g \in \Gamma\right\} \subsetneq \partial\mathbf{g},$$

is a subalgebra. Recall that,

$$\mathcal{X}^{\mathcal{D}}(g) = \left(\frac{d}{dt}\right)_{t=0} \mathcal{X}_t^{\mathcal{D}}(g). \tag{10}$$

Since Γ is discrete, any $g \in \Gamma$ is a connected component of $\widetilde{G}$, $\mathcal{X}_t^{\mathcal{D}}(e) = e$, and $\mathcal{X}_t^{\mathcal{D}}$ is a continuous map. Thus, g is a fixed point of $(\mathcal{X}_t^{\mathcal{D}})_{t\in\mathbb{R}}$. So, any $\mathcal{D} \in \mathfrak{aut}(G)$ determines a vector field $\mathcal{X}^{\mathcal{D}}$ on $\widetilde{G}$, which is projected to the homogeneous space $G \cong \widetilde{G}/\Gamma$. However, the converse is not always true.

We end this section giving a naive idea of the size of the normalizer.

Remark 4. *Depending of the structure of the Lie group G, the algebra of the derivation $\partial\mathbf{g}$ can be small, $\partial\mathbf{g} = Inn\mathbf{g}$, when the Lie algebra is semisimple, or big, $\partial\mathbf{g} = End\mathbf{g}$, for instance when $\mathbf{g}$ is Abelian. Thus, the dimension of $\partial\mathbf{g}$ goes from n up to n^2.*

3.1. *The vector fields in the normalizer*

In this section, we show the shape of the dynamics inside the normalizer, i.e., the left-invariant, linear and affine vector fields. We start with the group $G = \mathbb{R}^n$. Any left invariant (or right-invariant) vector field Y_b is just determined by a constant function $f(x) = b \in \mathbb{R}^n$, as follows,

$$Y_b(x) = \sum_{i=1}^{n} b_i \left(\frac{\partial}{\partial x_i}\right)_x \in T_x\mathbb{R}^n. \tag{11}$$

In fact, the vector fields determined by constant functions are invariant by the group $(\mathbb{R}^n, +)$. For each $x \in \mathbb{R}^n$, the Jacobian matrix DL_x corresponding to the translation $L_x : \mathbb{R}^n \to \mathbb{R}^n$, and defined by $L_x(z) = x + z$, is the identity matrix at any $z_0 \in \mathbb{R}^n$. Geometricaly, $DL_x(z_0) : T_{z_0}\mathbb{R}^n \to T_{x+z_0}\mathbb{R}^n$, transform a basis from $T_{z_0}\mathbb{R}^n$ to a corresponding basis of $T_{x+z_0}\mathbb{R}^n$. In particular, $DL_x(0) : T_0\mathbb{R}^n \to T_x\mathbb{R}^n$ determines the left-invariant vector field, $X^b(x) = DL_x(0)(b)$.

Any linear vector field is defined by a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. In other words, linear vector fields on $\mathbb{R}^n$ are in correspondence with the vector space $\mathfrak{gl}_n(\mathbb{R})$, of all real matrix of order n.

Since the Lie algebra of $\mathbb{R}^n$ is the own $\mathbb{R}^n$, and $\mathbb{R}^n$ is an Abelian group, it turns out that any linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is a derivation. Therefore,

$$\mathfrak{N}_{\mathbb{R}} = \mathbb{R}^n \otimes_s \mathfrak{gl}_n(\mathbb{R}). \tag{12}$$

Thus, tipically an affine element of $\mathfrak{n}_{\mathbb{R}}$ has the shape $b + A$.

When G is a Lie group, each $Y \in \mathbf{g}$ determines a left-invariant vector field $\widetilde{Y}$ as follows. For each $g \in G$ consider the automorphism $L_g : G \to G$, defined by $L_g(h) = gh$, and its derivative $(dL_g)_e : T_eG \to T_gG$. By definition, the value of the left-invariant vector field $\widetilde{Y}$ on $g \in G$ reads

$$\widetilde{Y}_g = (dL_g)_e(Y_e) \in T_gG.$$

We denote the vector field $\widetilde{Y}$ just by Y, and Y_e by Y. In particular,

$$T_gG = Span\left\{\left(Y^1\right)_g, \left(Y^2\right)_g, ..., (Y^n)_g\right\} \tag{13}$$

Furthermore, T_gG is a Lie algebra isomorphic to $\mathbf{g}$, [26]. In fact, as a vector fields on the group G, we have

$$\widetilde{X}, \widetilde{Y} \in \mathbf{g} \Longrightarrow \left[\widetilde{X}, \widetilde{Y}\right] \in \mathbf{g}. \tag{14}$$

It follows that $X_e, Y_e \in \mathbf{g} \Longrightarrow [X_e, Y_e] \in \mathbf{g}$. Which gives to the tangent space at the identity element T_eG, a structure of Lie algebra isomorphic to $\mathbf{g}$.

In the sequel we follow [11]. A vector field $\mathcal{X}$ is called a *linear vector field* if its flow $\{\mathcal{X}_t : t \in \mathbb{R}\}$ is a 1-parameter group of $Aut(G)$, the group of G-automorphisms, [11]. Precisely,

$$\mathcal{X}_t(gh) = \mathcal{X}_t(g)\mathcal{X}_t(h), \text{ for all } \quad t \in \mathbb{R}, g, h \in G. \tag{15}$$

Associate to $\mathcal{X}$ there exists a $\mathbf{g}$-derivation, i.e., a linear transformation $\mathcal{D} : \mathbf{g} \to \mathbf{g}$ which respects the Leibniz rule, i.e.,

$$\mathcal{D}[X, Y] = [\mathcal{D}X, Y] + [X, \mathcal{D}Y], \text{ for all } \quad X, Y \in \mathbf{g},$$

The relationship between $\mathcal{X}$ and $\mathcal{D}$ is given by the following identity, see [26],

$$\mathcal{X}_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}}Y), \text{ for all } \quad Y \in \mathbf{g}. \tag{16}$$

Where the exponential map $\exp : \mathbf{g} \to G$ is the usual one, as we explain ahead.

Therefore, an affine element in $\mathfrak{N}_G$ has the shape

$$Y + \mathcal{X}, Y \in \mathbf{g}, \text{ and } \mathcal{X} = \mathcal{X}^{\mathcal{D}}, \mathcal{D} \in \partial\mathbf{g}, \tag{17}$$

3.2. The solutions of elements in the normalizer

In this section, we show the solutions of the ordinary differential equations associated to the different classes of vector fields in the normalizer.

The following analysis is globally valid in Euclidean spaces, and locally true on Lie groups.

Let $X \in \mathrm{X}^\infty(\mathbb{R}^n)$ be a vector field in $\mathbb{R}^n$ determined by the function $f = (f_1, f_2, \ldots, f_n)$ as follows $X(x) = \sum\limits_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}_x$. The differential equation induced by X, reads

$$\dot{x}\,(t) = X(x(t)) = \sum_{i=1}^{n} f_i(x(t)) \frac{\partial}{\partial x_i}_{x(t)} \in T_x\mathbb{R}^n,$$

on $\mathbb{R}^n$. From that we obtain a system of differential equation of first order

$$\dot{x_i}\,(t) = f_i(x(t)) \in \mathbb{R}, i = 1, 2, \ldots, n.$$

By the usual existence and uniqueness of solutions of ordinary differential equations, [13], for any initial condition $x_0 \in \mathbb{R}^n$, there exists a maximal real interval $\mathbb{I}(x_0)$ containing x_0, and an unique solution given by $\gamma(x_0,\cdot) : \mathbb{I}(x_0) \to \mathbb{R}^n$, with

$$\gamma(x_0,t) = (\gamma_1(x_0,t), \gamma_2(x_0,t), \dots, \gamma_n(x_0,t)), \text{ such that}$$
$$\dot{\gamma_i}\,(x_0,t) = f_i(\gamma(x_0,t)), i = 1,2,\dots,n, t \in \mathbb{I}(x_0), \text{ and } \gamma(x_0,0) = x_0.$$

For instance, for $Y_b(x) = \sum\limits_{i=1}^{n} b_i \frac{\partial}{\partial x_i}{}_x \in T_x\mathbb{R}^n$, the solution $\gamma(x_0,\cdot) : \mathbb{R} \to \mathbb{R}^n$ reads as $\gamma(x_0,t) = x_0 + tb$. Geometrically, the lines generated by the value of $Y_b(x)$, $x \in \mathbb{R}$, are parallels.

Any matrix A in $gl_n(\mathbb{R})$ defines in $\mathbb{R}^n$ a linear vector field determing the differential equation $X^A x\,(t) = Ax\,(t)$. The solution with initial condition $x_0 \in \mathbb{R}^n$, reads

$$\gamma\,(t) = e^{tA}x_{0,}t \in \mathbb{R}. \tag{18}$$

Which can be computed directly through the exponential map of matrices

$$e : \mathfrak{g}l_n\,(\mathbb{R}) \to GL_n\,(\mathbb{R})\,, \tag{19}$$

given by the well-known series

$$e^A = \sum_{k \geq 0} \frac{1}{k!}A^k = Id + A + \frac{1}{2!}A^2 + \ldots + \frac{1}{k!}A^k + \ldots, \text{ with } A^0 = Id. \tag{20}$$

Finally, the affine element $b + A \in \mathfrak{n}_{\mathbb{R}} = \mathbb{R}^n \otimes_s \mathfrak{g}l_n(\mathbb{R})$, determines the differential equation

$$\dot{x}\,(t) = Ax\,(t) + b.$$

The solution with initial condition x_0 read as follows

$$\gamma(x_0,t) = X_t^A\,(x_0) = \exp(tA)(x_0 + \int\limits_0^t \exp(-sA)\,b\;ds) \in \mathbb{R}^n. \tag{21}$$

On the other hand, for a connected Lie group G, we show the solution of the differential equations associated to the any element in the normalizer. First, consider $Y \in \mathbf{g}$. As we saw, the value of the left-invariant vector field $\widetilde{Y}$ on $g \in G$, is given by

$$\widetilde{Y}_g = (dL_g)_e(Y_e) \in T_gG.$$

Therefore, the solution with an arbitrary initial condition $g \in G$, is computed through the solution starting at the identity element, [18]. Precisely,

$$Y_t\,(g) = \exp\,(tY)\,g, t \in \mathbb{R}. \tag{22}$$

To be more clear, the map $\exp : \mathbf{g} \to G$, $\exp Y, Y \in \mathbf{g}$, is defined as the solution $\gamma\,(t)$ of the differential equation induced by Y on G, with initial condition e and evaluated at the time $t = 1$, i.e., $\exp Y = \gamma\,(1)\,.$

Recall, a vector field $\mathcal{X}$ on G is said to be *linear* if its flow $(\mathcal{X}_t)_{t \in \mathbb{R}}$ is a 1-parameter subgroup of $Aut(G)$. Associated to $\mathcal{X}$ there is a derivation $\mathcal{D}$ of $\mathbf{g}$ defined by the formula

$$\mathcal{D}Y = [\mathcal{X}, Y](e), \text{ for all } Y \in \mathbf{g}.$$

The relation between $\mathcal{X}_t$ and $\mathcal{D}$ is given by the formula

$$(d\mathcal{X}_t)_e = \mathrm{e}^{t\mathcal{D}}, \text{ for all } t \in \mathbb{R}. \tag{23}$$

From a very well known commutative diagram, [26], we obtain,

$$\mathcal{X}_t(\exp Y) = \exp(\mathrm{e}^{t\mathcal{D}} Y), \text{ for all } t \in \mathbb{R}, Y \in \mathbf{g}.$$

The solution of a linear vector field $\mathcal{X}$ can be computed directly through the exponential map. Since, we consider just connected groups, any element g of G can be described as a product of exponentials. Precisely, there are real numbers $t_1, t_{2,...,}t_m$, and $Y_1, Y_2, ...Y_m \in \mathbf{g}$, such that

$$g = \exp(t_1 Y_1)\exp(t_1 Y_2)...\exp(t_1 Y_m). \tag{24}$$

Then, we apply the homomorphims property of $\mathcal{X}_t$ and the formula in (23), for any element of the product.

When the derivation is inner, which means that there exits $Y \in \mathbf{g}$, such that $\mathcal{D} = -\operatorname{ad}(Y)$. Or more general, when the Lie algebra of G is semi-simple, then any derivation is inner. It follows that the solution comes from conjugation,

$$\mathcal{X}_t\,(g) = \exp(tY)g\exp(-tY), t \in \mathbb{R}, g \in G. \tag{25}$$

We end this section, by introducing an analytical formula appears in reference [11], which gives the shape of the solution of any arbitrary affine vector field $X = Y + \mathcal{X}^{\mathcal{D}}$ in the normalizer $\mathfrak{N}_G$, of a connected arbitrary Lie group G.

Theorem 5. *The analytical solution associated to $X = Y + \mathcal{X}^{\mathcal{D}} \in \mathfrak{N}_G$, reads*

$$X_t(g) = X_t^{\mathcal{D}}(g)\exp(\sum_{n\geq 1}(-1)^{n+1}t^n d_n(Y, \mathcal{D}).$$

Here, for any natural number

$$n \geq 1, \text{ the map } d_n : \mathbf{g} \otimes_{\mathbf{s}} \partial\mathbf{g} \to \mathbf{g} \tag{26}$$

is an homogeneous polynomial of degree n, defined by the recurrence formula

$$(n+1)d_{n+1}(Y, \mathcal{D}) = \frac{1}{2}[Y, d_n(Y, \mathcal{D})] + d_n(Y, \mathcal{D})$$

$$+ \sum_{2\leq 2p\leq n} K_{2p} \sum_{k_1,...,k_{2p}>0,\ k_1+\cdots+k_{2p}=n} [d_{k_1}(Y, D), [\cdots d_{k_{2p}}(Y, \mathcal{D}), Y]\cdots].$$

The coefficients K_2 are rational numbers. Furthermore, the vector field X is complete, which means the associated interval $\mathbb{I}\,(g)$, for any $g \in G$, is $\mathbb{R}$.

Remark 6. *We describe the first polynomials in Theorem 5.*

$$d_1(Y, D) = Y, d_2(Y, D) = \frac{1}{2}D(Y) \tag{27}$$

$$d_3(Y, D) = \frac{1}{12}[Y, D(Y)] + \frac{1}{6}D^2(Y), d_4(Y, D) = \frac{1}{24}[Y, D^2(Y)] + \frac{1}{24}D^3(Y). \tag{28}$$

3.3. *The matrix group case*

In this chapter, we analyze the case when G is a matrix Lie group. Let us consider first $GL_n(\mathbb{R})$, the set of all invertible real matrices of order n. Since $GL_n(\mathbb{R}) = (\det^{-1}(0))^c$ is an open set, it turns out that the tangent space $T_eGL_n(\mathbb{R}) = \mathfrak{gl}_n(\mathbb{R})$, is the vector space of all real matrices of order n.

In fact, for any $P \in GL_n(\mathbb{R})$, the curve $\gamma(t) = e^{tP} \in GL_n(\mathbb{R})$ for any real time t, satisfy: $\gamma(0) = e$ and, $(\frac{d}{dt})_{t=0}\gamma(t) = P$.

Moreover, take $P \in GL_n(\mathbb{R})$ and $A \in \mathfrak{gl}_n(\mathbb{R})$. The differential curve

$$\gamma_P(t) = e^{tA}P, \tag{29}$$

satisfy $\gamma_P(0) = P$, and $(\frac{d}{dt})_{t=0}\gamma_P(t) = AP$.

Therefore, for any $P \in GL_n(\mathbb{R})$, the tangent space of $GL_n(\mathbb{R})$ at the point P, is given by

$$T_PGL_n(\mathbb{R}) = \mathfrak{gl}_n(\mathbb{R})P. \tag{30}$$

Thus, any left-invariant vector field on $GL_n(\mathbb{R})$, is determined by a matrix in $\mathfrak{gl}_n(\mathbb{R})$. Precisely, any matrix $A \in \mathfrak{gl}_n(\mathbb{R})$ induces the matricial differential equation

$$\dot{P}(t) = AP(t), P(t) \in GL_n(\mathbb{R}), t \in \mathbb{R}. \tag{31}$$

According with our previous analysis, the solution with initial condition matricial $P_0 \in GL_n(\mathbb{R})$, is given by $\exp(tA)P_0$.

So, this solution is obtained by the left-translation of P_0 by the solution of

$$\dot{P}(t) = AP(t), \tag{32}$$

through the identity element. On the other hand, the flow $\{\mathcal{X}_t : t \in \mathbb{R}\}$ of a *linear vector field* $\mathcal{X}$ is a 1-parameter group of $Aut(G)$, which is a subgroup of $GL_n(\mathbb{R})$. Furthermore, $\mathcal{X}$ is computed through the following identities,

$$\mathcal{X}_g = \left(\frac{d}{dt}\right)_{t=0} \mathcal{X}_t(g), \mathcal{X}_t(\exp Y) = e^{t\mathcal{D}}Y, \text{and } \mathcal{X}_t(gh) = \mathcal{X}_t(g)\mathcal{X}_t(h). \tag{33}$$

Remark 7. *The same analysis can be done for any matricial Lie subgroup G contained in $GL_n(\mathbb{R})$. In this situation, the Lie algebra will be a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$. And, everything works out as before.*

If the Lie algebra $\mathfrak{g}$ is semi-simple, any derivation $\mathcal{D}$ is inner. It turns out that there exists a matrix $Y = Y(\mathcal{D}) \in \mathfrak{gl}_n(\mathbb{R})$, such that $\mathcal{D} = [, Y]$.

Therefore, $\mathcal{D}$ is easily computed by matrix multiplication,

$$\mathcal{D} = -ad(Y) \Longrightarrow \mathcal{X}_g = gY - Yg, g \in G.$$

Finally, we mention that for the Torus $\mathbb{T}^m$, $Aut(\mathbb{T}^m) = SL(m, Z)$ is a discrete group of determinant 1.

Any linear vector field $\mathcal{X}$ on the Torus is trivial, i.e., $\mathcal{X}_g = 0$. In fact, the 1-parameter group of automorphisms $\{\mathcal{X}_t : t \in \mathbb{R}\}$ is discrete. Because of that, when a linear vector field is involved we never consider $\mathbb{T}^m$.

We end this section by considering two classes of semi-simply Lie groups, where $\partial\mathbf{g}$ coincided with $\mathbf{g}$.

Example 8. *The compact case. The Lie algebra of the rotational group of $\mathbb{R}^3$,*

$$SO_n(\mathbb{R}) = \{P \in O_n(\mathbb{R}) \mid \det(P) = 1\} \tag{34}$$

is defined by

$$\mathfrak{so}_n(\mathbb{R}) = \{A \in \mathbf{gl}_n(\mathbb{R}) \mid A + A^T = 0\}.$$

Here, A^T denotes the transpost of A.

Example 9. *The non compact case. The Lie algebra of the group*

$$SL_n(\mathbb{R}) = \{P \in GL_n(\mathbb{R}) \mid \det(P) = 1\} \tag{35}$$

is given by

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathbf{gl}_n(\mathbb{R}) \mid tr(A) = 0\}.$$

In fact, it is well know that the derivative of the determinant function at the identity element is the trace.

Thus, in both examples the corresponding normalizer is given by

$$\mathfrak{N}_G \cong \mathbf{g} \otimes_s \mathbf{g}. \tag{36}$$

4. Examples on low dimensional Lie groups

In this section we give examples of the dynamic inside of the normalizer of Lie groups of dimension 2 and 3. We establish the group, its Lie algebra, the Lie algebra of derivations, the invariant and linear vector fields, the corresponding differential equations and its solutions. Finally, through Theorem 5, we compute the solution of a ordinary differential equation on the 3-dimensional Heisenberg Lie group.

Example 10. *Here, we follow [4]. Consider the 2-dimensional connected and simply connected solvable Lie group $G = \mathbb{R} \times_\rho \mathbb{R}$, where $\rho_x = e^x$. Under this semi-direct structure, the product in the simply connected Lie group G reads*

$$(x_1, y_1) * (x_2, y_2) = (x_1 + x_2, y_1 + e^{x_1} y_2).$$

The Lie algebra of G is given by the semi-direct product $\mathfrak{g} = \mathbb{R} \times_\theta \mathbb{R}$, where $\theta = e_{\mathbb{R}}$ is the identity. It follows that the bracket in $\mathbf{g}$ reads as

$$[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = (0, \alpha_1\beta_2 - \alpha_2\beta_1). \tag{37}$$

The exponential map is explicitily given by,

$$\exp(a, b) = \begin{cases} (0, b), & \text{if } a = 0 \\ \left(a, \frac{1}{a}(e^a - 1)b\right) & \text{if } a \neq 0 \end{cases}.$$

The structures $Aut(\mathbf{g})$ and $Aut(G)$ are in bijection with the direct product between $\mathbb{R}$ and $\mathbb{R}^*$, as follows

$$P(\alpha, \beta) = (\alpha, a\alpha + b\beta), \psi(x, y) = (x, (e^x - 1)a + by),$$

respectively. Here, $(a, b) \in \mathbb{R} \times \mathbb{R}^*$.

In particular, the algebra of derivations $\partial\mathbf{g}$ has dimension 2..

For the parameters $\alpha, \beta \in \mathbb{R}^2$, the left-invariant vector field reads

$$Y(x, y) = (\alpha, e^x \beta).$$

On the other hand, any linear vector field on G is determined by

$$\mathcal{X}(x, y) = (0, by + (e^x - 1)a). \tag{38}$$

And, the corresponding 1-parameter group of automorphisms of $\mathcal{X}$ is given by

$$\mathcal{X}_t(x,y) = \begin{cases} (x, y + t(e^x - 1)a) & \text{if } b = 0, \\ \left(x, e^{tb}y + \frac{1}{b}(e^{tb} - 1)(e^x - 1)a\right) & \text{if } b \neq 0. \end{cases}$$

It is worth saying that on a homogeneous space of G we obtain a concrete model to analyze a time optimal problem in a 2-dimensional cylinder, [4].

Example 11. *We follow [7]. On $\mathbb{R}^3$ let us consider the canonical basis .*

$$\left\{Y^i = e_i : i = 1, 2, 3\right\}. \tag{39}$$

The semi-direct product $\mathbf{g} = \mathbb{R} \otimes_A \mathbb{R}^2$ induced by the matrix $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, give rises through the general rule

$$[(z_1, v_1), (z_2, v_2)] = (0, z_1 A v_2 - z_2 A v_1), \tag{40}$$

a structure of a solvable non nilpotent 3-dimensional Lie algebra. In fact, a short computation shows that

$$[Y^1, Y^3] = Y^3, \text{and } \left[Y^1, Y^2\right] = 0. \tag{41}$$

The associated connected and simply connected Lie group has the shape,

$$G = \mathbb{R} \otimes_\rho \mathbb{R}^2. \tag{42}$$

Let $Y = (a, w)$ an arbitrary element of the Lie algebra $\mathbf{g}$. It turns out that, the left-invariant vector fields induced by Y is given by the formula

$$Y(t, v) = (a, \rho_t w), (t, v) \in G. \tag{43}$$

where $\rho_t = e^{t\theta}$.

On the other hand, any linear vector field on G, reads as

$$\mathcal{X}(t, v) = (0, \mathcal{D}^* v + \Lambda_t \xi), \tag{44}$$

where, $\mathcal{D}^*$ is defined by the formula $\mathcal{D}(0, v) = (0, \mathcal{D}^* v)$, and

$$\Lambda_t = \begin{pmatrix} t & 0 \\ 0 & e^t - 1 \end{pmatrix} \tag{45}$$

Example 12. *Here, we follow [11]. Through Theorem 5, we compute the solution of a vector field on the 3-dimensional nilpotent Heisenberg Lie group G.*

Let us consider the Lie algebra

$$\mathbf{g} = \mathbb{R}Y^1 + \mathbb{R}Y^2 + \mathbb{R}Y^3, \tag{46}$$

with the rules: all the Lie brackets vaniches except $[Y^1, Y^2] = Y^3$. In particular, $\mathbf{g}$ is nilpotent.

The corresponding connected lie group is $G = \mathbb{R}^3$, with the product

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2).$$

The Lie algebra of derivation reads

$$\partial\mathbf{g}=\left\{\begin{pmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & d_{11}+d_{22}\end{pmatrix} : d_{ij}\in\mathbb{R}\right\}.$$

In this case, the dimension of $\partial\mathbf{g}$ *is* 6 *and* $\mathfrak{N}_G \cong \mathbf{g}\otimes_s \partial\mathbf{g}$ *is* (18)*. We select a left-invariant vector field* Y^2 *and a linear vector field* $\mathcal{X}^{\mathcal{D}}$ *with derivation* $\mathcal{D}$ *such that* $d_{ij}=0$*, except* $d_{1,2}=1$*.*

To compute the solution we denote

$$\zeta\,(t)=\sum_{n\geq 1}(-1)^{n+1}t^n d_n(Y^2,D), X_t(x)=\mathcal{X}_t^{\mathcal{D}}(x)\exp(\zeta\,(t)). \tag{47}$$

It turns out that the 1*-parameter group of automorphism is given by*

$$\mathcal{X}_t^{\mathcal{D}}(x_1,x_2,x_3)=(x_1+x_2t+\frac{1}{2}x_2^2t,x_2,tx_2+x_3). \tag{48}$$

Consider an affine vector field $X=Y^2+\mathcal{X}^{\mathcal{D}}$ *in the normalizer* $\mathfrak{N}_G$*.*

In coordinates, the differential equation induced by X reads as

$$\dot{x}_1=x_2+\frac{1}{2}x_2^2+x_3;\dot{x}_2=1,\dot{x}_3=x_2. \tag{49}$$

Just observe that $\mathcal{D}$ *is nilpotent since* $\mathcal{D}^2=0$*. Therefore, the homogeneous polynomial* d_n *in the series of Theorem* 5*, are nulls for any* $n\geq 4$*.*

The non-null homogeneous polynomial are given by

$$d_1=Y^2,d_2=\frac{1}{2}(Y^1+Y^3),d_3=-\frac{1}{12}Y^1. \tag{50}$$

Therefore,

$$\zeta\,(t)=td_1-t^2d_2+t^3d_3, \tag{51}$$

and

$$\exp\zeta\,(t)=\exp((-\frac{t^3}{12}-\frac{t^2}{2})Y^1+tY^2-\frac{t^2}{2}Y^3). \tag{52}$$

Finally, by applying the exponential rules, the solution of the affine vector field X with initial conition $x=(x_1,x_2,x_3)$ is obtained as follows

$$X_t(x)=(x_1+(x_2+\frac{1}{2}x_2^2+x_3)t+(x_2-\frac{1}{2})t^2-\frac{t^3}{3},x_2+t,tx_2+x_3-\frac{t^2}{2}). \tag{53}$$

5. Control systems on groups. A challenge

In this chapter we show that very well known control system on Lie groups, are strictly related to $\mathfrak{N}_G$. Moreover, we explain how to extend the theory on Lie groups to more general set up. After that, we propose a challenges to research.

According with the algebraic classification of the normalizer $\mathfrak{N}_G$, the following classes of systems are generated by elements in the normalizer.

1. A linear control systems on a Euclidean space, [27], is determined by

$$\dot{x}(t)=A\,(x(t))+\sum_{j=1}^{m}u_j(t)b^j,x(t)\in\mathbb{R}^n,u\in\mathcal{U}, \tag{54}$$

$$G=\mathbb{R}^n, A\in\partial\mathbb{R}^n=\mathfrak{gl}_n(\mathbb{R}),\ b^j\in\mathbf{g}=\mathbb{R}^n. \tag{55}$$

2. A bilinear control system on a Euclidean space [15], is defined by

$$\dot{x}(t)=Ax(t)+\sum_{j=1}^{m}u_i(t)A^jx(t), x(t)\in\mathbb{R}^n, u\in\mathcal{U}, \tag{56}$$

$$G=\mathbb{R}^n,\ A, A^j\in\partial\mathbb{R}^n=\mathfrak{gl}_n(\mathbb{R}). \tag{57}$$

3. An invariant control systems on a Lie group G, [16], is induced by

$$\dot{x}(t)=Y\ (x(t))+\sum_{j=1}^{m}u_i(t)Y^j\ (x(t))\,, x(t)\in G, u\in\mathcal{U}, \tag{58}$$

$$G,\ Y,\ Y^1,\dots,\ Y^m\in\mathbf{g}. \tag{59}$$

4. A linear control systems on a Lie group G, [11], is defined by

$$\dot{x}(t)=\mathcal{X}\ (x(t))+\sum_{j=1}^{m}u_it)Y^j\ (x(t))\,, x(t)\in G, u\in\mathcal{U}, \tag{60}$$

$$G,\ \mathcal{X}\in\partial\mathbf{g},\ Y^1,\dots,\ Y^m\in\mathbf{g}. \tag{61}$$

The following references show that these classes of control systems have been used as a model for many relevant concrete applications, in aerospace, ingeneering, chemistry, biology, medicine, etc. See, [3,14–17,21,23,24,28].

To extend the control system theory from groups to control systems on arbitrary finite dimensional manifolds, we establish the Jouan Equivalence Theorem.

Let M be a smooth finite dimensional differential manifold, and consider an affine control system Σ_M of the form

$$\dot{x}(t)=Z\ (x(t))+\sum_{j=1}^{m}u_i(t)Z^j\ (x(t))\,, x(t)\in M, u\in\mathcal{U}, \tag{62}$$

where $Z, Z^1,\dots,Z^m$ are smooth vector fields on M, and $\mathcal{U}$ as before.

Theorem 13. *An affine control system Σ_M on a manifold M is equivalent by diffeomorphism to a linear control system on a Lie group or a homogeneous space, if and only the vector fields are complete, and*

$$Span_{\mathcal{LA}}\left\{Z, Z^1,\dots,Z^m\right\}<\infty. \tag{63}$$

Therefore, through Theorem 13, it is possible to extend the control theory in $\mathfrak{N}_G$, to a more general control systems set up.

Challenge

Let us consider a general affine control system Σ_{Aff} on the normalizer $\mathfrak{N}_G$, as follows

$$\dot{x}(t)=(\mathcal{X}+Y)\ \ (x(t))+\sum_{j=1}^{m}u_i(t)\left(\mathcal{X}^j+Y^j\right)(x(t))\,, x(t)\in G, u\in\mathcal{U}, \tag{64}$$

with $\mathcal{U}$ the piecewise admissible control functions with values in a closed subset Ω in $\mathbb{R}^m$. Here, $\mathcal{X}+Y, \mathcal{X}^1+Y^1,\dots,\mathcal{X}^m+Y^m$ belongs to $\mathfrak{N}_{G.}$

According to our knowledge, there exists just one published article for the general class Σ_{Aff}, [10]. In the mentioned paper, the authors work on a very particular case of affine and bilinear control systems on a Lie group. However, it is just the beginning. And, we are far from understanding the complexity of Σ_{Aff}.

As usual, the fundamental problems are:

To characterize the controllability property, i.e., the possibility to connect any two arbitrary elements in the group by a finite concatenation of solutions of the system in a positive time.

To study the existence, uniqueness, and topological properties of the so-called control sets, which are special subsets of the group where controllability holds in its interior.

To establish the Pontryagin Maximum Principle and its Hamiltonian equations for time and quadratic optimal problems for Σ_{Af}.

For the class of linear control systems on Lie groups, the reference section shows some relevant results of all three problems. So, it is already a starting point.

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